HW #4

1. Let f, g be polynomials with coeficients in a commutative ring R. Suppose the leading coefficient of f is a unit (i.e., the coefficient of the highest term is a unit). Show that there are polynomials q and r with coefficients in R such that $g = fq + r$ with either $r = 0$ or the degree of r is less than the degree of f. This says the Division Algorithm holds when dividing by a polynomial with unit leading term. (This problem was also given to you last quarter – more or less.)

2. Let R be a domain. Show that $R[t]$, the ring of polynomials with coefficients in R is a a euclidean domain if and only if R is a field.

3.(*) Let $R = \mathbf{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbf{Z}\}$ a subring of $\mathbf{C}$ with d a positive square-free integer. Let $N : R \to \mathbf{Z}$ by $\alpha = a + b\sqrt{-d} \mapsto \alpha\overline{\alpha} = a^2 + db^2$. Show all of the following.
   a. The field of quotients of R is $Q[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbf{Q}\}$.
   b. $N : R \setminus \{0\} \to \mathbf{Z}$ is a monoid homomorphism.
   c. $R^\times = \{\alpha \in R \mid N(\alpha) = 1\}$ and compute this group for all d.
   d. The element α is irreducible in R if $N(\alpha)$ is a prime. Is the converse true?
   e. Suppose $d \geq 3$, then 2 is irreducible but not prime in R.

4. Show that $\mathbf{Z}[\sqrt{-2}]$ is a euclidean domain.

5. Let $R = \mathbf{Z}[\sqrt{-1}]$. Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be a factorization of an integer $n > 1$. Show that the following are equivalent:
   a. n is a sum of two squares.
   b. $n = N(\alpha)$ for some $\alpha \in R$.
   c. If $p_i \equiv 3 \pmod 4$ then e_i is even.

6.(*) Let $R = \mathbf{Z}[\sqrt{-5}]$. Show the all of the following:
   a. The elements $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are all irreducible but no two are associates. In particular, R is not a UFD.
   b. None of the elements $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are prime.

7. Let $R = \mathbf{Z}[\sqrt{-5}]$. Let $\mathfrak{P} = (2, 1 + \sqrt{-5})$. Show
   a. $\mathfrak{P}^2 = (2)$ in R.
   b. $\mathfrak{P}$ is a maximal ideal.
   c. $\mathfrak{P}$ is not a principal ideal.

8. Determine all prime elements, up to units, in $\mathbf{Z}[\sqrt{-1}]$.