Math 115AH Homework #1

Problem 1. Let A and B be **FINITE** sets, i.e., each has a finite number of elements. Suppose each has the same number of elements. Let $f : A \to B$ be a map. Show that f is a bijection iff f is a surjection iff f is an injection. Is this true if A is not finite? Prove or give a counterexample.

Problem 2. Show that the map $T : \mathbf{R}[t]_n \to \mathbf{R}^{n+1}$ defined by

$$T(a_0 + a_1 t + \cdots + a_n t^n) = (a_0, ..., a_n)$$

has the following properties:

(a) If $p, q \in \mathbf{R}[t]_n$ and $\alpha \in \mathbf{R}$ then T is **LINEAR**, i.e.,

$$T(\alpha p + q) = \alpha T(p) + T(q).$$

(b) T is injective.

(c) T is surjective.

When a map between vector spaces satisfies all three of these properties, i.e., is a linear bijection, we say that the map is an **ISOMORPHISM** of vector spaces.

Problem 3. Construct a field with three elements. Show it is a field. Can you construct a field with 4 elements, 5 elements, 6 elements?
[Hint: This is not easy. If you can construct a field with 3 elements you should be able to construct a field with 5 elements. The other two are harder. One thing to consider is that $1 + 1 + 1 \cdots + 1$ must equal 0 at some point – why?]

Problem 4. Which of the following W are subspaces of $C[-1, 1]$.
[You must always justify your answers.]

(a): $W = \{f \in C[-1, 1] \mid f(-1) = f(1)\}$:

(b): $W = \{f \in C[-1, 1] \mid f(-1) = -f(1)\}$:

(c): $W = \{f \in C[-1, 1] \mid f \text{ is non-decreasing }\}$:

(d): $W = \{f \in C[-1, 1] \mid f(-1) = 0 \text{ and } f(1) = 0\}$:

(e): $W = \{f \in C[-1, 1] \mid f(-1) = 0 \text{ or } f(1) = 0\}$:

Problem 5. Let A be a fixed 2×2 matrix with real coefficients. Which of the following W are subspaces of $\mathbf{R}^{2\times 2}$.
[You must always justify your answers.]

(a): $W = \{B \in \mathbf{R}^{2\times 2} \mid AB = BA\}$:

(b): $W = \{B \in \mathbf{R}^{2\times 2} \mid AB \neq BA\}$:

(c): $W = \{B \in \mathbf{R}^{2\times 2} \mid AB = 0 \text{ the zero matrix }\}$:

Problem 6. Let V be a vector space over F and W_i subspaces of V for all $i \in I$. Prove that $\cap_I W_i$ is a subspace of V.

Problem 7,8. Do two problems in the book from Section 2.1 p. 33-34.

Problem 9,10. Do two problems in the book from Section 2.2 p. 39-40.