Sampling of Homological Ring Theory

We give some definitions and theorems (no proofs) in homological ring theory.

Question I. What is the relationship between free and projective modules?

1. $\mathbf{Z}/(2) \coprod \mathbf{Z}/(3) \cong \mathbf{Z}/(6)$ in ${}_{\mathbf{Z}/(6)}\mathfrak{M}$ but $\mathbf{Z}/(2)$ is not $\mathbf{Z}/(6)$-free since it does not have enough elements.
2. Examples when all projectives over R are free.
   i. R is a division ring.
   ii. R is a PID.
   iii. (Kaplansky) R is a local ring.
3. (Bass) If R is commutative and noetherian ring then any **non** finitely generated R-projective is free.
4. **Serre Conjecture**. Let $R = k[t_1, \ldots, t_n]$ with k a field. Let $P \in {}_R\mathfrak{M}$ be a finitely generated projective R-module.
   i. (Hilbert-Serre) P is **stably free**, i.e., there exist positive integers n, m such that $P \coprod R^m \cong R^n$ in ${}_R\mathfrak{M}$.
   ii. (Quillen-Suslin) P is free.

   [**Remark.** Let $R = \mathbf{C}[t_1, t_2, t_3]/({t_1}^2 + {t_2}^2 + {t_3}^2)$ or $R = D[t_1, t_2]$ with D a division ring. Then there exist non-free projective R-modules.]

Question II. Let R be a domain. What is the relationship between $M \in {}_R\mathfrak{M}$ being **torsion-free**, i.e.,

$$rm = 0 \text{ with } r \neq 0 \quad \Longrightarrow \quad m = 0$$

and projective?

1. If $P \in {}_R\mathfrak{M}$ is projective (R is a domain) then P is a submodule of a free so torsion-free.
2. $\mathbf{Q} \in \mathcal{A}b$ is torsion-free but not projective. It is flat. Indeed, flats in ${}_R\mathfrak{M}$ are always torsion-free.
3. Finitely generated torsion-free modules in ${}_R\mathfrak{M}$ are always free if and only if R is a **Prüfer** domain, i.e., a domain in which every finitely generated ideal is projective. Examples:
   i. **Valuation Rings**, i.e., domains R in which every non-zero element or its inverse in the quotient field of R lies in R. In fact, R is Prüfer if and only if $R_{\mathfrak{p}}$ is a valuation ring for each prime ideal $\mathfrak{p}$ in R.
   ii. **Dedekind** domains. These are just the noetherian Prüfer rings.
   iii. The ring of entire functions on an open Riemann surface.
4. If R is Prüfer then an R-module is flat if and only if it is torsion-free.

Question III. When are all $M \in {}_R\mathfrak{M}$ projective?

This is easy. This is when R is a semi-simple artinian ring, i.e., $R \in {}_R\mathfrak{M}$ is a sum of **irreducible** R-modules (i.e., modules $M \neq 0$ which have no proper submodules).

Question IV. When are all submodules of a projective module in ${}_R\mathfrak{M}$ always projective?

1. If $R = \mathbf{Z}/(p^2)$ with p a prime then

$$0 \to \mathbf{Z}/(p) \to R \xrightarrow{p} \mathbf{Z}/(p) \to 0$$

is exact in ${}_R\mathfrak{M}$ but is not split exact. (Why?) So $\mathbf{Z}/(p)$ is not projective but a submodule of a free (hence projective). Thus submodules of projectives need not be projective.
2. A ring R is called **left hereditary** if every left ideal of R is projective. If R is left hereditary, every submodule of a projective R-module is projective.
example. R is a dedekind domain, e.g., $\mathbf{Z}[\sqrt{-5}]$ which has non-free projectives.
3. A finitely submodule of an R-projective is always projective if R if a **left semi-hereditary** ring, i.e., a ring in which every finitely generated left ideal in R is projective.
examples. Prúfer domains, left hereditary rings, and the Small ring

$$R := \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix}$$

which is right noetherian, right hereditary, left semi-hereditary but not left noetherian.

Question V. When is every flat R-module in ${}_R\mathfrak{M}$ projective, i.e., P is projective in ${}_R\mathfrak{M}$ if and only if $\otimes_R P$ is exact?

1. $\mathbf{Q} \in {}_\mathbf{Z}\mathfrak{M}$ is flat and not projective as it is not $\mathbf{Z}$-free and R is a PID.
2. (Bass) Every $M \in {}_R\mathfrak{M}$ flat is projective if and only if R is a **left perfect** ring, i.e., R satisfies the descending chain condition on *right* principal ideals, equivalently, every $0 \neq M \in {}_R\mathfrak{M}$ contains an irreducible R-submodule.
3. $M \in {}_R\mathfrak{M}$ is called **finitely presented** or **fp** if there exist positive integers m and n and an exact sequence

$$R^m \to R^n \to M \to 0 \text{ in } {}_R\mathfrak{M}.$$

Every finitely presented flat module is projective. In particular, if R is left noetherian every finitely generated flat is projective.

Question VI. When is $P = \prod_\alpha P_\alpha$ projective if and only if P_α is projective for all α?

1. It is easy to see that if P is projective so is each P_α.
2. $\prod_{i=1}^{\infty} \mathbf{Z}$ is not $\mathbf{Z}$-free hence not $\mathbf{Z}$-projective.
3. (Chase) R is called **right coherent** if every finitely generated right ideal of R is finitely presented.
   i. R is right coherent if and only if

$$F_\alpha \in {}_R\mathfrak{M} \text{ is flat for all } \alpha \quad \Longrightarrow \quad \prod F_\alpha \text{ is flat.}$$

   ii. R is right coherent and left perfect if only if Question VI always has a positive answer. [If R is commutative this is equivalent to R being artinian, e.g., $\mathbf{Z}/(4)$.]

Question VII. What is the relationship between P being projective in ${}_R\mathfrak{M}$ and h_P being exact?

1. $\mathbf{Z}$ is $\mathbf{Z}$-free but $h_{\mathbf{Z}}$ is not exact:
$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/(n) \to 0$$
is exact but $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/(n), \mathbf{Z}) = 0$ in ${}_R\mathfrak{M}$ as $\mathbf{Z}/(n)$ is torsion and $\mathbf{Z}$ is torsion-free. Since $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$ and multiplication by n is not an isomorphism, $h_{\mathbf{Z}}$ is not exact.
2. If R is **quasi-Frobenius**, i.e., right and left noetherian and h_R is exact (e.g., $\mathbf{Z}/(n)$ or $k[G]$ with k an arbitrary field and G an arbitrary finite group) then
   i. If P is projective then h_P is exact.
   ii. If P is finitely generated and h_P is exact then P is projective.