# Brauer Trees

Sam Qunell

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# Recall from Talk 2:

- We fix a DVR $\mathcal{O}$ of characteristic zero with quotient field $K$ and residue field $k$. We assume as usual that $k$ is a finite field of characteristic $p > 0$ and that $K$ is complete with respect to its valuation
- Let $G$ be a finite group with $p||G|$. We assume $K$ is large enough that the group algebra $KG$ is split semisimple.

# Recall More from Talk 2:

The "$e$" map of the cde triangle was defined as follows:

- For each projective indecomposable $kG$-module $P$, there is a unique projective indecomposable $\mathcal{O}G$-module $P'$ for which $P' \otimes_{\mathcal{O}} k \simeq P$.
- We therefore can define a map $P \to P' \otimes_{\mathcal{O}} K$.
- $e$ is the induced map on (split) Grothendieck groups.

# Recall Even More from Talk 2:

The transpose of the decomposition matrix is a matrix whose columns are indexed by projective indecomposable $kG$-modules and whose rows are indexed by simple $KG$-modules. It is given by $D^{\intercal} := (d_{ij})$ where

$$e([P_j]) = \sum_{[S_i] \in \mathrm{Irr}(KG)} d_{ij}[S_i].$$

# Blocks

- We can also do this blockwise.
- Fix now $B$ a block subalgebra of $kG$ with cyclic defect group.
- This implies, among other things, that $B$ has finitely many indecomposables.

# The Brauer Tree of $B$

The *Brauer Tree* of $B$ is the decorated graph defined as follows.

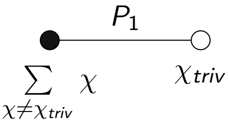
- One edge for each projective indecomposable of $B$.
- Vertices are labelled by (distinct) simple characters of $KG$ for this block. At most one "exceptional" vertex is labelled instead by a sum of (distinct) simple characters.
- The edge for $P_j$ is adjacent to the vertices with labels $X$ and $Y$ if $[P_j] = X + Y$.

# Notes

- It is not obvious that this defines a tree, let alone a graph!
- The simple ordinary characters for the exceptional vertex are called "exceptional characters" and have certain special properties. We will darken the exceptional vertex and leave others hollow.
- We can read the transposed decomposition matrix right off of the tree.

# Example

Take $G = C_p$. There is only one block, and $C_p$ is its defect group.




Indeed, $D^{\intercal} = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}$

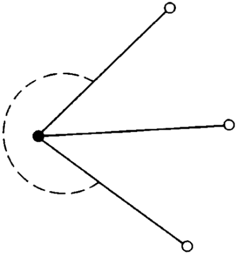
# Cartan Matrices

- Recall the Cartan matrix $C$ which records how many times the simple $kG$ or $B$ modules appear as composition factors of projective indecomposables.
- We saw in talk 2 that $C = D * D^{\intercal}$
- The Brauer tree can therefore be used to compute $C$. We will show that we can say more about the *structure* of projective indecomposables.

# Brauer Trees Again

A *Brauer tree* is a finite, connected, acyclic graph along with the following data.

- At most one vertex is "exceptional" and is labelled by some integer $m \geq 2$.
- For each vertex, we fix a circular ordering of the edges adjacent to that vertex.



# Prerequisites

Fix a $k$-algebra $R$.

- For an $R$-module $M$, the *socle* of $M$, denoted $\mathrm{soc}(M)$, is the submodule generated by the simple submodules of $M$. It is the unique maximal semisimple submodule of $M$.
- A module $M$ is called *uniserial* if its submodules can be totally ordered by inclusion $0 = M_0 \subset M_1 \cdots \subset M_n = M$.

# Brauer Tree Algebras

A *Brauer tree algebra* $R$ is a $k$-algebra whose projective indecomposable modules have structure described by an algorithm determined by some fixed Brauer tree. We first require a few properties of the algebra.

- There is a bijection between simple $R$-modules up to isomorphism and edges in the tree. We label the edges by the corresponding simple.
- For $P_i$ projective indecomposable with corresponding simple $S_i$, we have $S_i \simeq \mathrm{soc}(P_i) \subseteq \mathrm{rad}(P_i)$.

# Brauer Tree Algebra Algorithm

For each projective indecomposable $P_i$, we require that $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is the direct sum of two uniserial modules constructed as follows.

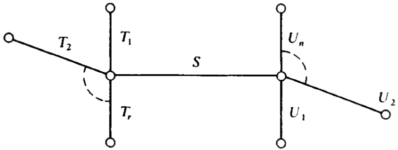
- There are two vertices $v$ and $w$ adjacent to edge $S_i$. There is one summand associated to each vertex, denoted $M_v$ and $M_w$.
- $M_v$ is uniserial with ordered composition factors determined by our circular ordering.
- If $v$ is not exceptional, then we take each edge around $v$ after $S_i$ in order exactly once.
- If $v$ is exceptional with multiplicity $m$, then we do $m - 1$ complete loops around $v$, and then take each edge around $v$ after $S_i$ in order exactly once.

# Brauer Tree Algebra Comments

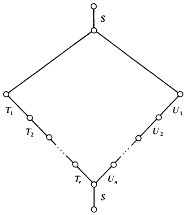
- Either of $M_v$ or $M_w$ could be zero depending on adjacency.
- The circular ordering of edges around a vertex $v$ is usually depicted as the counterclockwise ordering under some fixed planar embedding.

# Example

Sample Brauer tree:




Loewy diagram for $P_S$:



# Results

Theorem: Let $k$ be an algebraically closed field of characteristic $p > 0$, and $G$ a finite group with $p||G|$. If $B$ is a block of $kG$ with cyclic defect group, then $B$ is a Brauer tree algebra.
Moreover, this will be the same Brauer tree described in the previous section, where we match projective indecomposables with their corresponding simples.

## Applications & More

- Actually *all* indecomposables for such a block can be constructed this way and counted. If $m$ is the exceptional multiplicity and $e$ is the number of edges, then there are $e(em+1)$ indecomposables in the block.
- Any indecomposable module in such a block has a socle that has no repeated simple summands.
- Computation of Brauer trees is a big area that is still somewhat active. Raphael has a few papers here.

# p-local Theory

- The goal of p-local representation theory is to understand how much of the representation theory of $G$ is determined by that of its p-local subgroups $N_G(P)$.
- Recall Brauer's first main theorem. For $D$ a $p$-subgroup, there is a bijection $Bl(kN_G(D)|D) \to Bl(kG|D)$ denoted by $B \to B^G$.
- The strongest result we could hope for is a Morita equivalence $B - rep \to B^G - rep$.

# Morita Equivalence

Theorem: Let $G$ and $H$ be two finite groups with blocks $E \in Bl(kG)$ and $B \in Bl(kH)$. If the defect groups of $E$ and $B$ are both cyclic of the same order then the blocks $B$ and $E$ are Morita equivalent iff they have the same Brauer tree.

# Outline

- Any finite-dimensional algebra is Morita equivalent to a basic algebra, i.e. one whose projective indecomposable summands are all nonisomorphic
- Any basic algebra is of the form $kQ/I$ where $Q$ is a quiver and $I$ is an "admissible" ideal. This $Q$ is unique.
- Many classes of finite-dimensional algebras, including hereditary algebras and Brauer graph algebras, can be described in terms of placing restrictions on $Q$ and $I$.
- The corresponding $Q$ and $I$ can actually be determined from the Brauer tree.

# Next Time

It is not always true that $b$ and $b^G$ have the same Brauer trees, so we do not always have Morita equivalence. However, derived representation categories may still be equivalent.

# References I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# References II


- Jay Taylor, *https://www.math.ias.edu/files/broues-conjecture-I.pdf*.
- ———, *https://www.youtube.com/watch?v=oTPpb208pvs*.

[Alp86] [Taya] [Tayb] [Jan69] [Dad66] [Bra41] [Sch17]