Workshop 5

1. Let Γ be a free abelian group of rank n on basis $\mathcal{C}$ and Γ' a subgroup of rank n on basis $\mathcal{B}$. Let $T : \Gamma' \to \Gamma$ be the inclusion. Show that $[\Gamma : \Gamma'] = \det([T_{\mathcal{B},\mathcal{C}}])$.

2. List precisely one representative from each similarity class of 3×3-matrices defined over each of the following fields: $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$.

3. Describe all 3×3 complex matrices A up to similarity satisfying $A^2 + 2A - 3 = 0$.

4. Let $f(x) = (x-1)^3(x-2)^2$. Find representatives up to conjugacy for the set of matrices A in the group $GL_3(\mathbf{C})$ of 3×3 invertible complex matrices such that $f(A) = 0$.

5. Let D be a non-commutative division ring containing the real numbers $\mathbf{R}$ in its center and finite dimensional over $\mathbf{R}$ as a left $\mathbf{R}$-vector space. Prove all five of the following:

(a) There is an embedding of the complex numbers $\mathbf{C}$ into D. In particular, we may assume that $\mathbf{C}$ is a subset of D.

(b) Let $T : D \to D$ be the (left) $\mathbf{R}$-linear map $x \mapsto xi$, where $i = \sqrt{-1}$. Determine the eigenspaces of T and show that D is a direct sum of these eigenspaces. [Hint: What polynomial does T satisfy?]

(c) Show that one of the eigenspaces in D is $\mathbf{C}$.

(d) Show that each of the eigenspaces in (b) is 2-dimensional.

(e) Show that there is an element $x \in D$ such that $x^2 = -1$, $xi = -ix$, and $1, i, x, ix$ is an $\mathbf{R}$-basis for D. D is called the *ring of quaternions*.