Matrix Representations

We give a review of the **matrix representation** of a linear transformation relative to a pair of ordered bases and what happens when we change bases. We do this in somewhat more generality then in linear algebra. All proofs, however, are the same, hence omitted. Throughout R will be a commutative ring.

Setup.

Let R be a commutative ring.
V and W are free R-modules of rank n and m respectively.
$\mathcal{B} := \{v_1, ..., v_n\}$ is an ordered basis for V.
$\mathcal{B}'$ is a second ordered basis of V.

$\mathcal{C} := \{w_1, ..., w_n\}$ is an ordered basis for W.
$\mathcal{C}'$ is a second ordered basis of W. .

$T : V \to W$ is an R-homomorphism. .

Let v be an element in the free R-module V. Then there exist unique scalars, $\alpha_1, ..., \alpha_n$ in R such that

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

The scalars $\alpha_1, ..., \alpha_n$ are called the *coordinates* of v relative to the (ordered) basis $\mathcal{B}$.

Let

$$[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

the *coordinate matrix* of v relative to the basis $\mathcal{B}$ and

$$V_{\mathcal{B}} := \{\, [v]_{\mathcal{B}} \mid v \in V\} = R^{n \times 1}.$$

We have an R-isomorphism

$$V \to V_{\mathcal{B}} \text{ given by } v \mapsto [v]_{\mathcal{B}}.$$

Similarly, if w is a vector in the vector space W, then there exist unique scalars, $\beta_1, ..., \beta_m$ in R such that

$$w = \beta_1 w_1 + \cdots + \beta_n w_n.$$

Let

$$[w]_{\mathcal{C}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$$

the *coordinate matrix* of w relative to the basis $\mathcal{C}$ and

$$W_{\mathcal{C}} := \{ [w]_{\mathcal{C}} \mid w \in W \} = R^{m \times 1}.$$

Examples.

1.

$$[v_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots \ , [v_n]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

2. Suppose that $n = 3$. Then

$$[2v_1 - 3v_3]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \qquad [7v_2 - v_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix}$$

We turn to the R-homomorphism $T : V \to W$. By the Universal Property of Free Modules, there exists a unique matrix

$$[T]_{\mathcal{B},\mathcal{C}} \in R^{m \times n}$$

called the *matrix representation* of T relative to the ordered bases $\mathcal{B}, \mathcal{C}$ that satisfies

$$[T]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}} \quad \forall v \in V.$$

If $V = W$ and $\mathcal{B} = \mathcal{C}$, we let $[T]_{\mathcal{B}} = [T]_{\mathcal{B},\mathcal{C}}$.

The matrix $[T]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$ is computed as follows. Write Tv_j in the $\mathcal{C}$ basis. Then the coordinate matrix of Tv_j relative to the $\mathcal{C}$ basis is the jth column of the matrix $[T]_{\mathcal{B},\mathcal{C}}$, i.e., if

$$Tv_j = \beta_{1j} w_1 + \cdots + \beta_{mj} w_m \text{ then } [T(v_j)]_{\mathcal{C}} = \begin{pmatrix} \beta_{1j} \\ \vdots \\ \beta_{mj} \end{pmatrix}$$

and this is the jth column of $[T]_{\mathcal{B},\mathcal{C}}$.

Definition. A matrix of the form $[Id]_{\mathcal{B},\mathcal{B}'}$ is called a *change of basis matrix*.

It arises by writing the elements in the $\mathcal{B}$ basis in terms of the elements in the $\mathcal{C}$ basis.

The Main results are:

Proposition. Let V be an R-free module of rank n. Let $\mathcal{B}$ and $\mathcal{B}'$ be ordered bases for V. Then $[Id]_{\mathcal{B},\mathcal{B}'}$ is an invertible matrix and

$$[Id]^{-1}_{\mathcal{B},\mathcal{B}'} = [Id]_{\mathcal{B}',\mathcal{B}}$$

Remark. It can be shown that

$$\mathbf{GL}_n R = \{[T]_{\mathcal{B},\mathcal{C}} \,|\, \mathcal{B}, \mathcal{C} \text{ bases for } V\}.$$

Since R is commutative, determinants exist. It can also be shown that

$$\mathbf{GL}_n R = \{A \in \mathbf{M}_n R \,|\, \det(A) \in R^\times\}$$

Theorem. Let V, W and X be finitely generated free R-modules. Let $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be ordered bases for V, W and X, respectively. Let $T : V \to W$ and $S : W \to X$ be R-homomorphisms. Then

$$[S \circ T]_{\mathcal{B},\mathcal{D}} = [S]_{\mathcal{C},\mathcal{D}}[T]_{\mathcal{B},\mathcal{C}}$$

where the right hand side is matrix multiplication.

Theorem. Let V be finitely generated free R-module. $\mathcal{B}$ an ordered basis for V. Then

$$\text{End}_R V \to \mathbf{M}_n R \text{ via } T \mapsto [T]_{\mathcal{B}}$$

is a ring isomorphism and induces a group isomorphism

$$\text{Aut}_R V \to \mathbf{GL}_n R.$$

Change of Basis Theorem. Let V and W be finitely generated R-free modules. Let $\mathcal{B}$ and $\mathcal{B}'$ be ordered bases for V and $\mathcal{C}$ and $\mathcal{C}'$ be ordered basis for W. Let $T : V \to W$ be an R-homomorphism. Then

$$[T]_{\mathcal{B}',\mathcal{C}'} = [Id]_{\mathcal{C},\mathcal{C}'}[T]_{\mathcal{B},\mathcal{C}}[Id]_{\mathcal{B},\mathcal{B}'} = [Id]^{-1}_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B},\mathcal{C}}[Id]_{\mathcal{B},\mathcal{B}'}.$$

The Change of Basis Theorem states that the following diagram commutes

$$\begin{array}{ccc} V_{\mathcal{B}} & \xrightarrow{[T]_{\mathcal{B},\mathcal{C}}} & W_{\mathcal{C}} \\ {\scriptstyle [Id]_{\mathcal{B},\mathcal{B}'}}\big\downarrow & & \big\downarrow{\scriptstyle [Id]_{\mathcal{C},\mathcal{C}'}} \\ V_{\mathcal{B}'} & \xrightarrow[{[T]_{\mathcal{B}',\mathcal{C}'}}]{} & W_{\mathcal{C}'} \end{array}$$

Note that the inverses of the change of bases matrices go in the reverse direction. One can fill in more of the diagram. For example, the maps from the diagonals can also be read off. E.g.,

$$[T]_{\mathcal{B},\mathcal{C}'} = [T]_{\mathcal{B}',\mathcal{C}'}[Id]_{\mathcal{B},\mathcal{B}'}$$

$$[T]_{\mathcal{B}',\mathcal{C}} = [Id]^{-1}_{\mathcal{C},\mathcal{C}'}[T]_{\mathcal{B}',\mathcal{C}'} = [Id]_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B}',\mathcal{C}'}$$

Warning. Usually T is not an isomorphism, so $[T]_{\mathcal{B},\mathcal{C}'}$ is not invertible. So you cannot reverse arrows having T in them. If T is an isomorphism, then the matrix representation of T^{-1} is the inverse of the corresponding matrix representation of T.

Historically, one defined equivalence relationships between matrices. We define these and tell how they are related to R-homomorphisms of finitely generated free R-modules.

Definition. If $A, B \in \mathbf{M}_n R$, we say that they are *similar* and write $A \sim B$ if there is an invertible matrix $C \in \mathbf{GL}_n R$ such that $A = C^{-1}BC$.

Clearly, $\sim$ is an equivalence relation. An important problem is to find good representatives for the classes under this equivalence relation when R is a nice ring, e.g., a PID. This leads to the study of Rational Canonical Forms and Jordan Canonical Forms in linear algebra.

Theorem. Let $A, B \in \mathbf{M}_n R$. Then $A \sim B$ in $\mathbf{M}_n R$ if and only if there exist a free R-module V of rank n with bases $\mathcal{B}, \mathcal{B}'$ and $T \in \mathrm{End}_R V$ such that $A = [T]_{\mathcal{B}}$ and $B = [T]_{\mathcal{B}'}$.

Definition. If $A, B \in R^{m\times n}$, we say that they are *equivalent* write $A \simeq B$ if there is an invertible matrices $P \in \mathbf{GL}_m R$ and $Q \in \mathbf{GL}_n R$ such that $A = PBQ$.

Clearly, $\simeq$ is an equivalence relation. An important problem is to find good representatives for the classes under this equivalence relation when R is a nice ring, e.g., a PID. This leads to the study of Smith Normal Forms in linear algebra.

Theorem. Let $A, B \in R^{m\times n}R$. Then $A \simeq B$ in $R^{m\times n}$ if and only if there exist free R-modules V, W of rank n, m, respectively, with bases $\mathcal{B}, \mathcal{B}'$ for V and bases $\mathcal{C}, \mathcal{C}'$ for W and an R-homomorphism $T : V \to W$ such that such that $A = [T]_{\mathcal{B},\mathcal{C}}$ and $B = [T]_{\mathcal{B}',\mathcal{C}'}$.