Workshop III: Module and Ring Theory Problems.

1. Let R be a commutative artinian ring, i.e., every descending chain of ideals in R stabilize (equivalently, the Minimal Principal holds). Prove that
   a. Every prime ideal in R is maximal..
   b. R is *semilocal*, i.e., there are finitely many maximal ideals.

2. Let M be an R-module. Recall that M is called a *simple* or *irreducible* R-module, if $M \neq 0$ and if $0 < N \subset M$ is a submodule then $N = M$. Let F be an algebraically closed field. Let A be a finite dimensional F-algebra (i.e, a (not necessarily commutative) ring that is a F-vector space so that $F \cong F1_A$ is central in A). Let M and N be finitely generated A-modules. Prove
   a. If M and N are irreducible then

$$Hom_A(M,N) \cong \begin{cases} F & \text{if } M \cong N \text{ (as } A\text{-modules)} \\ 0 & \text{otherwise} \end{cases}$$

   b. If M has the property that every submodule is a direct summand and if, in addition, $Hom_A(M,M) \cong F$ then M is irreducible.

3. Let R be an ring and M an R-module. Show
   a. $\mathrm{Hom}_R(R,M) \cong M$ as R-modules. More generally, $\mathrm{Hom}_R(R^n,M) \cong M^n$ as R-modules.
   b. If R is a commutative noetherian ring and M a finitely generated R-module then $\mathrm{End}_R(M)$ is a finitely generated R-module and hence a noetherian ring.

4. If R is a commutative noetherian ring then there exists an N such that if $x \in R$ is nilpotent then $x^N = 0$.

5. Let R be a left noetherian ring. Let M be a finitely generated R-module. Show that any R-epimorphism $M \to M$ is an R-isomorphism. State and prove a corresponding result if R is artinian.