Math 110B Take-home Midterm

Instructions: Do all ten problems in Parts I and Parts II. Parts in problems do NOT count equally.

You may use books, the web, and other (non-human) sources, although you should make a serious effort to do each problem by yourself without looking anything up.

If you copy a proof from any source, you must reference that source and the pages, web address (or description) that you are using. If the sources's proof uses results that we have not done, you must write up those results also. You must also fill in the details that the source leaves out. This includes the text for the class. Failure to site a source in a proof, even if a partially copied solution will result in a negative score equal to the value of that problem. **You must write or type up your results. No cutting and pasting is allowed.**

Part I

Let R be a (commutative) ring below.

1. A *zero divisor* in a ring is an element x so that $xy = 0$ for some non-zero element y in the ring. If $f(t) \in R[t]$ is a zero divisor prove that there exists a non-zero element b in R so that $bf = 0$.
2. Let $S \subset R$ be a multiplicative set such that $0 \notin S$. We say that S is *saturated* if $ab \in S$ implies that $a, b \in S$. Prove that
   i. S is a saturated multiplicative set if and only if $R \setminus S$ is a union of prime ideals.
   ii. The set of zero divisors in a commutative ring is a union of prime ideals.
3. Let $S \subset R$ be a subrng (it does not have to have a 1 nor be an ideal).
   i. Let $\mathfrak{A}_1, ..., \mathfrak{A}_n$ be ideals in R, at least $n-2$ of which are prime. Suppose that $S \subset R$ is contained in $\mathfrak{A}_1 \cup ... \cup \mathfrak{A}_n$. Then one of the $\mathfrak{A}_j$'s contains S.
   ii. If $\mathfrak{P}_1, ..., \mathfrak{P}_n$ are prime ideals in R and $\mathfrak{B}$ is an ideal properly contained in S such that $S \setminus \mathfrak{B} \subset \mathfrak{P}_1 \cup ... \cup \mathfrak{P}_n$ then S lies in one of the $\mathfrak{P}_i$'s.
4. Prove each of the following is a euclidean domain:
   i. $\mathbf{Z}\left[\dfrac{-1-\sqrt{-3}}{2}\right]$.
   ii. $\mathbf{Z}[\sqrt{3}\,]$.
5. Let F be a field. Prove both of the following:
   i. There exist infinitely many monic irreducible polynomials in $F[t]$.
   ii. If F is algebraically closed then it must be infinite.