

Workshop IV: Miscellaneous Problems.

1. Let R be a PID and let M be a R -module. Suppose that M has a presentation

$$0 \rightarrow R^m \rightarrow R^n \rightarrow M \rightarrow 0$$

i.e., this is an exact sequence. Define the *rank* of M to be $n - m$. Show

- i. The rank of M is independent of the presentation and coincides with the rank of the free module M/M_t . Write the rank of M as $\text{rk}(M)$.

- ii. Let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be a short exact sequence of finitely generated R -modules. Show that $\text{rk}(M) = \text{rk}(M') + \text{rk}(M'')$.

2. Show that every finitely generated torsion module over a PID has a composition series.

3. Let R be a domain that is not a field. Let K be the quotient field of R . Show that K is R -torsion-free but not R -free.

4. Let R be a commutative ring and $\mathfrak{Q} < R$ and ideal. We say that \mathfrak{Q} is a *primary ideal* if whenever $ab \in \mathfrak{Q}$ and $a \notin \mathfrak{Q}$ then there exists a positive integer such that $b^n \in \mathfrak{Q}$. Let $\mathfrak{B} < R$ be an ideal. Show all of the following

- i. Every prime ideal is primary.

- ii. \mathfrak{B} is primary if and only if $\text{zd}(R/\mathfrak{B}) \subset \text{nil}(R/\mathfrak{B})$.

- iii. If \mathfrak{B} is primary then $\sqrt{\mathfrak{B}}$ is prime.

- iv. What are the primary ideals of \mathbf{Z} ? Prove your assertion.

5. Let V be a not necessarily finite dimensional vector space over a field F and $T : V \rightarrow V$ a linear transformation. Let $p \in F[t] \setminus F$. Show that if λ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T)$ and if μ is an eigenvalue of $p(T)$ then there exists an eigenvalue λ of T such that $\mu = p(\lambda)$.