Group Action Problems

Notation. If \( H \subset G \) are groups then we view the coset space \( G/H \) as a \( G \)-set via \( a \cdot bH = abH \) unless otherwise stated.

Prove all of the following. [Note one problem often depends on another.]

1. Let \( G \) be a group. Define what it means to be a homomorphism (respectively, monomorphism, epimorphism, isomorphism) of \( G \)-sets. Show that the \( G \)-action on a \( G \)-set \( S \) is transitive iff \( S \) is isomorphic to \( G/G_s \) for any \( s \in S \).

2. Let \( G \) be a group and \( X \) and \( Y \) be \( G \)-sets and \( x \in X \). Let \( \phi : X \to Y \) to be a homomorphism of \( G \)-sets. Show that \( G \cdot x \subset G \phi(x) \) with equality if \( \phi \) is an isomorphism.

3. Let \( G \) be a group and let \( H \) and \( K \) be subgroups. Show that the \( G \)-sets \( G/H \) and \( G/K \) are isomorphic iff \( H \) and \( K \) are conjugate in \( G \), i.e., there exists a \( g \in G \) such that \( H = gKg^{-1} \).

4. Let \( S \) be a transitive \( G \)-set. We say that \( G \) acts doubly transitively or 2-transitively on \( S \) if given any two pairs of elements \( (x_1, x_2) \) and \( (y_1, y_2) \) in \( S \times S \) with \( x_1 \neq x_2 \) and \( y_1 \neq y_2 \) then there exists some \( g \in G \) such that \( gx_i = y_i \) for \( i = 1, 2 \). [One has the obvious generalization to \( n \)-transitive.] Suppose \( S \) has more than two elements. Show that \( G \) acts doubly transitively on \( S \) iff \( G \cdot s \) acts transitively on \( S \setminus \{s\} \) for all \( s \in S \).

5. Let \( S \) be a \( G \)-set and \( s \in S \). Suppose that \( G \) acts doubly transitively on \( S \). Show that \( G_s \) is a maximal subgroup of \( G \) (i.e., if \( G_s \subset H \subset G \) is a subgroup then \( H = G_s \) or \( H = G \)).

6. Let \( G \) be a group and let \( H \) and \( K \) be subgroups. A double coset of \( H,K \) in \( G \) is a subset of \( G \) of the form \( HxK \), with \( x \in G \). Prove that the family of all double cosets of \( H,K \) of \( G \) partition \( G \). [Define an appropriate equivalence relation on \( G \).] If \( G \) is finite and \( Hx_iK \), \( 1 \leq i \leq n \), partition \( G \) show \( |G : K| = \sum_{i=1}^{n} |H \cap x_iKx_i^{-1}| \). [Note Lagrange’s Theorem is the case that \( K = 1 \).]

The following is not a group action problem:

7. If \( A \) and \( B \) are additive groups then

\[
\text{Hom}(A, B) := \{ \varphi : A \to B \mid \text{group homomorphism} \}
\]

is a group under addition of functions. Compute (and justify) all of the following:

i. \( \text{Hom}(\mathbb{Z}, \mathbb{Q}) \) and \( \text{Hom}(\mathbb{Q}, \mathbb{Z}) \).

ii. \( \text{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \) and \( \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \).

iii. \( \text{Hom}(\mathbb{Q}, \mathbb{Z}/n\mathbb{Z}) \) and \( \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \).