Mathematics 110BH

HW #2

1. Let R be a domain with finitely many elements. Show that the characteristic of R is p for some (positive) prime p and R is a field.

2. (*) Let $\varphi : R \to S$ be a ring homomorphism of commutative rings. Show that if $\mathcal{B}$ is an ideal (respectively, prime ideal) of S then $\varphi^{-1}(\mathcal{B})$ is an ideal (respectively, prime ideal) of R. Give an example, where $\mathcal{B}$ is a maximal ideal of S but $\varphi^{-1}(\mathcal{B})$ is not a maximal ideal of R.

3. (**) Let R be a commutative ring and S a *multiplicative set* in R, i.e., a subset of R containing 1 satisfying whenever a and b are elements of S then so is ab. Let

$$\mathcal{F} := \{(r, s) \mid r \in R, s \in S\}.$$

Define $\sim$ on $\mathcal{F}$ by $(r, s) \sim (r', s')$ if there exists an element $s'' \in S$ such that

$$s''(rs' - sr') = 0.$$

Do all of the following:

a. Prove that $\sim$ is an equivalence relation on $\mathcal{F}$.
Denote the equivalence class of (r, s) under $\sim$ by $\frac{r}{s}$ and let $S^{-1}R := \{\frac{r}{s} \mid r \in R,\ s \in S\}$ be the set of equivalence classes. Define

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + sr'}{ss'} \text{ and}$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

b. Prove that these operations make $S^{-1}R$ into a commutative ring.
c. Prove that the map $\varphi : R \to S^{-1}R$ by $\varphi(r) = \frac{r}{1}$ is a ring homomorphism. Determine the kernel of φ.
d. Suppose that $0 \notin S$. Show that the kernel of φ above does not contain any element of S.
e. Prove that every element of the form $\frac{s}{s'}$ with $s, s' \in S$ is a unit in $S^{-1}R$.

4. Let R be a commutative ring and S a multiplicative set in R. Let $\varphi : R \to S^{-1}R$ be given by $r \mapsto \frac{r}{1}$. Show that this satisfies the following *universal property*. If $\psi : R \to R'$ is a ring homomorphism with R' commutative and $\psi(S)$ a subset of the unit group of R',

then there exists a unique ring homomorphism $\theta : S^{-1}R \to R'$ such that

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S^{-1}R \\ & \underset{\psi}{\searrow} & \downarrow \theta \\ & & R' \end{array}$$

5. Let $\mathfrak{p}$ be a prime ideal of commutative ring R. Show that $S = R \backslash \mathfrak{p}$ is a multiplicative set. Write $R_{\mathfrak{p}} = S^{-1}R$ where $S^{-1}R$ is as in Problem 3. Determine all maximal ideals of $S^{-1}R$

6. If R is a non-commutative rng satisfying $x^3 = x$ for all x in R then R is commutative.

7. Let R be a commutative ring of prime characteristic $p > 0$. Prove that the map $R \to R$ by $x \mapsto x^p$ is a ring homomomorphism. It is called the *Frobenius homomorphism*.