Homework: The Tensor Product

All rings R are commutative with one unless otherwise stated.
Some of this generalizes easily to the non-commutative case.
Prove all of the following. Do at least Problems # 1,2,3,5,6,7.

Definition. Let M_1, M_2, N be R-modules. A map $f : M_1 \times M_2 \to N$ is called R-*bilinear* if for each $m_1 \in M_1$ and each $m_2 \in M_2$ the maps $f(m_1, *) : M_2 \to N$ by $m \mapsto f(m_1, m)$ and $f(*, m_2) : M_1 \to N$ by $m \mapsto f(m, m_2)$ are R-homomorphisms.

1. Let M_1, M_2, N be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-bilinear map $g : M_1 \times M_2 \to T$ with the following properties:
   i. Given any R-module N and R-bilinear map $f : M_1 \times M_2 \to N$, there exists a unique R-homomorphism $f' : T \to N$ and a commutative diagram

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{g} & T \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & N. \end{array}$$

   ii. If (T', g') is another such pair then there exists a unique R-isomorphism $j : T \to T'$

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{g} & T \\ & {\scriptstyle g'}\searrow & \downarrow{\scriptstyle j} \\ & & T' \end{array}$$

   commutes. We denote (T, g) by $M_1 \otimes_R M_2$ (or by $M_1 \otimes M_2$ when R is clear) and call it the *tensor product* of M_1 and M_2 over R. The elements $g(m_1, m_2)$ in $M_1 \otimes_R M_2$ are denoted by $m_1 \otimes m_2$, so every element in $M_1 \otimes_R M_2$ is an R-linear combination of such elements.
2. Let F be a field and $F \subset K$ another field with V, W F-vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Show
   i. $V \otimes_F W$ is an F-vector space of dimension $\dim V \cdot \dim W$ on basis $\{v \otimes w \mid v \in \mathcal{B} \text{ and } w \in \mathcal{C}\}$.
   ii. K is an F-vector space and $K \otimes W$ is both an F-vector space and a K-vector space. The K-structure is induced by: for all $\alpha, \beta \in K$ and for all $w \in W$,

$$\alpha(\beta \otimes w) = (\alpha\beta) \otimes w.$$

   As a K-vector space it has dimension also $\dim W$.
3. Compute $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ for m, n non-negative integers.
4. Let A, B, C be R-modules. Show that there exist isomorphisms:
   i. $A \otimes_R B \to B \otimes_R A$.
   ii. $(A \otimes_R B) \otimes_R C \to A \otimes_R (B \otimes_R C)$. [So we write this as $A \otimes_R B \otimes_R C$.]
   iii. $R \otimes_R A \to A$.

5. Let A, B, C be R-modules.
   i. There exists a canonical isomorphism
   $$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)).$$
   ii. If $\phi : R \to T$ is a ring homomorphism then T is an R-module by the pullback and $T \otimes_R A$ is both an R- and T-module. The T-structure is induced by (cf. Problem 2 ii): for all $\alpha, \beta \in T$ and for all $a \in A$,
   $$\alpha(\beta \otimes a) = (\alpha\beta) \otimes a.$$
6. Let A, B, C, D be R-modules. If $f : A \to B$ is an R-homomorphism then this induces an R-homomorphism $f \otimes 1 : A \otimes_R D \to B \otimes_R D$ induced by $a \otimes d \mapsto f(a) \otimes d$. [Remember every element in $A \otimes B$ is an R-linear combination of such elements.] Then
   $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
   exact implies that
   $$A \otimes_R D \xrightarrow{f\otimes 1} B \otimes_R D \xrightarrow{g\otimes 1} C \otimes_R D \to 0$$
   is exact. This means that $* \otimes_R D : \mathcal{M}_R \to \mathcal{M}_R$ is a *right exact functor*. If $* \otimes_R D$ always takes short exact sequences in $\mathcal{M}_R$ to short exact sequences then we say D is an R-*flat* module. Show if D is R-projective then it is R-flat.
7. Let $S \subset R$ be a multiplicative set and M, N be R-modules. Let $S^{-1}M = \{\frac{m}{s} \mid s \in S \,\text{and}\, m \in M\}$ where $\frac{m}{s} = \frac{m'}{s'}$ if there exists $s'' \in S$ such that $s''(s'm - sm') = 0$ in M. Then $S^{-1}M$ is an $S^{-1}R$-module by $\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}$ and $\frac{r}{s'}\frac{m}{s} = \frac{rm}{s's}$ for all $s, s' \in S,\ r \in R,\ m, m' \in M$. Moreover,
   i. There is a canonical $S^{-1}R$-isomorphism between the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$.
   ii. $S^{-1}R$ is a flat R-module.
   iii. There exists a natural $S^{-1}R$-isomorphism
   $$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N).$$
   In particular, if $\mathfrak{p}$ is a prime ideal then there exists an isomorphism of $R_\mathfrak{p}$-modules
   $$M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \cong (M \otimes_R N)_\mathfrak{p}.$$
8. Let M be an R-module. Then the following are equivalent:
   i. $M = 0$.
   ii. $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$ of R.
   iii. $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ of R.
9. Let $\phi : M \to N$ be an R-homomorphism. Then the following are equivalent:
   i. ϕ is injective.
   ii. $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for each prime ideal $\mathfrak{p}$.
   iii. $\phi_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is injective for each maximal ideal $\mathfrak{m}$.

   Similarly, with 'injective' replaced by 'surjective' throughout.
10. Let M be an R-module. Then the following are equivalent:
   i. M is a flat R-module.
   ii. $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p}$ of R.
   iii. $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.