The Krull-Akizuki Theorem

We begin with a few remarks about *artinian* rings and modules, i.e., those satisfying the *descending chain condition.* Recall that a module is said to be of *finite length* (resp., *length* n) if it has a finite composition series (resp., of length n).

Lemma. *Let M be an R-module. Then M has finite length if and only if it is both noetherian and artinian.*

Proof. If M has finite length, all proper chains of submodules of M have length bounded by the length of M by the Jordan-Hölder Theorem. Conversely, suppose that M is both noetherian and artinian. Since it is noetherian, there exists a maximal proper submodule $M_1 < M$, i.e., M/M_1 is *irreducible* (= no proper submodules) by the Correspondence Principle. Since M is noetherian so is M_1. Continuing gives a descending chain of modules $M = M_0 > M_1 > M_2 > ...$ with M_i/M_{i+1} irreducible. This must stop, since M is also artinian. □

Proposition. *Let V be a vector space over a field K. Then the following are equivalent:*
(1) *V is finite dimensional.*
(2) *V has finite length.*
(3) *V is noetherian.*
(4) *V is artinian.*
Moreover, the dimension of V is just its length.

Proof. Exercise.

Corollary. *Suppose $\mathfrak{m}_1, ..., \mathfrak{m}_n$ are maximal ideals in R (not necessarily distinct). Suppose further that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Then R is noetherian if and only if R is artinian.*

Proof. Each $\mathfrak{m}_1 \cdots \mathfrak{m}_i/\mathfrak{m}_1 \cdots \mathfrak{m}_{i+1}$ is an $(R/\mathfrak{m}_{i+1})$-vector space so is artinian if and only if it is noetherian. We know that $\mathfrak{m}_1 \cdots \mathfrak{m}_i$ is noetherian (resp., artinian) if and only if $\mathfrak{m}_1 \cdots \mathfrak{m}_{i+1}$ and $\mathfrak{m}_1 \cdots \mathfrak{m}_i/\mathfrak{m}_1 \cdots \mathfrak{m}_{i+1}$ are. The result now follows since $R \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset ... \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. □

Proposition. *Let R be an artinian ring. Then R is semi-local of dimension zero. In particular,* $\mathrm{rad}(R) = \mathrm{nil}(R)$*.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Then $R/\mathfrak{p}$ is an artinian domain. Thus to show that $\dim(R)$ is zero we need only show that any artinian domain is a field. But if R is an artinian domain and $x \neq 0$ in R then the descending chain

$$Rx \supset Rx^2 \supset Rx^3 \supset ...$$

must stabilize. Hence $Rx^n = Rx^{n+1}$ for some n. In particular, $yx^{n+1} = x^n$ in the domain R. Thus $yx = 1$ and R is a field.

Now suppose that R is an arbitrary artinian ring. If $\mathrm{Spec}(R) = \mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, ...\}$, the descending chain

$$\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset ...$$

stabilizes, so $\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap ... \cap \mathfrak{m}_{n+1} = \mathrm{rad}(R) = \mathrm{nil}(R)$, some n. In particular, if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n \subset \mathfrak{m}$ implies that $\mathfrak{m} = \mathfrak{m}_i$, some i. ☐

Theorem. (Akizuki) *Let R be a ring. Then the following are equivalent.*

(1) *R is artinian.*
(2) *R is noetherian of dimension zero.*
(3) *Every finitely generated R-module has finite length.*

Proof. Every module over an artinian (resp., noetherian) ring R is artinian (resp., noetherian), since it is a quotient of R^n for some n. Thus by the above, we know that R satisfies both (1) and (2) if and only if R satisfies (3). So we need only show that R satisfies (1) if and only if R satisfies (2).

Suppose that (1) holds, i.e., that R is artinian. Then R is semi-local of dimension zero. Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$. By the corollary above, it suffices to show that $0 = \prod_{1=1}^{n} \mathfrak{m}_i^k$ for some k. But $\prod_{i=1}^{n} \mathfrak{m}_i \subset \bigcap_{1=1}^{n} \mathfrak{m}_i = \mathrm{rad}(R) = \mathrm{nil}(R)$ by the proposition. So it suffices to prove the following

Claim. If R is artinian then $\mathrm{nil}(R)$ is nilpotent:

By the descending chain condition, $(\mathrm{nil}(R))^k = (\mathrm{nil}(R))^{k+i}$ for some k and all i. Let $\mathfrak{A} := (\mathrm{nil}(R))^k$. We must show that $\mathfrak{A} = 0$. Suppose not. Let

$$\mathcal{S} = \{\mathfrak{B} < R \,|\, \mathfrak{B} \text{ is an ideal of } R \text{ with } \mathfrak{A}\mathfrak{B} \neq 0\}.$$

Since $\mathfrak{A} \in \mathcal{S}$ and R is artinian, there exists $\mathfrak{B} \in \mathcal{S}$ minimal. In particular, there exists an x in $\mathfrak{B}$ such that $x\mathfrak{A} \neq 0$. By minimality, $\mathfrak{B} = Rx$. Since $(x\mathfrak{A})\mathfrak{A} = x\mathfrak{A}^2 = \mathfrak{A} \neq 0$, we also have $x\mathfrak{A} = \mathfrak{A}$ by minimality. Choose $y \in \mathfrak{A}$ such that $x = xy$. Then $x = xy^N$ for all positive integers N. But $y \in \mathfrak{A} \subset \mathrm{nil}(R)$, so $y^N = 0$ for some N and hence $x = 0$ also. This is a contradiction. Thus $\mathfrak{A} = 0$ and the Claim is established.

Now suppose that R is noetherian of dimension zero. By the corollary above, it suffices to show that there exist maximal ideals $\mathfrak{m}_1, ..., \mathfrak{m}_n$ in R, not necessarily distinct, so that $0 = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Since R is noetherian the zero ideal contains a finite product of prime ideals. Since $\dim(R) = 0$, these prime ideals are maximal. The result follows. ☐

Corollary. *Let R be a domain. Then the following are equivalent.*

(1) *R is noetherian of dimension at most one.*
(2) *If $0 < \mathfrak{A} < R$ is a ideal then $R/\mathfrak{A}$ has finite length.*
(3) *If $0 < \mathfrak{A} < R$ is a ideal then $R/\mathfrak{A}$ is artinian.*

Proof. If (1) holds and $0 < \mathfrak{A} < R$ is a ideal then $R/\mathfrak{A}$ is noetherian of dimension zero so (2) holds. Clearly, (2) implies (3), so we need only show that (3) implies (1).

If (3) holds then $R/\mathfrak{A}$ is a noetherian ring of dimension zero for any ideal $0 < \mathfrak{A} < R$. In particular, if $0 \neq x \in \mathfrak{A}$ then $\mathfrak{A}/Rx$ is a finitely generated ideal in R/Rx. It follows that $\mathfrak{A}$ is finitely generated as an ideal in R, i.e., R is noetherian. If $0 < \mathfrak{p}_1 \subset \mathfrak{p}_2$ is a chain of primes in R then $\mathfrak{p}_2/\mathfrak{p}_1 = 0$ in the artinian domain, hence field, $R/\mathfrak{p}_1$. Thus R has dimension at most one. ☐

Lemma. *Let M be a non-trivial R-module and $\mathfrak{p}$ a prime ideal containing $\operatorname{ann}_R(M)$ and a minimal such prime ideal. Then $\mathfrak{p}$ consists of zero divisors of M. In particular,*

$$\bigcup_{\operatorname{Min}(R)} \mathfrak{p} \subset \operatorname{zd}(R).$$

[There is no noetherian condition or finite generation condition.]

Proof. Let S be the multiplicative set in R defined by

$$S := \{ab \mid a \in R \setminus \mathfrak{p} \text{ and } b \in R \setminus \operatorname{zd}(M)\}.$$

Claim. $S \cap \operatorname{ann}_R(M) = \emptyset$.

Suppose not. Then there exists an a in $R\setminus\mathfrak{p}$ and b in $R\setminus\operatorname{zd}(M)$ such that $abM = 0$. Since b is not a zero divisor on M, we have $aM = 0$ and hence $a \in \operatorname{ann}_R(M) \subset \mathfrak{p}$, a contradiction. This establishes the Claim.

Thus there exists a prime $\mathfrak{P}$ containing $\operatorname{ann}_R(M)$ such that $\mathfrak{P}$ excludes S and is maximal with respect to this property. Since 1 lies in R but not in $\mathfrak{p}$ or $\operatorname{zd}(M)$, we have

$$(R \setminus \operatorname{zd}(M) \cdot (R \setminus \mathfrak{p}) \supset (R \setminus \operatorname{zd}(M) \cap (R \setminus \mathfrak{p}).$$

Thus

$$\mathfrak{P} \subset \operatorname{zd}(M) \cap \mathfrak{p} \subset \mathfrak{p}.$$

The minimality condition on $\mathfrak{p}$ implies that $\mathfrak{p} = \mathfrak{P}$, so $\mathfrak{p} \subset \operatorname{zd}(M)$ as desired.
For the last statement, let $M = R$. Then every prime contains $0 = \operatorname{ann}_R(R)$. It follows from the first part that if $\mathfrak{p}$ is a *minimal* prime ideal, i.e., a prime ideal properly containing no other prime ideal, then $\mathfrak{p}$ consists of zero divisors of the ring . ☐

We need the lemma in the following special case.

Corollary. *If* $\dim(R) = 0$ *then* $R^\times = R \setminus \operatorname{zd}(\mathrm{R})$.

Proof. We have

$$\operatorname{zd}(R) \supset \bigcup_{\operatorname{Min}(R)} \mathfrak{p} = \bigcup_{\operatorname{Spec}(R)} \mathfrak{p} = \bigcup_{\operatorname{Max}(R)} \mathfrak{p}.$$

Clearly, $R^\times \cap \operatorname{zd}(\mathrm{R}) = \emptyset$, so this is an equality. ☐

Lemma 2. *Let R be a noetherian domain of dimension one. Let a and c be non-zero elements of R. Let*

$$\mathfrak{A} = \bigcup_{n=0}^{\infty} Rc : Ra^n := \{x \in R \mid \exists n \ni xa^n \in Rc\}.$$

Then

$$\mathfrak{A} + Ra = R.$$

Proof. Let $\mathfrak{A}_k = Rc : Ra^k := \{x \in R \,|\, xa^k \in Rc\}$. Since $\mathfrak{A}_k \subset \mathfrak{A}_{k+1}$, for all k, we know that $\mathfrak{A}$ is an ideal. Since R is noetherian, there exists an integer n such that $\mathfrak{A}_n = \mathfrak{A}_{n+i} = \mathfrak{A}$ for all positive integers i. Since c lies in $\mathfrak{A}_k$ for all k, we have $c \in \mathfrak{A}$. In particular, $\mathfrak{A}$ is not trivial.

Let $\bar{\ } : R \to R/\mathfrak{A}$. Since $\mathfrak{A} > 0$ and R is a domain, it is clear that $\dim(R/\mathfrak{A}) = 0$. (We do not need Akizuki's Theorem.) By the corollary above, it suffices to establish the following

Claim. $\overline{a}$ is not a zero divisor in $R/\mathfrak{A}$:

If this were false then there would exist a $y \in R \setminus \mathfrak{A}$ such that $\overline{ay} = 0$, i.e., $ay \in \mathfrak{A} = \mathfrak{A}_n$. Then we would have $(ay)a^n \in Rc$ and hence $y \in \mathfrak{A}_{n+1} = \mathfrak{A}_n = \mathfrak{A}$, a contradiction. This establishes the Claim. $\square$

Krull-Akizuki Theorem. *Let A be a noetherian domain of dimension one. Let K be the quotient field of A and let L/K be a finite field extension. Let B be a ring such that $A \subset B \subset L$. Then B is a noetherian domain of dimension at most one and if $\mathfrak{B}$ is a non-trivial ideal of B then $B/\mathfrak{B}$ is a finitely generated $(A/\mathfrak{B} \cap A)$-module of finite length.*

Remarks.

1. If $L = K$ in the theorem then clearly K is the only field between A and K, i.e., the only such B with $\dim(B) = 0$ is K.
2. If $\mathfrak{B}$ is zero and $B = K$ then K is not a finitely generated A-module when $A < K$.

Proof. We first make two reductions.

Reduction 1. We may assume that $K = L$:

Certainly, we may assume that L is the quotient field of B and, in fact, $L = K(x_1, ..., x_n)$ for some $x_i \in B$. Choose $0 \neq c \in A$ such that each cx_i is integral over A. Let $C = A[cx_1, ..., cx_n]$. Then C is integral over A and a finitely generated A-module. Thus C is a noetherian domain of dimension one with quotient field L and contained in B. If $\mathfrak{C}$ is an ideal in C then $C/\mathfrak{C}$ is integral over $A/\mathfrak{C} \cap A$ and a finitely generated $(A/\mathfrak{C} \cap A)$-module. So $C/\mathfrak{C}$ and $A/\mathfrak{C} \cap A$ are noetherian rings of the same dimension. In particular, one is artinian if and only if the other is. Consequently, if $\mathfrak{B}$ is a non-trivial ideal in B then by clearing denominators and multiplying by an appropriate power of c, we see that $B \cap C$ is a non-trivial ideal in C, In particular, $B/\mathfrak{B}$ is a finitely generated $(A/\mathfrak{B} \cap A)$-module if it is a finitely generated $(C/\mathfrak{B} \cap C)$-module and has finite length as an $(A/\mathfrak{B} \cap A)$-module if it has finite length as a $(C/\mathfrak{B} \cap C)$-module. This completes the reduction.

Reduction 2. It suffices to show that the (A/Aa)-module B/Ba is finitely generated for any $0 \neq a \in A$:

To show that B is noetherian of dimension at most one, it suffices, by our previous work, to show that $B/\mathfrak{B}$ is artinian for any non-zero ideal $\mathfrak{B}$ of B. Let $0 < \mathfrak{B}$ be an ideal of B. We must also show that $B/\mathfrak{B}$ has finite length over $A/\mathfrak{B} \cap A$.

Claim. $0 < \mathfrak{B} \cap A$:

Let $\mathfrak{B}'$ be any non-trivial finitely generated A-submodule of $\mathfrak{B}$. Then there exists $0 \neq c \in A$ such that $c\mathfrak{B}'$ lies in the domain A by the first reduction. This establishes the Claim.

Let $0 \neq a$ lie in $\mathfrak{B} \cap A$. Then A/Aa is artinian by the corollary to Akizuki's Theorem. By assumption, B/Ba is a finitely generated (A/Aa)-module. Since $B/\mathfrak{B}$ is a cyclic (B/Ba)-module, it is also finitely generated as an (A/Aa)-module, hence has finite length over the artinian ring A/Aa so also over the artinian ring $A/\mathfrak{B} \cap A$. This also implies that $B/\mathfrak{B}$ is artinian.

So to finish we are in the following situation. We have $0 \neq a \in A$ is fixed and we must show that B/Ba is finitely generated as an (A/Aa)-module. We do this in a number of steps. Note as before, that A/Aa is an artinian ring.

Step 1. Let $x \in B\,(\subset K)$. Then there exists a positive integer n such that $x \in Aa^{-n} + Ba$:

Write $x = \dfrac{b}{c}$ with $b, c \in A$ and $c \neq 0$. Set

$$\mathfrak{B} = \bigcup_{n=0}^{\infty} Ac : Aa^n := \{y \in A \,|\, ya^n \in Ac, \text{ some } n\}.$$

By Lemma 2, we have $\mathfrak{B} + Aa = A$ so $1 = y + za$, some $y \in \mathfrak{B}$ and $z \in A$. Consequently, $x = yx + zax$. Since $y \in \mathfrak{B}$, by definition, there exists an integer n such that $ya^n \in Ac$, so

$$x = yx + zax = \frac{ya^n}{a^n}\frac{b}{c} + zax \text{ lies in } Aa^{-n} + Ba$$

as needed.

Step 2. Let $\mathfrak{A}_n := (Ba^n \cap A) + Aa$, an ideal of A. Then there exists a positive integer m such that $\mathfrak{A}_m = \mathfrak{A}_{m+i}$ for all positive integers i:

Each $\mathfrak{A}_n$ contains a so is non-trivial. Moreover, it is clear that the $\mathfrak{A}_n$ form a descending chain of ideals. Since A/Aa is artinian, the descending chain of ideals

$$\ldots \supset \mathfrak{A}_n/Aa \supset \mathfrak{A}_{n+1}/Aa \supset \ldots$$

stabilizes and hence so does the chain of $\mathfrak{A}_n$'s.

Step 3. Let m be the integer in Step 2. Then $B \subset Aa^{-m} + Ba$:

Let $x \in B$ be fixed. Then by Step 1, there exists a minimal positive integer n so that $x \in Aa^{-n} + Ba$. If we show that $m \geq n$ then $a^m \in Aa^n$ and $Aa^{-n} \subset Aa^{-m}$ as needed. So we may assume that $n > m$. Write $x = ra^{-n} + ba$, with $r \in A$ and $b \in B$. Then

$$r = a^n(x - ba) \in Ba^n \cap A \subset (Ba^n \cap A) + Aa = \mathfrak{A}_n$$

and $\mathfrak{A}_n = \mathfrak{A}_{n+1} = \mathfrak{A}_m$. Hence $r = b_1a^{n+1} + r_1a$, for some $r_1 \in A$ and $b_1 \in B$ so $x = ra^{-n} + ba = (b_1a^{n+1} + r_1a)a^{-n} + ba$ lies in $Aa^{-n+1} + Ba$. This contradicts the minimality of n, so completes the step.

Step 4. B/Ba is a finitely generated (A/Aa)-module (and hence we are done):

By Step 3, we know that $B/Ba \subset (Aa^{-m} + Ba)/Ba$. Moreover, we know that the A-module $(Aa^{-m} + Ba)/Ba \cong Aa^{-m}/Aa^{-m} \cap Ba$ is cyclic hence noetherian. Thus B/Ba is a finitely generated A-module as needed. ☐

Corollary. *Let A be a dedekind domain with quotient field K. If L/K is a finite field extension then A_L is also a dedekind domain.*