HW #6

1. Let R be a UFD, F its quotient field. Let f, g be non-constant polynomials over R. Write $f = C(f)f_1$ and $g = C(g)g_1$ with f_1 and g_1 primitive polynomials in $R[t]$. Show
   a. If $f|g$ in $F[t]$ then $f_1|g_1$ in $R[t]$.
      In particular, if f and g are primitive then $f|g$ in $F[t]$ if and only if $f|g$ in $R[t]$.
   b. Suppose that f, g are primitive. Then f, g have a common factor over $F[t]$ if and only if they have a common factor over $R[t]$.

2.(*) Let $f = \sum_{i=0}^{n} a_i t^i$ be a polynomial with integer coefficients. Let $r = a/b$, with $b \neq 0$ and a and b relatively prime integers. If r is a root of f over $\mathbf{Q}$ then $b|a_n$ and if $a \neq 0$ then $a|a_0$. In particular, if f is monic then all rational roots of f (if any) are integers.

3. Let F be a field and $f \in F[t] \setminus F$. Describe the nilradical, $\mathrm{nil}(F[t]/(f))$.
   [Hint: You did this problem if $\mathbf{Z}$ replaces $F[t]$.]

4. Show that the irreducible polynomials over $\mathbf{R}[t]$ are either linear polynomials or quadratic polynomials of the form $at^2 + bt + c \in \mathbf{R}[t]$ with $a \neq 0$ and $b^2 - 4ac < 0$. [You may assume that we have proved that every non-constant polynominal $f \in \mathbf{C}[t]$ factors into a product of linear polynomials in $\mathbf{C}[t]$.]

5.(*) (Eisenstein's Criterion) Let R be a UFD and K be the quotient field of R. Let $0 \neq f = \sum_{i=0}^{n} a_i t^i \in R[t]$. Let p be an irreducible element in R. Suppose the coefficients of f satisfy:
   i. $p \mid a_i$ for all $0 \leq i < n$.
   ii. $p \nmid a_n$
   iii. $p^2 \nmid a_0$.

then f is irreducible in $K[t]$. In particular, if, in addition, f is primitive (e.g., if f is monic) then f is irreducible in $R[t]$.