Math 110B Take-home Midterm

Part II

Let R be a (possibly) non-commutative ring below.

1. A short exact sequence of R-modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called *split* if one of the following three equivalent conditions holds:

(a) There exists an R-homomorphism $f' : B \to A$ such that $f'f = Id_A$.
We say that f is a *split monomorphism*.

(b) $f(A)$ is a *direct summand* of B, i.e., $B = f(A) \oplus D$ for some R-module D.

(c) There exists an R-homomorphism $g' : C \to B$ such that $gg' = Id_C$.
We say that g is a *split epimorphism*.

i. Prove that these conditions are equivalent.

ii. Show the sequence above splits if C is R-free.

2. Let P be an free R-module on basis $\mathcal{B} = \{x_i\}_{i \in I}$ and $\mathfrak{A} < R$ a (2-sided) ideal. Show:

(i). $P/\mathfrak{A}P \cong \coprod_I Rx_i/\mathfrak{A}x_i \cong \coprod_I R/\mathfrak{A}$.

(ii.) Let $^{-} : R \to R/\mathfrak{A}$ be the canonical ring epimorphism. Let

$$\overline{\mathcal{B}} = \{\overline{x}_i := x_i + \mathfrak{A}P \mid i \in I\}.$$

Then $P/\mathfrak{A}P$ is a free $\overline{R}$-module on basis $\overline{\mathcal{B}}$ and $|\overline{\mathcal{B}}| = |\mathcal{B}|$.

(iii.) R is said to have satisfy the *invariant dimension property* or *IDP* if every basis for a finitely generated free R-module has the same number of elements. Let $\phi : R \to S$ be a ring epimorphism with $S \neq 0$. If S satisfies IDP so does R.

(iv). Any commutative ring satisfies IDP.

3. Let $M \neq 0$ be an R-module.

i. If M is a *simple* R-module, i.e., M has no proper submodules prove that $\mathrm{End}_R(M)$ is a division ring.

ii. Suppose that M is a *noetherian* R-module, i.e., the collection of submodules of M satisfies the *ascending chain condition.* Show that an R-endomorphism $f : M \to M$ is an isomorphism if it is surjective.

4. Let R be a euclidean domain. Let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by all matrices of the form $I + \lambda$ where λ is a matrix with precisely one non zero entry and this entry does not occur on the diagonal and I is the $n \times n$ identity matrix. Show that $SL_n(R) = E_n(R)$.

5. Let A be a finite abelian group and let $\hat{A} := \{\chi : A \to \mathbf{C}^\times \mid \chi \text{ a group homomorphism}\}$. It is easily checked that $\hat{A}$ is a group via $\chi_1\chi_2(x) := \chi_1(x)\chi_2(x)$. Show
   (i.) A and $\hat{A}$ have the same order and, in fact, are isomorphic.
   (ii.) If χ is not the identity element of $\hat{A}$ then $\sum_{a \in A} \chi(a) = 0$.