Non-commutative Algebra Problems

1. If V is a finite dimensional vector space over F, show that V is an irreducible $\mathrm{End}_F(V)$-module.

2. Let R be a CR ring with basic set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ and $B_1, \ldots, B_n$ its simple components where $B_i = \sum_{\mathfrak{A} \cong \mathfrak{A}_i} \mathfrak{A}$. Show if $0 \neq M$ is an R-module then $B_i M$ is a sum of irreducible submodules of M all isomorphic to $\mathfrak{A}_i$ and further that $M = \oplus B_i M$.

3. Let F be a field and G be a finite group whose order is divisible by $\operatorname{char} F = p > 0$. Show that $F[G]$ is not CR.

4. Let G be an infinite group and F a field. Show that $F[G]$ is not CR.

5. Show if G is the symmetric group S_3 that $\mathbf{Q}[G] \cong \mathbf{Q} \times \mathbf{Q} \times \mathbf{M}_2(\mathbf{Q})$.

6. Let G be a finite group and F an algebraically closed field of characteristic zero or p with p not dividing the order of G. Show that the number of linear (i.e., degree one) representations of G is the order of G^{ab}. In particular, G is abelian if and only if every irreducible character of G is linear.

7. Determine all irreducible representations of the quaternion group of order 8 over $\mathbf{Q}$ and over $\mathbf{C}$.

8. Let G be a finite group, F a field such that $\operatorname{char} F = 0$ or does not divide the order of G, $F[G]$ F-split, $\mathfrak{A}$ an irreducible $F[G]$-module, $B_{\mathfrak{A}}$ the simple component corresponding to $\mathfrak{A}$, and $f = 1_{B_{\mathfrak{A}}}$. Show that

$$f = \frac{\chi_{\mathfrak{A}}(1)}{|G|} \sum_G \chi_{\mathfrak{A}}(x^{-1})x.$$