Problem 1. Let $V = C[1,3]$ with $\langle f, g \rangle = \int_1^3 fg$. Let $f(x) = \frac{1}{x}$. Show that the constant polynomial $g$ nearest $f$ is $g = \frac{1}{2} \ln 3$. Compute $\|g - f\|^2$ for this $g$.

Problem 2. Let $V = C[0,2\pi]$ with $\langle f, g \rangle = \int_0^{2\pi} fg$. Let $W = \{1, \cos x, \sin x\}$. Let $f(x) = x$. Find $f(x)_W$.

Problem 3. Let $V = M_n \mathbb{C}$ with $\langle A, B \rangle = \text{tr}(AB^*)$. Let

$$S = \left\{ \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \mid \lambda_1, \ldots, \lambda_n \in \mathbb{C} \right\}$$

(i.e., the diagonal matrices). Determine $S^\perp$ and $\dim(S^\perp)$.

Problem 4. Let $V = C[-1,1]$ an inner product space via $\langle f, g \rangle = \int_{-1}^1 fg$. Let $W_{\text{odd}} := \{f \in V \mid f(x) = -f(-x)\}$. Show $W^\perp_{\text{odd}} = W_{\text{even}} := \{f \in V \mid f(x) = f(-x)\}$.

Problem 5. Let $V$ be an inner product space and $W \subset V$ a finite dimensional subspace. Show that $\langle v_W, x \rangle = \langle v, x_W \rangle$ for all $v, x \in V$.

Problem 6. Let $V$ be an inner product space and $S \subset V$ a subset. Show

a. $\text{Span}(S) \subset (S^\perp)^\perp$.

b. $\text{Span}(S) = (S^\perp)^\perp$ if $V$ is finite dimensional.

Problem 7. Let $V$ be a finite dimensional inner product space over $F$ and $W$ a subspace of $V$. Show that $P_W : V \to V$ defined by $v \to v_W$ is a linear operator and satisfies all of the following:

(i) $\text{im}(P_W) = W$ and $\ker(P_W) = W^\perp$. In particular, $V = \text{im}(P_W) \oplus \ker(P_W)$.

(ii) $P_W \circ P_W = P_W$.

(iii) If $W' \subset W^\perp$ is a subspace, then $P_W \circ P_{W'} = 0$.

(iv) $1_V = P_W + P_{W^\perp}$.

Problem 8. Let $A$ be an $n \times n$ real matrix. Let $\text{row}(A)$ be the subspace of $\mathbb{R}^n$ spanned by the rows of $A$ and $\text{col}(A)$ be the subspace of $\mathbb{R}^n$ spanned by the columns of $A$ viewed in $\mathbb{R}^n$. Note that if $R_i$ is the $i$th row of $A$ and $v \in \mathbb{R}^{n \times 1}$, then $R_i v$ is the dot product of $R_i$ and $v$. Show that $\ker A = (\text{row}(A))^\perp$.

Problem 9. Let $A$ be an $m \times n$ real matrix. Show that $(A^t A)^t = A^t A$ and if $A$ has rank $n$, then $A^t A$ is an invertible $n \times n$ matrix (see Problem 8).

Problem 10. Let $V$ be an inner product space over $F$. Let $S = \{v_1, \ldots, v_n\}$ be an orthogonal set. Suppose that no $v_i = 0$. Let $v \in V$. Prove Bessel’s Inequality:

$$\sum_{i=1}^n \frac{|\langle v, v_i \rangle|^2}{\|v_i\|^2} \leq \|v\|^2.$$  

Moreover, show equality holds if and only if

$$v = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i.$$