Some Exact Sequences

0. Look up all terms below that are not defined.

1. Suppose that

$$0 \to A_i \to B_i \to C_i \to 0$$

are exact sequences of R-modules for all $i \in I$. Prove both of the following sequences are exact:

$$0 \to \prod_I A_i \to \prod_I B_i \to \prod_I C_i \to 0$$

$$0 \to \coprod_I A_i \to \coprod_I B_i \to \coprod_I C_i \to 0$$

If I is a *directed set* and all maps are compatible then sequence of direct limits

$$0 \to \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i \to 0$$

is exact and if they are inverse systems

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i$$

is exact. (The last map may not be surjective.)

2. Suppose that

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence of R-modules and D is and R-module show that

$$\mathrm{Hom}_R(A, D) \xleftarrow{f^*} \mathrm{Hom}_R(B, D) \xleftarrow{g^*} \mathrm{Hom}_R(C, D) \longleftarrow 0$$

is an exact sequence of abelian groups where, e.g., $f^*(h) = h \circ f$. We say that $\mathrm{Hom}_R(\ , D)$ is a *contravariant left exact functor* and

$$0 \to \mathrm{Hom}_R(D, A) \xrightarrow{f_*} \mathrm{Hom}_R(D, B) \xrightarrow{g_*} \mathrm{Hom}_R(D, C)$$

is an exact sequence of abelian groups where, e.g., $f_*(h) = f \circ h$. We say that $\mathrm{Hom}_R(\ , D)$ is a *(covariant) left exact functor.*

3. (Five Lemma) Suppose the following diagram of R-modules and R-homomorphisms is commutative and has exact rows:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

Show that if two of the maps α, β, γ are R-isomorphisms then all three are.

4. (Snake Lemma) Suppose the following diagram of R-modules and R-homomorphisms is commutative and has exact rows:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

show that there is a (natural) exact sequence

$$\ker\alpha \rightarrow \ker\beta \rightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \rightarrow \operatorname{coker}\beta \rightarrow \operatorname{coker}\gamma$$

where δ (called the *connecting homomorphism*) is defined by

$$\delta : c \mapsto f'^{-1}\beta g^{-1}c + \operatorname{im}\alpha$$

(so δ is well-defined). Moreover if f is monic so is $\ker\alpha \rightarrow \ker\beta$ and if g' is epic so is $\operatorname{coker}\beta \rightarrow \operatorname{coker}\gamma$.

5. An exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

of R-modules and R-homomorphisms is said to be *split exact* if one of the following equivalent conditions hold

a. g is a *split epimorphism*, i.e., there exists an R-homomorphism $C \xrightarrow{g'} B$ such that $g \circ g' = \mathrm{Id}_C$.
b. f is a *split monomorphism*, i.e., there exists an R-homomorphism $B \xrightarrow{f'} A$ such that $f' \circ f = \mathrm{Id}_A$.
c. There is a submodule $D \subset B$ such that $D \xrightarrow{g} C$ is an R-isomorphism and $B = f(A) \oplus D$.

Show these conditions are equivalent.

6. If

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is a split exact sequence of R-modules and D is an R-module, show that both $\mathrm{Hom}_R(\ ,D)$ and $\mathrm{Hom}_R(D,\ )$ applied to this sequence yield split exact sequences of abelian groups.