Commutative Algebra Problems

Let R be a commutative ring. Prove all of the following.

1. $R[[t]]$ is noetherian if and only if R is noetherian.
2. R is noetherian if every prime ideal is finitely generated.
3. Let P be a finitely generated free R-module of rank n and M a submodule of P. Show if M is R-free that the rank of M is at most n.
4. Let $\mathfrak{A} \subset R[t]$ be an ideal, $^{-} : R[t] \to R[t]/\mathfrak{A}$ be the canonical map, $S := R[t]/\mathfrak{A}$ and $s = \bar{t}$.
   a. $S \in \mathfrak{M}_R$ is generated by $\leq n$ elements if and only if there exists a monic $f \in \mathfrak{A}$ of degree $\leq n$. In this case, $S = \sum_{i=0}^{n-1} Rs^i$. In particular, $S \in \mathfrak{M}_R$ is finitely generated if and only if $\mathfrak{A}$ contains a monic polynomial.
   b. $S \in \mathfrak{M}_R$ is finitely generated and free if and only if $\mathfrak{A} = (f)$ for some monic $f \in R[t]$. If this is the case then there exists an n such that $\mathcal{B} := \{1, s, \ldots, s^{n-1}\}$ is an R-basis for S.
5. Let $d \in \mathbf{Z}$ be square-free and $L = \mathbf{Q}(\sqrt{d})$.
   a. If $d \equiv 2, 3 \bmod 4$ then $\mathbf{Z}_L = \mathbf{Z}[\sqrt{d}]$ and $\{1, \sqrt{d}\}$ is an integral basis.
   b. If $d \equiv 1 \bmod 4$ then $\mathbf{Z}_L = \mathbf{Z}\left[\frac{-1+\sqrt{d}}{2}\right]$ and $\left\{1, \frac{-1+\sqrt{d}}{2}\right\}$ is an integral basis.
6. Let $(R, \mathfrak{m})$ be a local ring and $M \in \mathfrak{M}_R$ be finitely generated. Let $^{-} : M \to M/\mathfrak{m}M$ be the canonical map. Let $\mathcal{S} := \{x_1, \ldots, x_n\}$ and $\overline{\mathcal{S}} := \{\overline{x}_1, \ldots, \overline{x}_n\}$. Then
   a. $\mathcal{S}$ generates M if and only if $\overline{\mathcal{S}}$ spans the $R/\mathfrak{m}$-vector space $\overline{M} = M/\mathfrak{m}M$.
   b. $\mathcal{S}$ is a minimal generating set for M if and only if $\overline{\mathcal{S}}$ is an $R/\mathfrak{m}$-basis for $\overline{M}$.
   c. Suppose that R is noetherian. Then the $R/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ has $R/\mathfrak{m}$-dimension at least $\dim R$. [If we have equality, R is called a *regular* local ring. It is the substitute for smoothness in geometry.]
7. If $\phi : A \to B$ is a integral then ${}^a\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map.
8. If R is noetherian and $R \underset{i}{\subset} A$ with $A \in Calg_R$ finitely generated (via i) then GD holds if and only if ${}^a i : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is an open map.
9. Let A be a Dedekind domain. Prove all of the following:
   a. If $0 < \mathfrak{A} < A$ an ideal. then every ideal in $A/\mathfrak{A}$ is principal.
   b. Every ideal in A can be generated by two elements.
   c. If A/R is integral and $0 < \mathfrak{A} < A$ an ideal then one of the two generators in (b) can be chosen in R.

10. Let $A \underset{i}{\subset} B$ be domains with F the quotient field of A. Suppose that $B \in Calg_A$ is finitely generated (via i). Let $0 \neq b \in B$ be fixed. Let L be an algebraically closed field. Then there exists an $a \in A$ with the following property: If $\phi : A \to L$ is a homomorphism with $\phi(a) \neq 0$ then there exists an extension $\psi : B \to L$ of ϕ (i.e., a homomorphism such that $\psi|_A = \phi$) with $\psi(b) \neq 0$.

    [Hint: Reduce to the case that $B = A[x]$ and do the cases x is transcendental over A and algebraic over A separately.]

    [This is another form of the Nullstellensatz as Zariski's Lemma is the case that $A = F$, L its algebraic closure, and $b = 1$].

11. A noetherian valuation ring is called a *discrete valuation ring* or *DVR*.

    Let $(A, \mathfrak{m})$ be a local domain. Then the following are equivalent:

    a. A is a DVR.

    b. A is a PID.

    c. $\mathfrak{m}$ is principal and every non-zero ideal of A is of the form $\mathfrak{m}^n$ for some n.

    d. $\mathfrak{m}$ is principal and $\bigcap_n \mathfrak{m}^n = \{0\}$.

    e. A is a local Dedekind domain.

12. Let A be a domain. Then show the following are equivalent.

    a. A is a Dedekind domain.

    b. A is noetherian and $A_{\mathfrak{m}}$ is a DVR for all maximal ideal $\mathfrak{m}$ in A.

    c. Every non-zero (integral) ideal in A is invertible.

    d. $I_A = \operatorname{Inv}(A)$.

13. Suppose that R is noetherian and $\mathfrak{A} < R$ is an ideal.

    a. If $\operatorname{ht}(\mathfrak{A}) = n \geq 1$ then there exist $a_1, \dots, a_n \in \mathfrak{A}$ such that $\operatorname{ht}(a_1, \dots, a_m) = m$ for all $1 \leq m \leq n$.

    b. If $\operatorname{ht}(\mathfrak{p}) = n \geq 1$ then there exist $a_1, \dots, a_n \in \mathfrak{p}$ such that $\operatorname{ht}(a_1, \dots, a_m) = m$ for all $1 \leq m \leq n$ and $\mathfrak{p} \in V(a_1, \dots, a_n)$ is minimal.

14. Suppose that R is noetherian and $\mathfrak{A} < R$ an ideal. If $\mathfrak{A}$ can be generated by s elements and $\mathfrak{p} \in V(\mathfrak{A})$ satisfies $\operatorname{ht}(\mathfrak{p}/\mathfrak{A}) = m$ in $R/\mathfrak{A}$ then $\operatorname{ht}(\mathfrak{p}) \leq s + m$

15. There exist $\emptyset \neq V_i \subset \operatorname{Spec}(R)$ for $i = 1, 2$ closed such that $\operatorname{Spec}(R)$ is the disjoint union of V_1 and V_2 if and only if R contains a non-trivial idempotent.

16. Let R be a noetherian ring with $\dim R > 1$. Then $\operatorname{Spec}(R)$ is infinite.