Mathematics 110BH

HW #3

1.(*) We consider three properties of commutative rings.

a. A commutative ring R is called *noetherian* if it satisfies the following condition (called the *ascending chain condition* or *ACC*): Any chain of ideals

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}_n \subset \cdots$$

(countable) in R *stabilizes*, i.e., there exists an integer N such that $\mathfrak{A}_{N+i} = \mathfrak{A}_N$ for all $i \geq 0$. Equivalently, there exist no infinite chains

$$\mathfrak{B}_1 < \mathfrak{B}_2 < \cdots < \mathfrak{B}_n < \cdots.$$

b. An ideal $\mathfrak{A}$ in a commutative ring R is called *finitely generated* or *fg* if there exist $a_1, ..., a_n \in \mathfrak{A}$ some n such that $\mathfrak{A} = Ra_1 + \cdots + Ra_n$.

c. We say a commutative ring satisfies the *maximal condition* if any non-empty collection of ideals in R has a maximal element (under set inclusion). [Such a maximal element, of course, need not be a maximal ideal.]

Prove that the following are equivalent for a commutative ring R:

i. R is noetherian.

ii. Every ideal of R is finitely generated.

iii. R satisfies the maximal condition.

[You may assume the Axiom of Choice.]

2. Let R be a noetherian domain. Let r be a non-zero non-unit in R. Prove that r is a product of finitely many irreducible elements.

3.(*) Let R be a domain satisfying the maximal condition. Show that any non-trivial ideal of R contains a finite product of non-zero prime ideals, i.e., if $0 < \mathfrak{A} < R$ is an ideal then there exist non-zero prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_n$ in R such that $\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n \subset \mathfrak{A}$.

4. An ideal $\mathfrak{C}$ in a commutative ring R is called *irreducible* if $\mathfrak{C} = \mathfrak{A} \cap \mathfrak{B}$ for some ideals $\mathfrak{A}$ and $\mathfrak{B}$ in R then either $\mathfrak{C} = \mathfrak{A}$ or $\mathfrak{C} = \mathfrak{B}$. Show if R has the maximal property then every ideal $\mathfrak{A} < R$ is a finite intersection of irreducible ideals of R, i.e., $\mathfrak{A} = \mathfrak{C}_1 \cap \cdots \cap \mathfrak{C}_n$, for some irreducible ideals $\mathfrak{C}_i$ in R.

5. We shall see that $\mathbf{Z}[t]$, the ring of polynomials with coefficients in $\mathbf{Z}$, is a UFD. It is not a PID. Show this by finding a non-principal maximal ideal in $\mathbf{Z}[t]$. Also find a non-zero prime ideal in $\mathbf{Z}[t]$ that is not maximal and prove it is such.

6. Rngs may not have maximal ideals. This problem constructs one. Let p be a prime number. Let $\mathbf{Z}_{p^\infty}$ be the (additive) subgroup of $\mathbf{Q}/\mathbf{Z}$ consisting of all elements having order some power of p, i.e., $x = \alpha + \mathbf{Z}$, $\alpha \in \mathbf{Q}$, lies in $\mathbf{Z}_{p^\infty}$ if and only if $p^r x = 0$ in $\mathbf{Q}/\mathbf{Z}$, i.e., $p^r\alpha$ is an integer, for some positive integer r.

i. Show that the set $\{\frac{1}{p^r} + \mathbf{Z} \,|\, r \text{ a non-negative integer } \}$ generates $\mathbf{Z}_{p^\infty}$.

ii. Show the subgroup $\left\langle \frac{1}{p^r} + \mathbf{Z} \right\rangle$ of $\mathbf{Z}_{p^\infty}$ is isomorphic to $\mathbf{Z}/p^r\mathbf{Z}$.

iii. Show that any proper subgroup of $\mathbf{Z}_{p^\infty}$ is $\left\langle \frac{1}{p^r} + \mathbf{Z} \right\rangle$ for some non-negative integer r and the subgroups of $\mathbf{Z}_{p^\infty}$ form a chain under set inclusion.

iv. Make $\mathbf{Z}_{p^\infty}$ into a rng by defining $x \cdot y = 0$ for all $x, y \in \mathbf{Z}_{p^\infty}$. Show that $\mathbf{Z}_{p^\infty}$ has no maximal ideals.