Commutative Algebra Problems

Prove the following

1. If $\mathfrak{A} < R$ is an ideal then $V(\mathfrak{A})$ is irreducible if and only if $\sqrt{\mathfrak{A}} \in \text{Spec} R$. In particular, $\text{Spec} R$ is irreducible if and only if nil $R \in \text{Spec} R$.

2. $\text{Spec} R$ is the disjoint union of two closed sets if and only if there exist an idempotent $e \neq 0, 1$ in $R$. In particular, if $R = (R, \mathfrak{m})$ is local then $\text{Spec} R$ is connected.

3. $V(p) \subset \text{Spec} R$ with $p \in \text{Min} R$ are the maximal irreducible subspaces (called irreducible components of $\text{Spec} R$). Then $\text{Spec} R = \bigcup_{\text{Min} R} V(p)$. If $R$ is noetherian then $|\text{Min} R| < \infty$.

4. Let $\phi : A \to B$ be a ring homomorphism. Let $\mathfrak{p} \in \text{Spec} B$ and $p = a\phi^{-1}(\mathfrak{p})$. Let $\phi_{\mathfrak{p}} : A_p \to B_{\mathfrak{p}}$ the induced ring homomorphism given by $\frac{a}{b} \to \frac{\phi(a)}{\phi(b)}$. Show that the following are equivalent:
   (a) $\phi$ is flat.
   (b) $\phi_{\mathfrak{p}}$ is flat for all $\mathfrak{p} \in \text{Spec} B$.
   (c) $\phi_{\mathfrak{m}}$ is flat for all $\mathfrak{m} \in \text{Max} B$.

5. If $A \subset B$ are domains with the same quotient field $F$ then the inclusion map $i$ is faithfully flat if and only if $A = B$. Suppose that $A < F$. Show that $F$ is $A$-flat but not $A$-projective. [You can use that a projective module over a local ring is free.]

6. Let $(B_{ij}, \psi_{ij})$ be a direct system of rings and $B$ the direct limit. For each $i$ let $\phi_i : A \to B_i$ be a ring homomorphism such that $\psi_{ij} \circ \phi_i = \phi_j$ whenever $i \leq j$ (i.e., a direct system of $A$-algebras). Then the $\phi_i$ induce $\phi : A \to B$. Show that
   $$a\phi(\text{Spec} B) = \bigcap_i a\phi_i(\text{Spec} B_i).$$
   [Let $p \in \text{Spec} A$. Then, using tensor products and direct limits commute, $a\phi^{-1}(p)$ is the spectrum of
   $$B \otimes_A k(p) \cong \lim_{\to} (B_i \otimes_A k(p))$$
   where $k(p)$ is the quotient field of $A/p$. Show $a\phi^{-1}(p) = \emptyset$ if and only if $B_i \otimes_A k(p) = 0$ for some $i$, i.e., if and only if $a\phi_i^{-1}(p) = \emptyset$.]

7. Let $\phi : A \to B$ be a ring homomorphism. Consider the following three statements:
   I. $a\phi$ is a closed map.
   II. $\phi$ satisfies GU.
   III. If $\mathfrak{P} \in \text{Spec} B$ then the induced map $\bar{\phi} : A/a\phi(\mathfrak{P}) \to B/\mathfrak{P}$ satisfies LO.
Then show:
(a) (I) ⇒ (II) ⇔ (III).
(b) If $A$ is noetherian and $B$ is of finite type over $A$ then (II) ⇒ (I).

8. Let $\varphi : A \to B$ be a ring homomorphism. Consider the following three statements:
I. $^a \varphi$ is an open map.
II. $\varphi$ satisfies GD.
III. If $\mathfrak{P} \in \text{Spec } B$ then the induced map $\bar{\varphi} : A_{\varphi(\mathfrak{P})} \to B_{\mathfrak{P}}$ satisfies LO.
Then show (I) ⇒ (II) ⇔ (III).

[To prove (I) ⇒ (III), observe that $B_q$ is the direct limit of the rings $B_s$ where $q \in D(s)$. By Exercise 6,

$$^a \varphi(\text{Spec } B_q) = \bigcap_{s \in D(q)} ^a \varphi_s(\text{Spec } B_s) = \bigcap_{s \in D(q)} ^a \varphi_s(Y_s),$$

where $Y = \text{Spec } B$ and $Y_s = \text{Spec } B_s$ with $\varphi_s$ the composition of $\varphi$ and the obvious localization map. Since $Y_s$ is an open neighborhood of $q$ in $Y$ and since $^a \varphi$ is open, it follows that $^a \varphi(Y_s)$ is an open neighborhood of $p$ in $X = \text{Spec } A$ and therefore contains $\text{Spec } A_p$.]

It is also true that if $A$ is noetherian and $\varphi$ of finite type, then (II) ⇔ (I). [Cf. Atiyah-MacDonald, Exercises 21-24 p. 87.]

9. Let $\varphi : B \to B'$ be an integral homomorphism of $A$-algebras. If $C$ is an $A$-algebra show that $\varphi \otimes 1_C : B \otimes_A C \to B' \otimes_A C$ is integral.

10. Let $A$ be a normal domain, $F$ its quotient field. Let $L/F$ be a finite galois extension. Let $\mathfrak{P} \in \text{Spec } A_L$ lie over $0 \neq p \in \text{Spec } A$. Let

$$G_{\mathfrak{P}} := \{ \sigma \in G(L/F) \mid \sigma(\mathfrak{P}) = \mathfrak{P} \}$$

be the Decomposition Group of $\mathfrak{P}$. Show
(a) The natural map $(A/p)_p \to (A_L \cap \mathfrak{P} / (\mathfrak{P} \cap L^G))/p$ is an isomorphism.
(b) The quotient field of $L(\mathfrak{P})$ of $A_L/\mathfrak{P}$ is a normal extension of the quotient field $F(p)$ of $A/p$.
(c) If $\bar{\sigma} : A_L \to A_L/\mathfrak{P}$ is the canonical map then we have a group epimorphism

$$G_{\mathfrak{P}} \to G(L(\mathfrak{P})/F(p)) \text{ by } \sigma \mapsto \bar{\sigma}$$

where $\bar{\sigma} : \mathfrak{P} \to \mathfrak{P} / \mathfrak{P}\sigma(\mathfrak{P}) = \mathfrak{P} \sigma(\mathfrak{P})$.
11. Let $M$ be a finitely generated $R$-module and $S \subset R$ be a multiplicative set. Then $\text{ann}_{S^{-1}R} S^{-1}M = S^{-1} \text{ann}_R M$.

12. Let $R$ be a domain with quotient field $F$. Suppose for all rings $R \subset A \subset F$ that $A$ is noetherian. Show that $\dim R \leq 1$.

13. Let $\varphi : A \to B$ be a ring homomorphism and $\mathfrak{p}$ a prime ideal in $A$. Show that the map

\[(^a \varphi)^{-1}(\mathfrak{p}) \to \text{Spec}(B \otimes_A k(\mathfrak{p})) \quad \text{by} \quad \mathfrak{q} \mapsto \mathfrak{q}^* := \mathfrak{q}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}\]

is a homeomorphism and

\[(B \otimes_A k(\mathfrak{p}))_{\mathfrak{q}^*} = B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}\]

14. Prove that any regular local ring is a domain.

15. Let $A \subset B$ be noetherian domains with $B$ a finitely generated $A$-algebra. Let $\mathfrak{q} \in \text{Spec} B$ and $\mathfrak{p} = A \cap \mathfrak{q}$. Show

\[\dim B_{\mathfrak{q}} + \text{tr deg}_{A/\mathfrak{p}} B/\mathfrak{q} \leq \dim A_{\mathfrak{p}} + \text{tr deg}_A B.\]