

Commutative Algebra Problems

Let R be a commutative ring. Prove all of the following.

1. $R[[t]]$ is noetherian if and only if R is noetherian.
2. R is noetherian if every prime ideal is finitely generated.
3. Let $\mathfrak{A} \subset R[t]$ be an ideal, $\bar{} : R[t] \rightarrow R[t]/\mathfrak{A}$ be the canonical map, $S := R[t]/\mathfrak{A}$ and $s = \bar{t}$.
 - a. $S \in \mathfrak{M}_R$ is generated by $\leq n$ elements if and only if there exists a monic $f \in \mathfrak{A}$ of degree $\leq n$. In this case, $S = \sum_{i=0}^{n-1} R s^i$. In particular, $S \in \mathfrak{M}_R$ is finitely generated if and only if \mathfrak{A} contains a monic polynomial.
 - b. $S \in \mathfrak{M}_R$ is finitely generated and free if and only if $\mathfrak{A} = (f)$ for some monic $f \in R[t]$. If this is the case then there exists an n such that $\mathcal{B} := \{1, s, \dots, s^{n-1}\}$ is an R -basis for S .
4. Let $d \in \mathbf{Z}$ be square-free and $L = \mathbf{Q}(\sqrt{d})$.
 - a. If $d \equiv 2, 3 \pmod{4}$ then $\mathbf{Z}_L = \mathbf{Z}[\sqrt{d}]$ and $\{1, \sqrt{d}\}$ is an integral basis.
 - b. If $d \equiv 1 \pmod{4}$ then $\mathbf{Z}_L = \mathbf{Z}\left[\frac{-1 + \sqrt{d}}{2}\right]$ and $\left\{1, \frac{-1 + \sqrt{d}}{2}\right\}$ is an integral basis.
5. Let (R, \mathfrak{m}) be a local ring and $M \in \mathfrak{M}_R$ be finitely generated. Let $\bar{} : M \rightarrow M/\mathfrak{m}M$ be the canonical map. Let $\mathcal{S} := \{x_1, \dots, x_n\}$ and $\overline{\mathcal{S}} := \{\bar{x}_1, \dots, \bar{x}_n\}$. Then
 - a. \mathcal{S} generates M if and only if $\overline{\mathcal{S}}$ spans the R/\mathfrak{m} -vector space $\overline{M} = M/\mathfrak{m}M$.
 - b. \mathcal{S} is a minimal generating set for M if and only if $\overline{\mathcal{S}}$ is an R/\mathfrak{m} -basis for \overline{M} .
 - c. Suppose that R is noetherian. Then the R/\mathfrak{m} -vector space $\mathfrak{m}/\mathfrak{m}^2$ has R/\mathfrak{m} -dimension at least $\dim R$. [If we have equality, R is called a *regular* local ring. It is the substitute for smoothness in geometry.]
6. If $\phi : A \rightarrow B$ is an integral then ${}^a\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a closed map.
7. If R is noetherian and $R \overset{C}{\underset{i}{\subset}} A$ with $A \in \text{Calg}_R$ finitely generated (via i) then GD holds if and only if ${}^a i : \text{Spec}(A) \rightarrow \text{Spec}(R)$ is an open map.
8. Let A be a Dedekind domain. Prove all of the following:
 - a. If $0 < \mathfrak{A} < A$ an ideal. then every ideal in A/\mathfrak{A} is principal.
 - b. Every ideal in A can be generated by two elements.
 - c. If A/R is integral and $0 < \mathfrak{A} < A$ an ideal then one of the two generators in (b) can be chosen in R .
9. Let $A \overset{C}{\underset{i}{\subset}} B$ be domains with F the quotient field of A . Suppose that $B \in \text{Calg}_A$ is finitely generated (via i). Let $0 \neq b \in B$ be fixed. Let L be an algebraically closed

field. Then there exists an $a \in A$ with the following property: If $\phi : A \rightarrow L$ is a homomorphism with $\phi(a) \neq 0$ then there exists an extension $\psi : B \rightarrow L$ of ϕ (i.e., a homomorphism such that $\psi|_A = \phi$) with $\psi(b) \neq 0$.

[Hint: Reduce to the case that $B = A[x]$ and do the cases x is transcendental over A and algebraic over A separately.]

[This is another form of the Nullstellensatz as Zariski's Lemma is the case that $A = F$, L its algebraic closure, and $b = 1$].

10. A noetherian valuation ring is called a *discrete valuation ring* or *DVR*.

Let (A, \mathfrak{m}) be a local domain. Then the following are equivalent:

- a. A is a DVR.
- b. A is a PID.
- c. \mathfrak{m} is principal and every non-zero ideal of A is of the form \mathfrak{m}^n for some n .
- d. \mathfrak{m} is principal and $\bigcap_n \mathfrak{m}^n = \{0\}$.
- e. A is a local Dedekind domain.

11. Let A be a domain. Then show the following are equivalent.

- a. A is a Dedekind domain.
- b. A is noetherian and $A_{\mathfrak{m}}$ is a DVR for all maximal ideal \mathfrak{m} in A .
- c. Every non-zero (integral) ideal in A is invertible.
- d. $I_A = \text{Inv}(A)$.

12. Suppose that R is noetherian and $\mathfrak{A} < R$ is an ideal.

- a. If $\text{ht}(\mathfrak{A}) = n \geq 1$ then there exist $a_1, \dots, a_n \in \mathfrak{A}$ such that $\text{ht}(a_1, \dots, a_m) = m$ for all $1 \leq m \leq n$.
- b. If $\text{ht}(\mathfrak{p}) = n \geq 1$ then there exist $a_1, \dots, a_n \in \mathfrak{p}$ such that $\text{ht}(a_1, \dots, a_m) = m$ for all $1 \leq m \leq n$ and $\mathfrak{p} \in V(a_1, \dots, a_n)$ is minimal.

13. Suppose that R is noetherian and $\mathfrak{A} < R$ an ideal. If \mathfrak{A} can be generated by s elements and $\mathfrak{p} \in V(\mathfrak{A})$ satisfies $\text{ht}(\mathfrak{p}/\mathfrak{A}) = m$ in R/\mathfrak{A} then $\text{ht}(\mathfrak{p}) \leq s + m$

14. There exist $\emptyset \neq V_i \subset \text{Spec}(R)$ for $i = 1, 2$ closed such that $\text{Spec}(R)$ is the disjoint union of V_1 and V_2 if and only if R contains a non-trivial idempotent.

15. Let R be a noetherian ring with $\dim R > 1$. Then $\text{Spec}(R)$ is infinite.