2-categorical affine symmetries and q-boson algebras

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Quantum groups

Quantum groups are Hopf algebras deforming the universal enveloping algebra of several classes of Lie algebras.

Representations of quantum groups are useful for many things, including producing solvable lattice models in statistical physics.

What are the symmetries of the symmetries?
Who acts on the actors?

Categorification

Some symmetries are more obvious from the categorical viewpoint.

For us, categorifying an algebra means finding a monoidal category whose Grothendieck group is the desired algebra.

Quantum groups and related structures can be categorified via representations of KLR (quiver Hecke) algebras.

Affinization

For $\mathfrak{g}$ a simple complex Lie algebra, we can obtain the affine Lie algebra $\hat{\mathfrak{g}}$ by taking a central extension ($+$ derivation) of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. It is known $\hat{\mathfrak{g}}$ is a Kac-Moody Lie algebra.

Drinfeld showed that for $\mathfrak{g}$ a Kac-Moody Lie algebra, the algebra $U_q(\mathfrak{g})$ admits an "affinization" $U_q(\hat{\mathfrak{g}})$ given by a loop presentation.

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A categorical understanding of this affinization process or of loop generators is still missing.

Representations of quantum affinizations are increasingly common.

Main Results

- We find new representations of $U_q^+(\hat{\mathfrak{sl}}_n)$ on $U_q^+(\mathfrak{sl}_n)$ by constructing and decategorifying certain 2-representations.
- The (1-)representation also exists at least in types D_4 and C_2.
- We also present new categorifications of related algebras in all symmetrizable Kac-Moody types.

Positive part quantum groups

Notation

Let $(C_{ij})_{i,j\in I}$ be a symmetric generalized Cartan matrix.
Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated to C.
Let q be an indeterminate. Denote by $\binom{n}{k}_q$ the quantum binomial coefficient in q.

Our algebras and categories work for symmetrizable C as well.

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Definition

Define the algebra $U_q^+(\mathfrak{g})$ as the $\mathbb{C}(q)$-algebra with generators E_i for $i \in I$ and quantum Serre relations for $i \neq j$

$$\sum_{k=0}^{1-C_{ij}} \binom{1-C_{ij}}{k}_q (-1)^k E_i^k E_j E_i^{1-C_{ij}-k} = 0$$

Examples of $U_q^+(\mathfrak{g})$

Example

$U_q^+(\mathfrak{sl}_2) \simeq \mathbb{C}(q)[E_1]$, the polynomial algebra in 1 variable.

Example

$U_q^+(\hat{\mathfrak{sl}}_2)$ is generated by two non-commuting variables E_1 and E_0 subject to the relations

$$E_1^3E_0 - (q^2+1+q^{-2})E_1^2E_0E_1 + (q^2+1+q^{-2})E_1E_0E_1^2 - E_0E_1^3 = 0$$

and

$$E_0^3E_1 - (q^2+1+q^{-2})E_0^2E_1E_0 + (q^2+1+q^{-2})E_0E_1E_0^2 - E_1E_0^3 = 0.$$

Adjoints

- $U_q^+(\mathfrak{g})$ carries an important symmetric bilinear form. On the generators, it is given by

$$(E_i, E_j) = \delta_{ij} \frac{1}{1-q^2}.$$

- This form is nondegenerate and gives maps E_i^* adjoint to (right multiplication by) E_i.
- Adjoints are more natural to categorify.

q-boson algebras

Kashiwara shows that the operators E_i and E_j^* satisfy "q-boson relations"

$$E_i^* E_j - q^{-C_{ij}} E_j E_i^* = \frac{\delta_{ij}}{1-q^2}.$$

Definition

Let $B(\mathfrak{g})$ be the *q-boson algebra* for the Kac-Moody Lie algebra $\mathfrak{g}$. It has generators E_i, F_i for $i \in I$. The E_i and F_i both satisfy quantum Serre relations. We also have

$$F_i E_j - q^{-C_{ij}} E_j F_i = \frac{\delta_{ij}}{1-q^2}.$$

The action of $B(\mathfrak{g})$ on $U_q^+(\mathfrak{g})$ is faithful.

Evaluation homomorphisms

- There are natural evaluation homomorphisms $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \to \mathfrak{g}$.
- Jimbo gives quantum evaluation homomorphisms $U_q(\hat{\mathfrak{sl}}_n) \to U_q(\mathfrak{gl}_n)$.
- These quantum evaluation homomorphisms cannot be defined outside of type A_n.
- We will be primarily interested in finding a homomorphism $U_q^+(\hat{\mathfrak{g}}) \to B(\mathfrak{g})$ and resulting action on $U_q^+(\mathfrak{g})$.

Half 2-Kac-Moody algebra

We need a monoidal category that categorifies $U_q^+(\mathfrak{g})$.
We will construct one in several steps.

Definition summary

For C a symmetric generalized Cartan matrix with Kac-Moody Lie algebra $\mathfrak{g}$, define the strict monoidal additive category $\mathcal{U}'^+(\mathfrak{g})$ generated by objects E_i for $i \in I$ and by morphisms $X_i : E_i \to E_i$ and $T_{ij} : E_i E_j \to E_j E_i$.
These morphisms are required to satisfy several KLR (quiver Hecke) relations that depend on C.

The X_i are like polynomial variables. The T_{ij} satisfy variants of the braid relations.

$\mathbb{Z}$-grading

Definition

Denote $\mathcal{U}_q^+(\mathfrak{g}) := Kar(\mathcal{U}'^+(\mathfrak{g})) - gr$. By definition, $\mathcal{U}_q^+(\mathfrak{g})$ is idempotent-closed and $\mathbb{Z}$-graded. Denote by q the shift functor. Morphisms are graded via $\deg(X_i) = 2$, $\deg(T_{ij}) = -C_{ij}$, and $\deg(\mathrm{Id}_M) = 0$ for any object M.

For example, for each $m \in \mathbb{Z}$ we have morphisms

$$X_i : q^{m+2}E_i \to q^m E_i.$$

Shifting by q in $\mathcal{U}_q^+(\mathfrak{g})$ categorifies multiplying by q in $U_q^+(\mathfrak{g})$.

Categorification

Theorem (Khovanov-Lauda)

$U_q^+(\mathfrak{g}) \simeq \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{U}_q^+(\mathfrak{g}))$. *This is an isomorphism of algebras.*

Categorification

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$U_q^+(\mathfrak{g}) \simeq \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{U}_q^+(\mathfrak{g}))$. *This is an isomorphism of algebras.*

- There is a $\mathbb{C}(q)$-valued q-semilinear form on K_0 defined by

$$([M],[N]) = \sum_{n \in \mathbb{Z}} q^n \dim(\mathrm{Hom}(q^n(M), N)).$$

- If we apply the duality $\bar{q} := q^{-1}$, $\bar{E}_i = E_i$ to the first argument, then we recover the previous symmetric bilinear form on $U_q^+(\mathfrak{g})$.

2-representations

Definition

A *2-representation* of $U_q^+(\mathfrak{g})$ or $\mathcal{U}_q^+(\mathfrak{g})$ is an additive graded strict-monoidal functor $\mathcal{U}_q^+(\mathfrak{g}) \to End_{\oplus,\mathbb{Z}}(\mathcal{D})$ for some additive $\mathbb{Z}$-graded category $\mathcal{D}$.

These yield (1-)representations of $U_q^+(\mathfrak{g})$ on $\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{D})$.

Example

$\mathcal{U}_q^+(\mathfrak{g})$ comes with endofunctors for tensoring on the right by E_i. This categorifies the right-multiplication representation of $U_q^+(\mathfrak{g})$.

Extending coefficients

We would like adjoints F_i to the E_i, but $\mathcal{U}_q^+(\mathfrak{g})$ is "too small".

Definition

We say that a coproduct of $\mathcal{U}_q^+(\mathfrak{g})$ objects is *locally finite* if it has the form

$$\coprod_{i\in\mathbb{Z}} q^i(M_1^{\oplus k_{1,i}} \oplus M_2^{\oplus k_{2,i}} \cdots \oplus M_n^{\oplus k_{n,i}})$$

for some $M_j \in \mathcal{U}_q^+(\mathfrak{g})$ and $k_{j,i} \in \mathbb{N}$. We say also that the coproduct is *left-bounded* if there exists some $m \in \mathbb{Z}$ for which $k_{j,i} = 0$ for all j whenever $i < m$.

Definition

Denote by $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$ the category of all left-bounded, locally finite coproducts (in fact, biproducts) of $\mathcal{U}_q^+(\mathfrak{g})$ objects.

Categorical q-boson relations

The functor for tensoring on the right by E_i does have a right adjoint F_i defined on $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$.

Theorem (Kang-Kashiwara)

There is an isomorphism of functors

$$F_i E_j \simeq q^{-C_{ij}} E_j F_i \oplus \delta_{ij} \bigoplus_{n \in \mathbb{N}} q^{2n} Id$$

This categorifies the q-boson relations.
Note that F_i is only right adjoint to E_i.

$U_q^+(\hat{\mathfrak{sl}}_2)$ 2-representation

Rouquier discovered a 2-representation of $\mathcal{U}_q^+(\hat{\mathfrak{sl}}_2)$ on $\mathcal{U}_q^+(\mathfrak{sl}_2)^{lblf}$ extending that of $\mathcal{U}_q^+(\mathfrak{sl}_2)$. Here, E_1 acts by right tensoring by E_1, and E_0 acts by the adjoint F_1.

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Can this be generalized to other types? It is not easy to determine the action of E_0 in general.

Objective

To produce a 2-representation of $\mathcal{U}_q^+(\hat{\mathfrak{g}})$ extending the right multiplication 2-representation of $\mathcal{U}_q^+(\mathfrak{g})$, we need the following.

- Additive, graded endofunctor E_0 of $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$.
- Natural transformations of appropriate grading

$$X_0 : E_0 \to E_0,$$
$$T_{00} : E_0 E_0 \to E_0 E_0,$$

and for each i a vertex for the Dynkin diagram of $\mathfrak{g}$,

$$T_{0i} : E_0 E_i \to E_i E_0,$$
$$T_{i0} : E_i E_0 \to E_0 E_i$$

satisfying the KLR relations in $\mathcal{U}_q^+(\hat{\mathfrak{g}})$.

Jimbo's evaluation homomorphism

- In Jimbo's evaluation homomorphisms $U_q(\hat{\mathfrak{sl}}_{n+1}) \to U_q(\mathfrak{gl}_{n+1})$, we map

$$E_i \to E_i,$$
$$E_0 \to K \cdot [F_n, [F_{n-1}, \dots [F_2, F_1]_q \dots]_q]_q.$$

where K is some semisimple element and $[A, B]_q = AB - qBA$.

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  where K is some semisimple element and $[A, B]_q = AB - qBA$.

- We show that there is a similar homomorphism $U_q^+(\hat{\mathfrak{sl}}_{n+1}) \to B(\mathfrak{sl}_{n+1})$ given by

$$\begin{aligned} E_i &\to E_i, \\ E_0 &\to [F_n, [F_{n-1}, \ldots [F_2, F_1]_q \ldots]_q]_q. \end{aligned}$$

E_0 categorification example

As an example, let $\mathfrak{g} = \mathfrak{sl}_3$. How can we categorify $[F_2, F_1]_q = F_2 F_1 - q F_1 F_2$?

We look for a natural transformation from $F_1 F_2$ to $F_2 F_1$ of degree 1. The adjoint of T_{12} works.

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We will take the cokernel in the larger category of bimodules of KLR algebras.

E_0 functor generally

For $n \geq 2$ and $\mathfrak{g} = \mathfrak{sl}_{n+1}$, we will iterate this procedure of taking cokernels of appropriately graded morphisms. This gives us a functor E_0 defined *a priori* on a larger category.

Theorem (Q.)

The E_0 functor obtained this way is well-defined on $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$. Its action on $\mathbb{C}((q)) \otimes_{\mathbb{Z}[[q]][q^{-1}]} K_0(\mathcal{U}_q^+(\mathfrak{g})^{lblf})$ is

$$[E_n^*, [E_{n-1}^*, \dots [E_2^*, E_1^*]_q \dots]_q]_q.$$

Main result 1

The needed natural transformations are all descended from X_i, T_{ij}, or from Kang-Kashiwara's q-boson isomorphism.

Theorem (Q.)

For $n \geq 2$, the described E_0, X_{00}, T_{00}, T_{0i}, and T_{i0} give a 2-representation of $\mathcal{U}_q^+(\hat{\mathfrak{sl}}_{n+1})$ extending the right-multiplication 2-representation of $\mathcal{U}_q^+(\mathfrak{sl}_{n+1})$ on $\mathcal{U}_q^+(\mathfrak{sl}_{n+1})^{lblf}$.

Slight modifications are made in the $\mathfrak{sl}_2$ case due to double arrows in the affine Dynkin diagram.

q-boson category

The previous results suggest the existence and importance of a (new) categorification of $B(\mathfrak{g})$.

We produce a generators-and-relations categorification $\mathcal{B}(\mathfrak{g})$ akin to $\mathcal{U}_q^+(\mathfrak{g})$. This makes it easier to define 2-representations, compute Hom spaces, and generalize.

Our construction works for any symmetrizable Kac-Moody Lie algebra.

Main result 2

Lemma

The universal property of $\mathcal{B}(\mathfrak{g})$ is as follows. A 2-representation of $\mathcal{B}(\mathfrak{g})$ is a 2-representation of $\mathcal{U}_q^+(\mathfrak{g})$ such that

- *The image of each E_i has a right dual F_i*
- *The image admits all left-bounded, locally finite direct sums*
- *The q-boson morphisms (to be defined) are invertible*

The category $\mathcal{B}(\mathfrak{g})$ acts on $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$ with objects E_i acting by $\otimes E_i$ and F_i acting by the right adjoint.

Theorem (Q.)

$\mathcal{B}(\mathfrak{g})$ acts faithfully on $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$.

Generators

- First, we define a monoidal category with shift functor q. The generating objects are E_i and F_i. We have morphisms X_i and T_{ij} just like in $\mathcal{U}_q^+(\mathfrak{g})$.
- We also require that F_i is right dual to E_i, so we have units $\eta_i : \mathbf{1} \to F_i E_i$ and counits $\epsilon_i : E_i F_i \to \mathbf{1}$.
- Diagrammatic techniques can be used to control the size of Hom spaces.

The q-boson morphism

- Recall there is an isomorphism of functors on $\mathcal{U}_q^+(\mathfrak{g})^{lblf}$

$$q^{-C_{ij}} E_j F_i \oplus \delta_{ij} \bigoplus_{n \in \mathbb{N}} q^{2n} \mathbf{1} \to F_i E_j.$$

- This "q-boson morphism" is

$$F_j E_i \epsilon_j \circ F_j T_{ji} F_j \circ q^{-C_{ij}} \eta_j E_i F_j \oplus \delta_{ij} \bigoplus_{n \in \mathbb{N}} F_i X_i^n \circ \eta_i$$

- Very similar morphisms are formally inverted in Rouquier's categorification of the (idempotented) full quantum group $U_q(\mathfrak{g})$.

Other additions

- We add all left-bounded locally finite coproducts (in fact, direct sums).
- We then localize at a class of morphisms containing these q-boson morphisms to obtain our desired category $\mathcal{B}(\mathfrak{g})$. This class of morphisms admits a calculus of right fractions.
- For decategorification, we also need to take an idempotent completion.

Theorem (Q.)

$$\mathbb{C}((q)) \otimes_{\mathbb{C}(q)} B(\mathfrak{g}) \simeq \mathbb{C}((q)) \otimes_{\mathbb{Z}[[q]][q^{-1}]} K_0(Kar(\mathcal{B}(\mathfrak{g}))).$$

Applications

- A quotient of $U_q^+(\mathfrak{sl}_{n+1})$ by a maximal $U_q(\hat{\mathfrak{b}})$-submodule is isomorphic to a *prefundamental representation*. This gives new proofs of the prefundamental character formulas.
- Main result 2 yields interesting bases of $B(\mathfrak{g})$ and of the corresponding *bosonic extension*.
- Main result 2 yields a graphical description of the bilinear form on $B(\mathfrak{g})$.

Next Steps

- Main result 1 suggests a possible categorification of prefundamental representations.
- We expect that there is a functor $\mathcal{U}_q^+(\hat{\mathfrak{sl}}_{n+1}) \to Ho(\mathcal{B}(\mathfrak{sl}_{n+1}))$ coming from main result 1.
- Current work includes adding all duals to $\mathcal{U}_q^+(\mathfrak{g})$ and investigating connections to the quantum Grothendieck ring of $U_q(\mathcal{L}(\mathfrak{g}))$.
- We are also investigating applications to the crystal basis of $U_q^+(\mathfrak{g})$.

Thank you for listening!

Main results:

- For $n \geq 2$, we construct a 2-representation of $U_q^+(\hat{\mathfrak{sl}}_n)$ on $U_q^+(\mathfrak{sl}_n)$. The representation of algebras is new for $n \geq 3$.
- We also construct a representation of $U_q^+(\hat{\mathfrak{so}}_8)$ on $U_q^+(\mathfrak{so}_8)$ and of $U_q^+(\hat{\mathfrak{sp}}_4)$ on $U_q^+(\mathfrak{sp}_4)$.
- We construct a generators-and-relations monoidal categorification of the q-boson algebras $B(\mathfrak{g})$ for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. It has a faithful 2-representation on a categorification of $U_q^+(\mathfrak{g})$.