HW #2

- 1. Let R be a domain with finitely many elements. Show that the characteristic of R is p for some (positive) prime p and R is a field.
- 2. (*) Let $\varphi: R \to S$ be a ring homomorphism of commutative rings. Show that if \mathcal{B} is an ideal (respectively, prime ideal) of S then $\varphi^{-1}(\mathcal{B})$ is an ideal (respectively, prime ideal) of R. Give an example, where \mathcal{B} is a maximal ideal of S but $\varphi^{-1}(\mathcal{B})$ is not a maximal ideal of R.
- 3. (**) Let R be a commutative ring and S a multiplicative set in R, i.e., a subset of R containing 1 satisfying whenever a and b are elements of S then so is ab. Let

$$\mathcal{F} := \{ (r, s) \mid r \in R, s \in S \}.$$

Define \sim on \mathcal{F} by $(r,s) \sim (r',s')$ if there exists an element $s'' \in S$ such that

$$s''(rs' - sr') = 0.$$

Do all of the following:

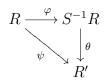
a. Prove that \sim is an equivalence relation on \mathcal{F} . Denote the equivalence class of (r,s) under \sim by $\frac{r}{s}$ and let $S^{-1}R := \{\frac{r}{s} \mid r \in R, s \in S\}$ be the set of equivalence classes. Define

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + sr'}{ss'} \text{ and}$$

$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

- b. Prove that these operations make $S^{-1}R$ into a commutative ring.
- c. Prove that the map $\varphi: R \to S^{-1}R$ by $\varphi(r) = \frac{r}{1}$ is a ring homomorphism. Determine the kernel of φ .
- d. Suppose that $0 \notin S$. Show that the kernel of φ above does not contain any element of S.
- e. Prove that every element of the form $\frac{s}{s'}$ with $s, s' \in S$ is a unit in $S^{-1}R$.
- 4. Let R be a commutative ring and S a multiplicative set in R. Let $\varphi: R \to S^{-1}R$ be given by $r \mapsto \frac{r}{1}$. Show that this satisfies the following universal property. If $\psi: R \to R'$ is a ring homomorphism with R' commutative and $\psi(S)$ a subset of the unit group of R',

then there exists a unique ring homomorphism $\theta:S^{-1}R\to R'$ such that



- 5. Let $\mathfrak p$ be a prime ideal of commutative ring R. Show that $S=R\backslash \mathfrak p$ is a multiplicative set. Write $R_{\mathfrak p}=S^{-1}R$ where $S^{-1}R$ is as in Problem 3. Determine all maximal ideals of $S^{-1}R$
- 6. If R is a non-commutative rng satisfying $x^3 = x$ for all x in R then R is commutative.
- 7. Let R be a commutative ring of prime characteristic p > 0. Prove that the map $R \to R$ by $x \mapsto x^p$ is a ring homomomorphism. It is called the *Frobenius homomorphism*.