Workshop II: Problems on Modules.

1. Let M be a *simple* or *irreducible* R-module, i.e, $M \neq 0$ and if $0 < N \subset M$ is a submodule then $N = M$. Show that $\mathrm{End}_R(M)$ is a division ring.

2. Let M be a finitely generated free abelian group of rank n. Let N be a submodule generated by n $\mathbf{Z}$-linearly independent elements. Show that M/N is a finite group generated by n [or possibly less] elements.

3. Show that if Q_i, $i \in I$, are R-modules and $Q = \prod_I Q_i$ then Q is R-injective if and only if each Q_i is R-injective.

4. Show that an R-module I is R-injective if and only if whenever there is an exact sequence of R-modules
$$0 \to I \to M \to N \to 0$$
with N an R-cyclic module, the sequence splits.

5. Let R be a commutative ring. Show that if F_i, $i \in I$, are R-modules and $F = \coprod_I F_i$ then F is R-flat if and only if each F_i is R-flat. In particular, any R-projective module is R-flat.

6. Show that a commutative ring R is noetherian if and only if every direct sum of R-injective modules is R-injective.

7. Show that a commutative ring is noetherian if and only if every prime ideal is finitely generated.