HW #8

1.(*) Let R be a PID, $d, e \in R \setminus \{0\}$. Let g be a gcd of d and e. Let $0 \neq p$ be a prime element [so $R/(p)$ is a field]. Let $N_i \subset M_i$, $i = 1, 2$, and N be R-modules. Show all the following R-isomorphisms:

a. $dR^n \cong R^n$.

b. $d(R/(e)) \cong R/(\frac{e}{g})$.

c. Let $R/(p)$ act on N/pN by

$$(r + (p))(x + pN) = rx + pN$$

i.e., the same action as R, since $p(N/pN) = 0$. Then N/pN becomes an $R/(p)$-vector space.

d. $(R/(d))/p(R/(d)) \cong \begin{cases} R/(p) & \text{if } p|d \\ 0 & \text{if } p \nmid d. \end{cases}$

e. $(M_1 \coprod M_2)/(N_1 \coprod N_2) \cong (M_1/N_1) \coprod (M_2/N_2)$.

2.(*) Let R be a domain, M an R-module, $m \in M$. We say m is an (R-) *torsion element* if there exists $r \neq 0$ in R such that $rm = 0$, i.e., $ann_R(m) \neq 0$. Let

$$M_t := \{m \in M \mid m \text{ is torsion }\}.$$

We say that M is *torsion-free* if $M_t = 0$. Show all of the following:

a. If $m \neq 0$ then m is not torsion if and only if Rm is free if and only if Rm is torsion-free.

b. $M_t \subset M$ is a submodule.

c. M/M_t is torsion-free.

3. Let R be a domain. Show that any R-free module is torsion-free. Show that $\mathbf{Q}$ is $\mathbf{Z}$-torsion-free but not $\mathbf{Z}$-free.