HW #7

1. Let M be an R-module and $M_i \subset M$ be R-submodules for all $i \in I$. We say that $\sum_{i\in I} M_i$ is an *internal direct sum* of the M_i if $M_i \cap \sum_{j\neq i} M_j = 0$ for all $i \in I$. If this is the case, we write $\oplus_{i\in I} M_i$ for $\sum_{i\in I} M_i$. Show that if this is the case then $\sum_{i\in I} M_i$ is R-isomorphic to the (external) direct sum $\coprod_{i\in I} M_i$.

2.(*) Suppose the following diagram of R-modules and R-homomorphisms is commutative and has exact rows:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

Show that if two of the maps α, β, γ are R-isomorphisms then all three are.

3.(*) Let M and $M_1, \ldots, M_n$ be R-modules. Show that $M \cong \coprod_{i=1}^n M_i$ if and only if there exist R-homomorphisms $\iota_i : M_i \to M$ and $\pi_i : M \to M_i$, $i = 1, \ldots, n$, satisfying all of the following:
   i. $\pi_i \iota_i = Id_{M_i}$ for $i = 1, \ldots n$.
   ii. $\pi_j \iota_i = 0$ for $i \neq j$.
   iii. $\iota_1 \pi_1 + \cdots + \iota_n \pi_n = Id_M$.

4. Let M be an R-module. Prove the following are equivalent:
   i. Every submodule of M is finitely generated (fg).
   ii. M satisfies ACC (the **ascending chain condition**), i.e., if $M_i \subseteq M$ are submodules and
   $$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$
   then there exists a positive integer N such that $M_N = M_{N+i}$ for all $i \geq 0$.
   iii. M satisfies the **Maximum Condition or Principle**, i.e., if $S \neq \emptyset$ is a collection of submodules of M then S contains a maximal element, that is a module $M_o \in S$ such that if $M_o \subseteq N$ with $N \in S$ then $N = M_o$.

   If M satisfies any of these equivalent conditions, we say that M is a **noetherian R-module**.

5. Let $N \subseteq M$ be R-modules. Show that M is an R-noetherian if and only if N and M/N are R-noetherian. In particular, show that if
$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$
is an exact sequence of R-modules and R-homorphisms with two of the modules M, M', M'' are R-noetherian then they all are.

6. Show if M, N are noetherian R-modules then so is $M \coprod N$.