

## Some Examples

Here are three examples of commutative rings  $R$  such that  $R_{\mathfrak{p}}$  is noetherian for all primes  $\mathfrak{p}$  of  $R$  but  $R$  is not noetherian. The second two examples are from Hutchins' *Examples of Rings* and have many additional properties.

**Example 1.** Let  $F$  be a field,  $R = \prod_{i=1}^{\infty} F$ . Then

$$\text{Max}(R) = \text{Spec}(R) = \{\mathfrak{m}_i := \prod_{j \neq i} F_j\}$$

is not noetherian and  $R_{\mathfrak{m}_i}$  has every ideal principal.

**Example 2.** [Example 58 from Hutchins']. Let  $p_1, p_2, \dots$  be the positive sequence of positive prime integers, and let  $F$  be the field containing  $\mathbf{Q}$  and generated over  $\mathbf{Q}$  by all the  $p_i$ th roots of unity, for all  $i > 0$ . Let  $R := \mathbf{Z}_F$ , the integral closure of  $\mathbf{Z}$  in  $F$ .

- a.  $R$  is normal and one dimensional.
- b. Let  $F_n$  be the subfield of  $F$  containing  $\mathbf{Q}$  and generated over  $\mathbf{Q}$  by all the  $p_i$ th roots of unity for  $1 \leq i \leq n$ . Let  $R_n := \mathbf{Z}_{F_n}$ . Then each  $R_n$  is a Dedekind domain and  $R_1 \subset R_2 \subset \dots$  and  $R$  is the union of this chain.
- c.  $R$  is Prüfer, i.e.,  $I_R = \text{Inv}(R)$ , since it is the integral closure of a Prüfer domain in an algebraic extension of its quotient field.
- d. In fact,  $R$  is an *almost Dedekind domain*, i.e.,  $R_{\mathfrak{p}}$  is a DVR for all non-zero prime ideals  $\mathfrak{p}$  in  $R$ . This is a rather deep fact.
- e. Each principal prime element of  $\mathbf{Z}$  is contained in infinitely many prime ideals of  $R$ . Thus  $R$  is not Noetherian; so it is an almost Dedekind domain which is not Dedekind. One proof involves study of the relationship between prime ideals of  $R_n$  and the prime ideals of  $R$ .
- f. this example is from Nakano, Idealtheorie in einem speziellen unendlichen algebraischen Zahlkörper, *J. Sci. Hiroshima Univ.*, Ser. A-1, 16, 425–439 (1953).

**Example 3.** [Example 166 from Hutchins']. Let  $p_1, p_2, \dots$  be the positive sequence of positive prime integers. Let  $R := \mathbf{Z}[t/p_1, t/p_2, \dots]$ . Then  $R$  has the following properties:

- a. For  $n > 0$ , let  $R_n := \mathbf{Z}[t/p_1 p_2 \dots p_n]$ . Each ring  $R_n$  is a 2-dimensional, regular UFD.  $R$  is the union of the ascending sequence  $R_1 \subset R_2 \subset \dots$  hence 2-dimensional and normal.
- b.  $R$  is not noetherian as the ideal  $(t/p_1, t/p_2, \dots)$  is not finitely generated.
- c. Maximal ideals of  $R$  are of two sorts:
  - i.  $(p, f)$  where  $p$  is a prime integer and the image of  $f$  is irreducible in  $\mathbf{Z}/[t/p]$
  - ii.  $R \cap f\mathbf{Q}[t]$  where  $f \in \mathbf{Q}[t]$  is irreducible (plus some other conditions).  
[ Maximal ideals of the form  $(t - k)$  for  $k \in \mathbf{Z}$  are  $(t + 1)$  and  $(t - 1)$ .]

It can be shown that each of these maximal ideals is finitely generated.

- d. If  $\mathfrak{m}$  is a maximal ideal of  $R$  then  $R_{\mathfrak{m}}$  is isomorphic to a localization of either  $\mathbf{Q}[t]$  or  $\mathbf{Z}/(p)[t/p]$ , where  $p$  is a prime integer. In either case,  $R_{\mathfrak{m}}$  is noetherian.
- e. Thus  $R$  is not noetherian or a polynomial ring over any subring of  $R$ , but it is locally noetherian, locally a polynomial ring, and locally a UFD.
- f.  $R$  is not a Krull domain. To see this, notice that the element  $t$  has infinitely many prime divisors, namely  $(p_1), (p_2), \dots$
- g. This example is from Eakin and Silver, Rings which are almost polynomial rings, *Trans. Amer. Math. Soc.*, **174**, 425–449 (1972).