Math 115AH Homework #2

**Problem 1.** Let $V \neq 0$ be a vector space over $F$. Suppose that $V$ can be spanned by one vector. Show that any subspace $W$ of $V$ is either the zero subspace or all of $V$. Discuss what can happen if $V$ can be spanned by two vectors.

**Problem 2.** Let $V \neq 0$ be a vector space over $F$. Suppose that $W$ and $X$ are subspaces of $V$. Is $W \cup X$ always a subspace of $V$? Prove if true and give a counterexample if not. If it is not true find a condition that will guarantee it to be true. Prove your assertion.

**Definition.** Let $V$ and $W$ be vector spaces over $F$. A map $T : V \to W$ is called **linear** or a **linear transformation** if for all $v_1, v_2 \in V$ and for all $\alpha \in F$, we have

$$T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2).$$

[We usually write $Tv$ for $T(v)$ (out of laziness).]

If $T : V \to W$ is a linear transformation, let

$$\ker(T) = N(T) := \{v \in V \mid Tv = 0\}$$

called the **kernel** or **null space** of $T$.

**Example.** Let $A$ be an $m \times n$ matrix in $F^{m \times n}$. Then $A : F^n \to F^m$ by $v \mapsto Av$ (matrix multiplication) is a linear transformation, by rules of matrix addition and multiplication.

**Problem 3.** Let $V$ and $W$ be vector spaces over $F$. Let $T : V \to W$ be a linear transformation. Prove all of the following:

a. $\ker(T) \subset V$ and $\im(T) \subset W$ are subspaces.

b. Suppose that $\ker(T) = N(T) = \{0\}$. Suppose that $\{v_1, \ldots, v_k\}$ is a linearly independent subset of $V$. Prove that $\{Tv_1, \ldots, Tv_k\}$ is linearly independent in $W$.

c. Let $A$ be an $m \times n$ matrix in $F^{m \times n}$. Suppose that $\ker(A) = N(A) = \{0\}$. Let $v_1, \ldots, v_k \in F^n$ be linearly independent. Then $Av_1, \ldots, Av_k$ are linearly independent in $F^m$.

d. Let $A$ be an invertible $n \times n$ matrix in $M_n(F)$. Let $v_1, \ldots, v_k \in F^n$ be linearly independent. Then $Av_1, \ldots, Av_k$ are also linearly independent in $F^n$. What would you conjecture if $k = n$. Can you prove your conjecture?

**Problem 4.** Let $V$ be a vector space over $F$ and $v_1, \ldots, v_n \in V$. Suppose that $v_1 \in \langle v_2, \ldots, v_n \rangle$. Then $\langle v_1, \ldots, v_n \rangle = \langle v_2, \ldots, v_n \rangle$. Moreover, $v_1, \ldots, v_n$ are linearly dependent (assuming they are distinct).

**Problem 5.** Suppose that $\alpha_1, \ldots, \alpha_n$ are distinct real numbers. Prove that the functions $e^{\alpha_1 x}, \ldots, e^{\alpha_n x}$ are linearly independent.

**Problem 6,7.** Do two problems from Section 2.2 p. 39 that you did not do for HW #1

**Problem 8,9.** Do two problems from Section 2.3 p. 48

**Problem 10.** Do one problem from Section 2.6 p.66