Module and Matrix Problems

All rings R are commutative with one unless otherwise stated.
Do at least Problems # 1,4,5,8,9,10

1. Show that the Vandermonde determinant

$$det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \prod_{i>j}(x_i - x_j)$$

2. Let $M = \prod M_i$ in $\mathcal{M}_R$.
   i. M is R-injective if and only if each M_i is injective.
   ii. M is injective if and only if every exact sequence $0 \to M \to N$ in $\mathcal{M}_R$ splits.
   iii. If $M \subset N$ in $\mathcal{M}_R$ and M is injective then $N = M \oplus M'$ in $\mathcal{M}_R$.
   iv. M is injective if and only if for all left ideals $\mathfrak{A} \subset R$ any R-homomorphism $\mathfrak{A} \to M$ extends to an R-homomorphism $R \to M$.
3. Let R be a PID and

$$R^n \xrightarrow{g} R^m \to M \to 0$$

be an exact sequence of R-modules. Show how a Smith Normal Form of g determines M even when g is not monic and m is not necessarily less than or equal to n.

4. If R is euclidean then SL_nR is generated by Type I matrices.
5. Let R be a PID, $d, e \in R \setminus \{0\}$, $0 \neq p$ a prime in R and $M \in \mathcal{M}_R$.
   i. $dR^n \cong R^n$ in $\mathcal{M}_R$ is free
   ii. $d\Big(R/(e)\Big) \cong R/\Big(e/gcd(e, d)\Big)$ in $\mathcal{M}_R$.
   iii. $R/(d)/p\Big(R/(d)\Big) = \begin{cases} R/(p) & \text{if } p|d \\ 0 & \text{otherwise.} \end{cases}$
   iv. The R and $R/(p)$ action on M/pM are the same. In particular, M/pM is an $R/(p)$-vector space.
   v. If $N_i \subset M_i$ then $(M_1 \oplus M_2)/(N_1 \oplus N_2) \cong (M_1/N_1) \oplus (M_2/N_2)$ in $\mathcal{M}_R$.
6. Let R be a domain, $\mathfrak{A} \subset R$ an ideal. Then $\mathfrak{A}$ is torsion-free but, in general, free if and only if $\mathfrak{A}$ is principal. In particular, R is a PID if and only if every submodule of a free module is free.
7. $\displaystyle\prod_{i=1}^{\infty} \mathbf{Z}$ is torsion-free but not free.
8. Two 3×3 matrices over any field are similar if and only if they have the same characteristic and minimal polynomials.
9. Find all $n \times n$ matrices A up to similarity which satisfy $A^n = I$ over the complexes, over the reals.

10. Compute the Jordan and Rational Canonical Forms of the matrix

$$\begin{pmatrix} -8 & 16 & -44 & 0 \\ 0 & 2 & 0 & 0 \\ \frac{5}{2} & -4 & 13 & 0 \\ -2 & 3 & -9 & 2 \end{pmatrix}$$