2-REPRESENTATIONS OF $\mathfrak{sl}_2$ FROM QUASI-MAPS

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1. Introduction

There are two main classes of constructions of 2-representations of Kac-Moody algebras [Rou2]. One is algebraic, for example via representations of cyclotomic quiver Hecke algebras, and the other uses constructible sheaves, for example on quiver varieties. We describe here a new type of 2-representations, using coherent sheaves.

One of our motivations is to develop an affinization of the theory of 2-representations of Kac-Moody algebras. One would want a theory of 2-representations of affinizations of symmetrizable Kac-Moody algebras, or rather of the larger Maulik-Okounkov algebras [MauOk].

A classical theme of (geometric) representation theory is that affinizations arise from degenerations. Shan, Varagnolo and Vasserot [ShaVarVas, VarVas] and the author have proposed that the affinization of the monoidal category associated to the positive part of a symmetric Kac-Moody algebra should be a full monoidal subcategory of the derived category of $\mathcal{O}$-modules on the derived cotangent stack of the moduli stack of representations of a corresponding quiver. A description of this category by generators and relations is missing, even in the case of $\mathfrak{sl}_2$.

This article stems from efforts to better understand 2-representations on categories of coherent sheaves. A number of constructions have been given by Cautis, Kamnitzer and Licata (cf e.g. [CauKaLi]). We study here a different geometrical framework. Feigin, Finkelberg, Kuznetsov, Mirković and Braverman [FeiFiKuMi, Bra] have provided a construction of Verma modules for complex semi-simple Lie algebras using based quasi-map spaces from $\mathbf{P}^1$ to flag varieties (zastavas). We consider here the case of $\mathfrak{sl}_2$, where the zastavas are smooth, and are mere affine spaces. We show that coherent sheaves on zastavas provide a 2-Verma module for $\mathfrak{sl}_2$ in the sense of Naisse-Vaz [NaiVa1]. Adding a superpotential and considering matrix factorizations, we obtain a realization of simple 2-representations of $\mathfrak{sl}_2$.

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2. Zastavas and correspondences

2.1. **Quasi-maps.** We fix a field k and we consider varieties over k. The space of maps $\mathbf{P}^1 \to \mathbf{P}^1$ of degree d sending ∞ to ∞ identifies with the space of pairs (g, h) of polynomials such that g and h have no common roots, $\deg(g(z) - z^d) < d$ and $\deg h < d$.

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The zastava space V_d of quasi-maps $\mathbf{P}^1 \to \mathbf{P}^1$ defined in a neighborhood of ∞ and sending ∞ to ∞ is the space of pairs as above, without the condition on roots. There is an isomorphism

$$\mathbf{A}^{2d} \xrightarrow{\sim} V_d,\ (a,b) \mapsto (g(z) = a_1 + a_2 z + \cdots + a_d z^{d-1} + z^d, h(z) = b_1 + b_2 z + \cdots + b_d z^{d-1}).$$

There is an action of $\mathbf{T} = \mathbf{G}_m \times \mathbf{G}_m$ on V_d. The first $\mathbf{G}_m$-action is by rescaling the variable z with weight -2. The second $\mathbf{G}_m$-action is by scalar action on $h(z)$ with weight 2.

2.2. **Correspondences.** Let $Y_d = V_d \times \mathbf{A}^1$. We extend the $\mathbf{T}$-action on V_d to an action on Y_d by letting $\mathbf{T}$ act on $\mathbf{A}^1$ by weight $(-2, 0)$.

We have a diagram of affine varieties with $\mathbf{T}$-actions

$$\begin{array}{ccc} & Y_d & \\ {\scriptstyle (g,h,z_0)\mapsto(g,h)} \swarrow \phi_d & & \psi_d \searrow {\scriptstyle (g,h,z_0)\mapsto((z-z_0)g,(z-z_0)h)} \\ V_d & & V_{d+1} \end{array}$$

We have functors

$$F_d = \psi_{d*} \circ \phi_d^* : D^b_{\mathbf{T}}(V_d\text{-qcoh}) \to D^b_{\mathbf{T}}(V_{d+1}\text{-qcoh})$$
$$E_d = \phi_{d*} \circ \mathbf{L}\psi_d^* : D^b_{\mathbf{T}}(V_{d+1}\text{-qcoh}) \to D^b_{\mathbf{T}}(V_d\text{-qcoh}).$$

2.3. **Universal Verma module.** Let $U_v(\mathfrak{sl}_2)$ be the quantum enveloping algebra of $\mathfrak{sl}_2$. It is the $\mathbf{Q}(v)$-algebra generated by e, f and $k^{\pm 1}$ subject to the relations

$$ke = v^2 ek,\ kf = v^{-2} fk,\ ef - fe = \frac{k - k^{-1}}{v - v^{-1}}.$$

The universal Verma module M_κ is the $U_v(\mathfrak{sl}_2)$-module over $\mathbf{Q}(v,\kappa)$ with basis $(m_d)_{d\geq 0}$, with

$$k(m_d) = \kappa v^{-2d} m_d,\ e(m_d) = \delta_{d,0} m_{d-1} \text{ and } f(m_d) = \frac{v^{d+1} - v^{-d-1}}{v - v^{-1}} \cdot \frac{\kappa v^{-d} - \kappa^{-1} v^d}{v - v^{-1}}$$

Specializing κ to v^λ gives the Verma module with highest weight λ.

2.4. **Geometric realization.** Let $\mathcal{C}$ be the category of bigraded vector spaces N such that $\dim(\bigoplus_{i\leq n, j\in\mathbf{Z}} N_{ij}) < \infty$ for all n. Taking bigraded dimension gives an isomorphism

$$K_0(\mathcal{C}) \xrightarrow{\sim} \mathbf{Z}((v)) \otimes \mathbf{Z}[t^{\pm 1}],\ N \mapsto \sum_{i,j} v^i t^j \dim N_{ij}.$$

We denote by $\mathcal{T}_d$ the full triangulated subcategory of $D^b_{\mathbf{T}}(V_d\text{-qcoh})$ generated by objects $N \otimes \mathcal{O}_{V_d}$ for $N \in \mathcal{C}$.

Given $N \in \mathcal{C}$ with class $P \in \mathbf{Z}((v)) \otimes \mathbf{Z}[t^{\pm 1}]$ and given $C \in D^b_{\mathbf{T}}(V_d\text{-qcoh})$, we write $P \cdot C$ for the object $N \otimes_k C$ of $D^b_{\mathbf{T}}(V_d\text{-qcoh})$ (well defined up to isomorphism).

We put $E = \bigoplus_{d\geq 0} E_d$ and $F = \bigoplus_{d\geq 0} t v^{-2d} F_d[1]$. The following proposition is an immediate consequence of Lemma 2.2 below. It is a variant of a result of Braverman and Finkelberg [BraFi].

Proposition 2.1. *The actions of* $[E]$ *and* $[F]$ *on* $M = \bigoplus_{d\geq 0} \mathbf{Q}(v,t)\otimes_{\mathbf{Z}[v,t]} K_0(\mathcal{T}_d)$ *give an action of* $U_v(\mathfrak{sl}_2)$ *and there is an isomorphism of representations*

$$\mathbf{Q}((v))\otimes_{\mathbf{Q}(v)} M_{tv^{-1}} \xrightarrow{\sim} M,\ m_d \mapsto (1-v^2)^d[\mathcal{O}_{V_d}].$$

2.5. **Modules.** Let $A_d = k[V_d] = k[a_1,\dots,a_d,b_1,\dots,b_d]$, a bigraded algebra with $\deg(a_i) = (2(d-i+1),0)$ and $\deg(b_i) = (2(d-i+1),-2)$.

Let $B_d = k[Y_d] = k[a_1,\dots,a_d,b_1,\dots,b_d,c]$, a bigraded algebra with $\deg(a_i) = (2(d-i+1),0)$, $\deg(b_i) = (2(d-i+1),-2)$ and $\deg(c) = (2,0)$.

There is a bigraded action of A_d on B_d by multiplication and a bigraded action of A_{d+1} on B_d given by multiplication preceded by the morphism of algebras

$$f : A_{d+1} \to B_d,\ a_i \mapsto a_{i-1} - ca_i \text{ and } b_i \mapsto b_{i-1} - cb_i$$

where we put $a_0 = b_0 = b_{d+1} = 0$ and $a_{d+1} = 1$ in B_d.

Via the equivalences $\Gamma : D^b_{\mathbf{T}}(V_d\text{-qcoh}) \xrightarrow{\sim} D^b_{bigr}(A_d\text{-Mod})$, the functors E_d and F_d become

$$F_d = B_d \otimes_{A_d} - : D^b_{bigr}(A_d\text{-Mod}) \to D^b_{bigr}(A_{d+1}\text{-Mod})$$
$$E_d = B_d \otimes^{\mathbf{L}}_{A_{d+1}} - : D^b_{bigr}(A_{d+1}\text{-Mod}) \to D^b_{bigr}(A_d\text{-Mod}).$$

Lemma 2.2. *We have* $[E_d(A_{d+1})] = \frac{1}{1-v^2}[A_d]$ *and* $[F_d(A_d)] = \frac{(1-t^{-2}v^{2(d+1)})(1-v^{2(d+1)})}{1-v^2}[A_{d+1}]$.

Proof. We have $B_d \simeq A_d \otimes k[c]$ as bigraded A_d-modules and the first statement follows.

The second statement follows from Lemma 2.3 below. □

Lemma 2.3. *There is an exact sequence of bigraded* A_{d+1}*-modules*

$$0 \to t^{-2}v^{2(d+1)}\frac{1-v^{2(d+1}}{1-v^2}A_{d+1} \to \frac{1-v^{2(d+1)}}{1-v^2}A_{d+1} \to B_d \to 0.$$

Proof. Let $C = k[a_1,\dots,a_d,c',b_1,\dots,b_d,c'']$. The morphism f is the composition of the following morphisms of algebras:

$$f_1 : k[a_1,\dots,a_{d+1},b_1,\dots,b_{d+1}] \to k[a_1,\dots,a_d,c',b_1,\dots,b_{d+1}]$$
$$b_i \mapsto b_i,\ a_i \mapsto \begin{cases} -c'a_1 & \text{for } i=1 \\ a_{i-1} - c'a_i & \text{for } 1 < i \leq d \\ a_d - c' & \text{for } i = d+1 \end{cases}$$
$$f_2 : k[a_1,\dots,a_d,c',b_1,\dots,b_{d+1}] \xrightarrow{\sim} k[a_1,\dots,a_d,c',b_1,\dots,b_{d+1}]$$
$$a_i \mapsto a_i,\ c' \mapsto c',\ b_i \mapsto \begin{cases} b_i & \text{for } i \leq d \\ b_{d+1} + c' & \text{for } i = d+1 \end{cases}$$
$$f_3 : k[a_1,\dots,a_d,c',b_1,\dots,b_{d+1}] \to C$$
$$a_i \mapsto a_i,\ c' \mapsto c',\ b_i \mapsto \begin{cases} -c''b_1 & \text{for } i=1 \\ b_{i-1} - c''b_i & \text{for } 1 < i \leq d \\ b_d - c'' & \text{for } i = d+1 \end{cases}$$

$$f_4 : C \to k[a_1, \ldots, a_d, b_1, \ldots, b_d, c]$$

$$a_i \mapsto a_i,\ b_i \mapsto b_i,\ c' \mapsto c,\ c'' \mapsto c.$$

The first morphism makes $k[a_1, \ldots, a_d, c', b_1, \ldots, b_{d+1}]$ into a free $k[a_1, \ldots, a_{d+1}, b_1, \ldots, b_{d+1}]$-module with basis $(1, c', \ldots, c'^d)$.

The third morphism makes C into a free $k[a_1, \ldots, a_d, c', b_1, \ldots, b_{d+1}]$-module with basis $(1, c'', \ldots, c''^d)$.

The last morphism makes $k[a_1, \ldots, a_d, b_1, \ldots, b_d, c]$ fit into an exact sequence of C-modules

$$0 \to C \xrightarrow{c'-c''} C \xrightarrow{f_4} k[a_1, \ldots, a_d, b_1, \ldots, b_d, c] \to 0.$$

Let L_1 (resp. L_0) be the free A_{d+1}-module with basis $(e_{ij})_{0 \le i,j \le d}$ (resp. $(f_{ij})_{0 \le i,j \le d}$). We have a commutative diagram of A_{d+1}-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{d_1} & L_0 & \xrightarrow{d_0} & k[a_1, \ldots, a_d, b_1, \ldots, b_d, c] & \longrightarrow & 0 \\ & & \sim \downarrow \alpha_1 & & \sim \downarrow \alpha_0 & & \| & & \\ 0 & \longrightarrow & C & \xrightarrow[c'-c'']{} & C & \xrightarrow[f_4]{} & k[a_1, \ldots, a_d, b_1, \ldots, b_d, c] & \longrightarrow & 0 \end{array}$$

where the structure of A_{d+1}-module on C comes from $f_3 \circ f_2 \circ f_1$, the one on $k[a_1, \ldots, a_d, b_1, \ldots, b_d, c]$ from $f_4 \circ f_3 \circ f_2 \circ f_1$ and where

$$d_0(f_{ij}) = c^{i+j}$$

$$d_1(e_{ij}) = \begin{cases} f_{i+1,j} - f_{i,j+1} & \text{for } 0 \le i, j < d \\ -(a_1 f_{0j} + a_2 f_{1j} + \cdots + a_{d+1} f_{dj} + f_{d,j+1}) & \text{for } i = d,\ 0 \le j < d \\ b_1 f_{i0} + b_2 f_{i1} + \cdots + b_{d+1} f_{id} & \text{for } j = d \end{cases}$$

$$\alpha_1(e_{ij}) = c'^i c''^j$$

$$\alpha_0(f_{ij}) = c'^i c''^j$$

Note that d_1 and d_0 are morphisms of bigraded modules with

$$\deg(e_{ij}) = \begin{cases} (2(i+j+1), 0) & \text{for } j \ne d \\ (2(i+d+1), -2) & \text{for } j = d \end{cases} \text{ and } \deg(f_{ij}) = (i+j, 0).$$

The A_{d+1}-module L_0 is generated by $\{d_1(e_{ij})\}_{0 \le i < d, 0 \le j \le d}$ and $(f_{i0})_{0 \le i \le d}$. It follows that the complex $0 \to L_1 \xrightarrow{d_1} L_0 \to 0$ is homotopy equivalent to a complex of the form

$$0 \to \bigoplus_{0 \le j \le d} A_{d+1} e_{d,j} \to \bigoplus_{0 \le i \le d} A_{d+1} f_{i,d} \to 0.$$

The lemma follows. □

3. 2-Representations

We construct now endomorphisms of F and F^2, leading to a structure of 2-representation.

• Let $\rho_d : Y_d \to \mathbf{A}^1$ be the projection map. It provides a morphim $k[X] = \Gamma(\mathcal{O}_{\mathbf{A}^1}) \to \Gamma(\mathcal{O}_{Y_d})$, hence a morphism $k[X] \to \mathrm{End}(F_d)$.

• There is an action of $\mathbf{G}_a$ on $V_d \times V_{d+1}$ given by $u \cdot \big((g,h),(g',h')\big) = \big((g,h),(g',h'+u)\big)$. It provides a map $\mathrm{Lie}(\mathbf{G}_a) = k \to \Gamma(\mathcal{T}_{V_d \times V_{d+1}})$ and we denote by ω' the image of 1, a vector field on $V_d \times V_{d+1}$.

The morphism $\phi_d \times \psi_d : Y_d \to V_d \times V_{d+1}$ is a closed immersion and we identify Y_d with its image. There is a canonical isomorphism

$$F_d \xrightarrow{\sim} \pi_{2*}\big(\mathcal{O}_{Y_d} \otimes \pi_1^*(-)\big) \tag{1}$$

where $\pi_1 : V_d \times V_{d+1} \to V_d$ is the first projection and $\pi_2 : V_d \times V_{d+1} \to V_{d+1}$ the second projection.

Let ω'' be the image of ω' by the composition of canonical maps

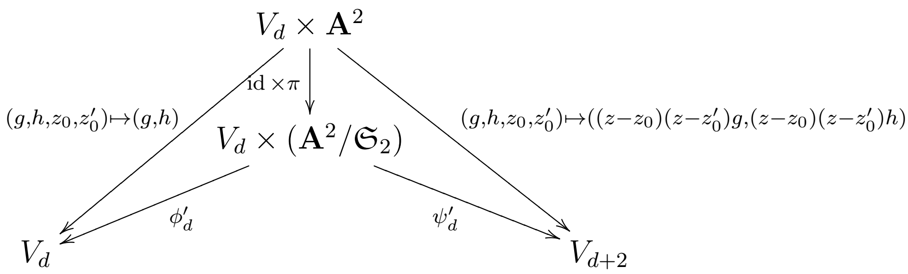
$$\Gamma(\mathcal{T}_{V_d \times V_{d+1}} \otimes \mathcal{O}_{Y_d}) \to \Gamma(\mathcal{N}_{Y_d/(V_d \times V_{d+1})}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_{V_d \times V_{d+1}}}(\mathcal{O}_{Y_d}, \mathcal{O}_{Y_d}).$$

Via the isomorphism (1), ω'' defines an element $\omega \in \mathrm{Hom}(F_d, F_d[1])$.

• There is an isomorphism

$$V_d \times \mathbf{A}^2 \xrightarrow{\sim} Y_d \times_{V_{d+1}} Y_{d+1},\ ((g,h,z_0,z_0') \mapsto \big((g,h,z_0),((z-z_0)g,(z-z_0)h,z_0')\big).$$

There is a commutative diagram



for some maps ϕ_d' and ψ_d', and where $\pi : \mathbf{A}^2 \to \mathbf{A}^2/\mathfrak{S}_2$ is the quotient map.

Consequently, we obtain an isomorphism

$$F_{d+1}F_d \xrightarrow{\sim} \psi'_{d*}((\mathcal{O}_{V_d} \boxtimes \pi_*(\mathcal{O}_{\mathbf{A}^2})) \otimes \phi_d'(-)). \tag{2}$$

Let ∂ be the endomorphism of $k[X_1, X_2] = \Gamma(\mathcal{O}_{\mathbf{A}^2})$ given by

$$\partial(P) = \frac{P(X_1,X_2) - P(X_2,X_1)}{X_2 - X_1}.$$

It induces an endomorphism of $\pi_*(\mathcal{O}_{\mathbf{A}^2})$, hence, via the isomorphism (2), an endomorphism T of $F_{d+1}F_d$.

The data of $((E_d)_d, X, T)$ above gives rise to a 2-representation of $\mathfrak{sl}_2^+$, but it does not extend to a 2-representation of $\mathfrak{sl}_2$. But the data of $((E_d)_d, X, T, \omega)$ gives rise to a 2-Verma module as defined by Naisse and Vaz (cf [NaiVa1] and [NaiVa2, Definition 4.1]).

Theorem 3.1. *The functors E_d, F_d and the natural transformations X, ω, T define a 2-Verma module for $\mathfrak{sl}_2$ on $\bigoplus_d \mathcal{T}_d$ equivalent to the universal 2-Verma module of* [NaiVa1, §5.2].

Proof. We show that our construction is equivalent to the Naisse-Vaz universal Verma module [NaiVa1, §5.2].

Let $\Omega_d = \mathrm{Ext}^*_{A_d}(A_d/(b_1,\ldots,b_d), A_d/(b_1,\ldots,b_d))$. The canonical Koszul isomorphism of graded algebras $\Lambda(b_1^*,\ldots,b_d^*) \xrightarrow{\sim} \mathrm{Ext}^*_{k[b_1,\ldots,b_d]}(k,k)$ induces an isomorphism of graded algebras

$$\iota : k[a_1,\ldots,a_d]\otimes\Lambda(b_1^*,\ldots,b_d^*) \xrightarrow{\sim} \Omega_d.$$

We denote by ω_i the image of $(-1)^{d+1-i}b^*_{d+1-i}$ in Ω_d and by x_i the image of $(-1)^i a_{d+1-i}$. We have $\Omega_d = k[x_1,\ldots,x_d]\otimes\Lambda(\omega_1,\ldots,\omega_d)$.

Consider the morphism of algebras $h : A_d\otimes A_{d+1}\to B_d,\ r\otimes s\mapsto rf(s)$. We put $a_i = a_i\otimes 1$, $b_i = b_i\otimes 1$, $a'_j = 1\otimes a_j$ and $b'_j = 1\otimes b_j$ for $1\le i\le d$ and $1\le j\le d+1$.

The morphism h is surjective, and its kernel is the ideal generated by $\tilde{b}_i = b'_i + \tilde{c}b_i - b_{i-1}$ where $\tilde{c} = a_d - a'_{d+1}$.

Let $R = k[a_1,\ldots,a_d,\tilde{c}] \subset A_d\otimes A_{d+1}$ and $I = \bigoplus_{1\le i\le d+1} R\tilde{b}_i$, an R-submodule of $L = \bigoplus_{1\le i\le d+1} Rb'_i \oplus \bigoplus_{1\le i\le d} Rb_i$. The morphism h restricts to a surjective morphism of algebras $S_R(L)\to B_d$ with kernel generated by L.

The orthogonal of I in $\mathrm{Hom}_R(L,R) = \bigoplus_{1\le i\le d+1} Rb'^*_i \oplus \bigoplus_{1\le i\le d} Rb^*_i$ is

$$I^\perp = \bigoplus_{1\le i\le d+1} R(b'^*_{i+1} + b^*_i - \tilde{c}b'^*_i).$$

Define

$$\Omega_{d,d+1} = \mathrm{Ext}^*_{A_d\otimes A_{d+1}}\Big(\big(A_d/(b_1,\ldots,b_d)\big)\otimes\big(A_{d+1}/(b_1,\ldots,b_{d+1})\big), B_d\Big).$$

The decomposition $A_d\otimes A_{d+1} = S_R(L)\otimes k[a'_1,\ldots,a'_d]$ induces an isomorphism

$$\Omega_{d,d+1} \xrightarrow{\sim} \mathrm{Ext}^*_{S_R(L)}(S_R(L)/(L), B_d)$$

hence

$$\Omega_{d,d+1} \xrightarrow{\sim} \mathrm{Ext}^*_{S_R(L)}(S_R(L)/(L), S_R(L)/(I)) \xrightarrow{\sim} \Lambda_R(\mathrm{Hom}_R(L,R)/I^\perp).$$

Let ω_i be the element of $\Omega_{d,d+1}$ corresponding to the image of $(-1)^{d+2-i}b'^*_{d+2-i}$ in $\mathrm{Hom}_R(L,R)/I^\perp$. Let y_i be the image of $(-1)^i a_{d+1-i}$ in $\Omega_{d,d+1}$ and ξ the image of $\tilde{c}$. We have $\Omega_{d,d+1} = k[y_1,\ldots,y_d,\xi]\otimes\Lambda(\omega_1,\ldots,\omega_{d+1})$. The actions of Ω_d and Ω_{d+1} on $\Omega_{d,d+1}$ are given by multiplication preceded by morphisms of algebras

$$\Omega_d\to\Omega_{d,d+1},\ x_i\mapsto y_i,\ \omega_i\mapsto\omega_i+\xi\omega_{i+1}$$

$$\Omega_{d+1}\to\Omega_{d,d+1},\ x_i\mapsto y_i+\xi y_{i-1},\ \omega_i\mapsto\omega_i.$$

Let $\mathcal{T}'_d$ be the full triangulated subcategory of $D^b_{bigr}(\Omega_d)$ generated by objects $N\otimes k[x_1,\ldots,x_d]$ for $N\in\mathcal{C}$.

There is an equivalence of triangulated categories $R\,\mathrm{Hom}^\bullet_{A_d}(A_d/(b_1,\ldots,b_d),-) : \mathcal{T}_d \xrightarrow{\sim} \mathcal{T}'_d$. This equivalence intertwines the action of E_d and F_d with the action of $\Omega_{d,d+1}\otimes_{\Omega_d} -$ and $\Omega_{d,d+1}\otimes_{\Omega_{d+1}} -$. This shows our construction is equivalent to that of Naisse and Vaz. $\square$

4. Finite-dimensional Simple modules

We fix now $n \geq 0$. We define a simple 2-representation of $\mathfrak{sl}_2$ by defining a superpotential on the universal 2-Verma module and considering matrix factorizations.

Let $(g, h) \in V_d$. We have

$$\frac{h(z^{-1})}{g(z^{-1})} = z\frac{b_d + b_{d-1}z + \cdots + b_1 z^{d-1}}{1 + a_d z + \cdots + a_1 z^d} = z\sum_{i\geq 0}(b_d v_{i,d} + \cdots + b_1 v_{i-d+1,d})z^i$$

for some polynomials functions $v_{i,d}$ of $a_1, \ldots, a_d$ with $v_{0,d} = 1$ and $v_{i,d} = 0$ for $i < 0$.

We define a morphism $W_{d,n} : V_d \times \mathbf{A}^n \to \mathbf{A}^1$ by

$$W_{d,n}\big((g,h),(\gamma_1,\ldots,\gamma_n)\big) = \sum_{i=0}^{n} \gamma_{i+1}(b_d v_{i,d} + \cdots + b_1 v_{i-d+1,d})$$

where we put $\gamma_{n+1} = 1$.

Note that $W_{d+1,n} \circ (\psi_d \times \mathrm{id}) = W_{d,n} \circ (\phi_d \times \mathrm{id})$ and we denote by $W_{d,n}$ that morphism $Y_d \times \mathbf{A}^n \to \mathbf{A}^1$. We endow $\mathbf{A}^n = \operatorname{Spec} k[\gamma_1, \ldots, \gamma_n]$ with an action of $(\mathbf{G}_m)^2$ with $\deg(\gamma_i) = (2(n+1-i), 0)$. This makes $W_{d,n}$ into a homogeneous map of degree $(2(n+1), -2)$.

We denote by $\mathcal{T}_{d,n}$ the homotopy category of $(\mathbf{G}_m)^2$-equivariant matrix factorizations of $W_{d,n}$ on $V_d \times \mathbf{A}^n$. The functors E_d and F_d of §2.2 and §2.4 extend to functors between the categories $\mathcal{T}_{d,n}$ and $\mathcal{T}_{d+1,n}$.

Proposition 4.1. *We have $\mathcal{T}_{d,n} = 0$ if $d > n$.*

The action of $[E]$ and $[F]$ on $\bigoplus_{d=0}^{n-1} \mathbf{Q} \otimes K_0(\mathcal{T}_{d,n})$ give an action of $U_q(\mathfrak{sl}_2)$ and the corresponding representation is simple of dimension $n+1$.

The data of (E, F, T, X) define a 2-representation of $\mathfrak{sl}_2$ on $\bigoplus_{d=0}^{n} \mathcal{T}_{d,n}$ equivalent to the homotopy category of bounded complexes of objects of the simple 2-representation $\mathcal{L}(n)$ of [Rou2, §4.3.2].

Proof. If $d > n$, then

$$W_{d,n} = b_d(\gamma_1 + \sum_{i=1}^{n}\gamma_{i+1}v_{i,d}) + b_{d-1}(\gamma_2 + \sum_{i=2}^{n}\gamma_{i+1}v_{i-1,d}) + \cdots + b_{d-n+1}(\gamma_n + v_{1,d}) + b_{d-n}.$$

As a consequence, the homotopy category of matrix factorizations $\mathcal{T}_{d,n}$ is 0.

Assume now $d \leq n$. We have

$$W_{d,n} = b_d(\gamma_1 + \sum_{i=1}^{n}\gamma_{i+1}v_{i,d}) + b_{d-1}(\gamma_2 + \sum_{i=2}^{n}\gamma_{i+1}v_{i-1,d}) + \cdots + b_1(\gamma_d + \sum_{i=d}^{n}\gamma_{i+1}v_{i-d+1,d}).$$

Let $P_n = k[x_1, \ldots, x_n]$, a graded algebra with $\deg(x_i) = 2$. Define

$$A_{d,n} = (A_d \otimes k[\gamma_1, \ldots, \gamma_n])/(\{b_r\}_{1\leq r\leq d} \bigcup \{\gamma_r + \sum_{i=r}^{n}\gamma_{i+1}v_{i-r+1,d}\}_{1\leq r\leq d}).$$

The inclusion map induces an isomorphism

$$k[a_1,\ldots,a_d]\otimes k[\gamma_{d+1},\ldots,\gamma_n]\xrightarrow{\sim} A_{d,n}.$$

Composing with the inverse of the isomorphism

$$k[a_1,\ldots,a_d,\gamma_{d+1},\ldots,\gamma_n]\xrightarrow{\sim} P_n^{\mathfrak{S}_d\times\mathfrak{S}_{n-d}},\ a_i\mapsto e_{d-i+1}(x_1,\ldots,x_d),\ \gamma_i\mapsto e_{n-i+1}(x_{d+1},\ldots,x_n)$$

we obtain an isomorphism of graded algebras

$$P_n^{\mathfrak{S}_d\times\mathfrak{S}_{n-d}}\xrightarrow{\sim} A_{d,n}.$$

We view $A_{d,n}$ as a $\mathbf{Z}$-graded algebra with $\deg(a_i)=2(d-i+1)$ and $\deg\gamma_i=2(n+1-i)$. We have an equivalence $D^b(A_{d,n}\text{-modgr})\xrightarrow{\sim}\mathcal{T}_{d,n}$.

Let $\mathcal{T}'_{d,n}$ be the homotopy category of $(\mathbf{G}_m)^2$-equivariant matrix factorizations of $W_{d,n}$ on $Y_d\times\mathbf{A}^n$. Let $B'_{d,n}=A_{d,n}\otimes k[c]$. We have an equivalence $D^b(B'_{d,n}\text{-modgr})\xrightarrow{\sim}\mathcal{T}'_{d,n}$. Let $B_{d,n}=B'_{d,n}\otimes_{A_{d+1}[\gamma_1,\ldots,\gamma_n]}A_{d+1,n}$. We have

$$B_{d,n}=B'_{d,n}/(\gamma_{d+1}+\sum_{i=d+1}^{n}\gamma_{i+1}\sum_{j=0}^{i-d}v_{j,d}c^{i-j-d})$$

and the inclusion map induces an isomorphism

$$k[a_1,\ldots,a_d,c,\gamma_{d+2},\ldots,\gamma_n]\xrightarrow{\sim} B_{d,n}.$$

Composing with the inverse of the isomorphism

$$k[a_1,\ldots,a_d,c,\gamma_{d+2},\ldots,\gamma_n]\xrightarrow{\sim} P_n^{\mathfrak{S}_d\times\mathfrak{S}_{n-d-1}}$$

$$a_i\mapsto e_{d-i+1}(x_1,\ldots,x_d),\ c\mapsto -x_{d+1},\ \gamma_i\mapsto e_{n-i+1}(x_{d+2},\ldots,x_n)$$

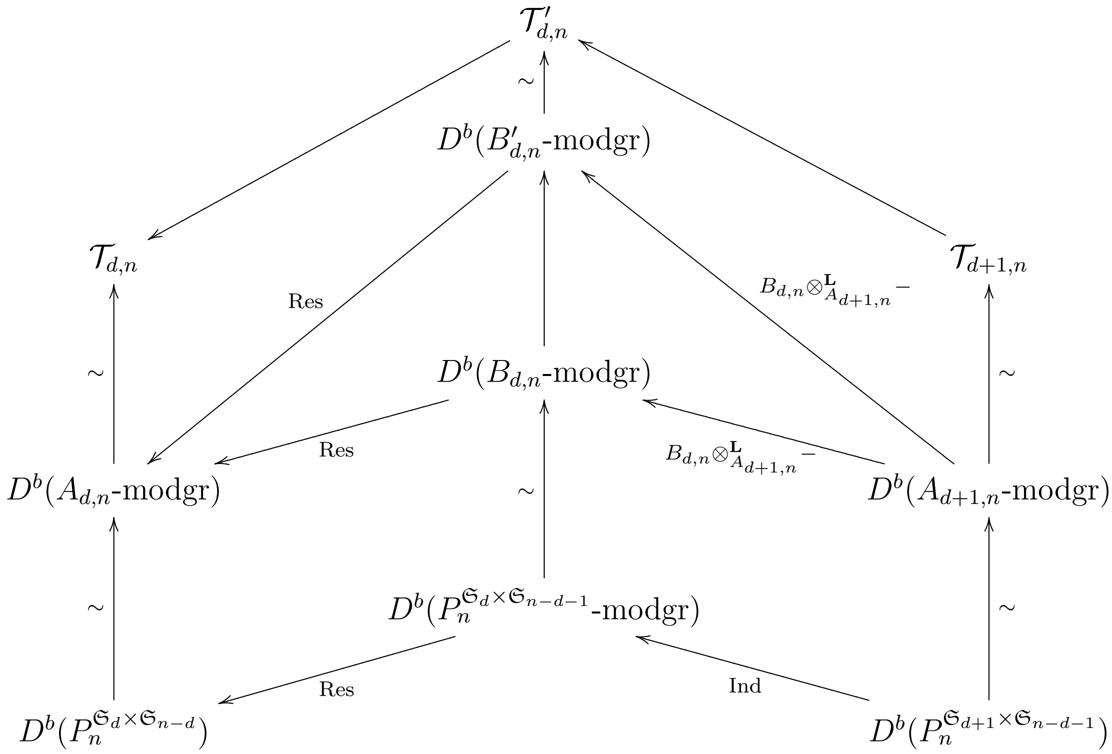
we obtain an isomorphism of graded algebras

$$P_n^{\mathfrak{S}_d\times\mathfrak{S}_{n-d-1}}\xrightarrow{\sim} B_{d,n}.$$

There is a commutative diagram

$$\begin{array}{ccc}
P_n^{\mathfrak{S}_{d+1}\times\mathfrak{S}_{n-d-1}} & \hookrightarrow & P_n^{\mathfrak{S}_d\times\mathfrak{S}_{n-d-1}} \\
\sim\downarrow & & \downarrow\sim \\
k[a_1,\ldots,a_{d+1},\gamma_{d+2},\ldots,\gamma_n] & \longrightarrow & k[a_1,\ldots,a_d,c,\gamma_{d+2},\ldots,\gamma_n] \\
\sim\downarrow & & \downarrow\sim \\
A_{d+1,n} & \longrightarrow & B_{d,n} \\
\uparrow & & \uparrow \\
A_{d+1}[\gamma_1,\ldots,\gamma_n] & \xrightarrow[f]{} & B_d[\gamma_1,\ldots,\gamma_n]
\end{array}$$

We deduce that there is a commutative diagram



It follows that $\mathcal{T}_{d,n}$ is equivalent to the bounded derived category of finitely generated graded $(k[a_1,\ldots,a_d]\otimes k[\gamma_{d+1},\ldots,\gamma_n])$-modules, where $\deg(a_i)=2(d-i+1)$ and $\deg(\gamma_i)=2(n+1-i)$. Similarly, the homotopy category of $(\mathbf{G}_m)^2$-equivariant matrix factorizations of $W_{d,n}$ on $Y_d\times\mathbf{A}^n$ is equivalent to the bounded derived category of finitely generated graded $(k[a_1,\ldots,a_d]\otimes k[\gamma_{d+1},\ldots,\gamma_n])$-modules, where $\deg(a_i)=2(d-i+1)$ and $\deg(\gamma_i)=2i$. We recover the usual construction of the (homotopy category of the) simple 2-representation $\mathcal{L}(n)$ of $\mathfrak{sl}_2$ (cf [Rou1, §5.2] and [Rou2, §4.3.2]). □

This construction is a Koszul dual counterpart of the construction of [NaiVa1, §7] based on adding a differential.

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