HW #9

1. Let R be a PID and $0 \neq M$ a cyclic R-module. Suppose that $M \cong R/(d)$. Let $d = p_1^{e_1} \cdots p_n^{e_n}$ with each $e_i \geq 1$ and the p_i non-associate primes. Show that
   (i) $M_{p_i} \cong R/(p_i^{e_i})$ for all i.
   (ii) $M = \bigoplus_{i=1}^{n} M_{p_i} \cong \coprod_{i=1}^{n} R/(p_i^{e_i})$.

2. Let $A \in \mathbf{M}_n(\mathbf{C})$. Show that A is similar to a diagonal matrix if and only if the minimal polynomial has no repeated roots.

3. Determine all abelian groups of order 400 up to isomorphism.

4. Show that every complex $n \times n$ matrix is similar to a matrix of the form $D + N$ where D is a diagonal matrix, N is a nilpotent matrix (i.e., $N^r = 0$ some $r > 0$), and $DN = ND$.

5. Let V be a finite dimensional real vector space, T an $\mathbf{R}$-endomorphism. Suppose that the minimal polynomial of T factors into linear terms over $\mathbf{R}$ with no repeated roots and all roots are non-negative. Then there exists an $\mathbf{R}$-endomorphism S of V such that $S^2 = T$.

6. Let $A, B \in \mathbf{M}_n\mathbf{C}$ be two diagonalizable matrices. Show that there is a matrix $P \in \mathbf{Gl}_n(\mathbf{C})$ such that **both** PAP^{-1} and PBP^{-1} are diagonal matrices if and only if $AB = BA$.
   [Hint: It is better to work with linear operators and use the Change of Basis Theorem. Recall that a linear operator $T : V \to V$ on a vector space V, is diagonalizable, i.e., has a matrix representation as a diagonal matrix relative to some basis of V if and only if V has a basis of eigenvectors for T.]