2-KAC-MOODY ALGEBRAS

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Abstract. We construct a 2-category associated with a symmetrizable Kac-Moody algebra and we study its 2-representations. This generalizes earlier work [ChRou] for $\mathfrak{sl}_2$. We relate categorifications relying on K_0 properties as in the approach of [ChRou] and 2-representations.

Contents

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1. Introduction

Over the past ten years, we have advocated the idea that there should exist monoidal categories (or 2-categories) with interesting "representation theory": we propose to call "2-representation theory" this higher version of representation theory and to call "2-algebras" those "interesting" monoidal additive categories. The difficulty in pinning down what is a

2-algebra (or a Hopf version) should be compared with the difficulty in defining precisely the meaning of quantum groups (or quantum algebras). The analogy is actually expected to be meaningful: while quantization turns certain algebras into quantum algebras, "categorification" should turn those algebras into 2-algebras. Dequantization is specialization $q \to 1$, while "decategorification" is the Grothendieck group construction — in the presence of gradings, it leads to a quantum object.

Our original example was the monoidal category $\mathcal{B}_W$ associated with Coxeter groups and braid groups [Rou1] and the motivation was to understand the Beilinson-Bernstein equivalence between the derived category of category $\mathcal{O}$ of a complex semi-simple Lie algebra and the derived category of sheaves on a flag variety (smooth on B-orbits) as an isomorphism of representations of $\mathcal{B}_W$. This was an attempt to recast Soergel's proof of that equivalence in a more conceptual framework. This was also meant to avoid the direct construction of a functor from the category $\mathcal{O}$ to the category of modules over the cohomology of the flag variety: that construction uses the longest element of the Weyl group and it doesn't generalize immediately to the affine case.

The starting point of the study of 2-representation theory of Lie algebras was the construction in 2003 of the theory of $\mathfrak{sl}_2$-categorifications with Joseph Chuang [ChRou].

A large part of geometric representation theory should, and can, be viewed as a construction of "irreducible" 2-representations as categories of sheaves.

In this paper, we define a 2-category $\mathcal{A}_{\mathfrak{g}}$ associated with a Kac-Moody algebra $\mathfrak{g}$. This generalizes the case of $\mathfrak{sl}_2$ that was considered and studied in a joint work with Joseph Chuang [ChRou]. Modulo some Hecke algebra isomorphisms, the generalization is quite natural. In type A, there is a very useful generalization of the presentation of [ChRou] (joint work with Joseph Chuang).

In [Rou2], we define and study tensor structures on the 2-category of 2-representations of $\mathcal{A}_{\mathfrak{g}}$ on dg-categories, with aim the construction of 4-dimensional topological quantum field theories. Our 2-categories associated with Kac-Moody algebras provide a solution to the question raised by Crane and Frenkel [CrFr] for a search of "Hopf categories".

A crucial feature of 2-representation theory is the construction of a machinery that produces new categories out of some given categories (with extra structure). We believe this should be viewed as an algebraic counterpart of the construction of moduli spaces as families of sheaves or other objects on a variety. The following oversimplified diagram explains how our algebraic constructions would reproduce the various counting invariants based on moduli spaces, bypassing the moduli spaces and the difficulties of their construction and the construction of their invariants

$$\begin{tikzcd}
\text{Variety} X \arrow[r, rightsquigarrow] \arrow[d] & \text{Moduli space } \mathcal{M} \text{ of objects on } X \arrow[d] \\
\text{Category of sheaves on } X \arrow[r, dashed] & \text{Category of sheaves on } \mathcal{M}
\end{tikzcd}$$

While our focus here is on classical algebraic objects (related in some way to 2-dimensional geometry), it is our belief that there should be 2-algebras associated with 3-dimensional geometry, possibly non-commutative, and that their higher representation theory would provide the proper algebraic framework for the various couting invariants (Gromov-Witten, Donaldson-Thomas,...).

The category $\mathcal{A}_{\mathfrak{g}}$ categorifies (a completion of) the $\mathbf{Z}$-form $U_{\mathbf{Z}}(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$. Consequently, a 2-representation of $\mathcal{A}_\Gamma$ on an exact or a triangulated category $\mathcal{V}$ gives rise to an action of $U_{\mathbf{Z}}(\mathfrak{g}_\Gamma)$ on $K_0(\mathcal{V})$. This gives a hint at the very non-semi-simplicity of the theory of 2-representations of $\mathcal{A}_\Gamma$. The presence of gradings actually gives rise to a categorification of the associated quantum group.

The Hecke algebras used in [ChRou] are replaced by Hecke algebras associated with graphs (or with Cartan matrices). They occur naturally as endomorphisms of correspondences for quiver varieties, as we will show in a sequel to this paper. In the type A cases, they occur when decomposing representations of (degenerate) affine Hecke algebras according to the spectrum of the polynomial subalgebra (and not just the center). They can be defined by generator and relations and they also have a simple construction as a subalgebra of a wreath product algebra.

We construct more generally a flat family of "Hecke" algebras over the space of matrices over $k[u,v]$ which are hermitian with respect to $u \leftrightarrow v$. They are filtered with associated graded algebra a wreath product of a polynomial algebra by a nil Hecke algebra. They satisfy the PBW property.

Consider a monoidal category or a 2-category defined by generators and relations. A difficulty in 2-representation theory is to check the defining relations in examples. The philosophy of [ChRou] was, instead of defining first the monoidal category, to describe directly what a 2-representation should be, using the action on the Grothendieck group. The main result of this paper is to provide a similar approach for Kac-Moody algebras. We show, under certain finiteness assumptions, that it is enough to check the relations $[e_i, f_j] = \delta_{ij} h_i$ on K_0. This is needed to show that the earlier definition of Chuang and the author of type A or $\tilde{A}$-categorifications coincides with the more general notion defined here.

The main results of this paper have been announced at seminars in Orsay, Paris and Kyoto in the Spring 2007. Certain specializations of the Hecke algebras associated with quivers and the resulting monoidal categories associated with "half" Kac-Moody algebras have been introduced independently by Khovanov and Lauda [KhoLau].

2. Preliminaries

2.1. **Notations and conventions.** Let k be a commutative ring. We write $\otimes$ for $\otimes_k$. Given M a graded k-module and i an integer, we denote by $M(i)$ the graded k-module given by $M(i)_n = M_{n+i}$. Given $n \in \mathbf{Z}$, we put $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$ and $[n]! = \prod_{i=1}^n [i]$ for $n \in \mathbf{Z}_{\geq 0}$.

Given $P = \sum_{i \in \mathbf{Z}} p_i v^i \in \mathbf{Z}_{\geq 0}[v^{\pm 1}]$ a Laurent polynomial with non-negative coefficients, we put $Pk = \bigoplus_{i \in \mathbf{Z}} k^{p_i}(-i)$. Given k' a k-algebra and M a k-module, we put $k'M = k' \otimes M$. We also put $PM = Pk \otimes M$.

Given A a k-algebra, γ an automorphism of A and M a right A-module, we denote by M_γ the right A-module $\gamma^* M$: it is equal to M as a k-module and the action of $a \in A$ on M_γ is given by $M_\gamma \ni m \mapsto m \cdot \gamma(a)$. Given M an (A, A)-bimodule, we put $M^A = \{m \in M \mid am = ma, \ \forall a \in A\}$.

An A-algebra is an algebra B endowed with a morphism of algebras $B \to A$. Given B an A-algebra, we say that a B-module is *relatively A-projective* if it is a direct summand of $B \otimes_A M$ for some A-module M.

We say that an endofunctor F of an additive category $\mathcal{C}$ is *locally nilpotent* if for every $M \in \mathcal{C}$, there is $n > 0$ such that $F^n(M) = 0$.

Categories are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc. and 2-categories are denoted by gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, etc.

We denote by $\mathrm{Ob}(\mathcal{A})$ or by $\mathcal{A}$ the set of objects of a category (or of a 2-category) $\mathcal{A}$. Given a an object, we will denote by a or $\mathbf{1}_a$ or 1_a the identity of a.

Given $F, G : \mathcal{A} \to \mathcal{B}$ two functors, a morphism $F \to G$ is the data of a compatible collection of arrows $F(a) \to G(a)$ for $a \in \mathcal{A}$ and we call these *natural morphisms*.

We denote by $\mathcal{S}ets$ (resp. $\mathcal{A}b$) the category of sets (resp. of abelian groups). We denote by A-Mod the category of A-modules, by A-mod the category of finitely generated A-modules and by A-free is full subcategory of free A-modules of finite rank. Here, module means left module.

We denote by $\mathfrak{Cat}$ (resp. $\mathfrak{Add}$, $\mathfrak{Lin}_k$, $\mathfrak{Ab}$, $\mathfrak{Tri}$) the strict 2-category of categories (resp. of additive categories, of k-linear categories, of abelian categories with exact functors, of triangulated categories). When k is a field, we denote by $\mathfrak{Ab}_k^f$ the 2-category of k-linear abelian categories all of whose objects have finite composition series and such that $k = \mathrm{End}(V)$ for any simple object V (1-arrows are k-linear exact functors).

Given Ω a finite interval of $\mathbf{Z}$, we denote by $\mathfrak{S}(\Omega)$ the symmetric group on Ω, viewed as a Coxeter group with generating set $\{s_i = (i, i+1)\}$ where i runs over the non-maximal elements of Ω. We denote by $w(\Omega)$ the longest element of $\mathfrak{S}(\Omega)$. Given E a family of disjoint intervals of Ω, we put $\mathfrak{S}(E) = \prod_{\Omega' \in E} \mathfrak{S}(\Omega')$ and we denote by $\mathfrak{S}(\Omega)^E$ (resp. ${}^E\mathfrak{S}(\Omega)$) the set of minimal length representatives of $\mathfrak{S}(\Omega)/\mathfrak{S}(E)$ (resp. $\mathfrak{S}(E) \setminus \mathfrak{S}(\Omega)$). We put $\mathfrak{S}_n = \mathfrak{S}[1, n]$.

2.2. **2-Categories.** We set up in this section the appropriate formalism for 2-representation theory. At first, we recall the more classical setting of representation theory as a study of functors.

2.2.1. *Categories.* Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. We denote by $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ the category of functors $\mathcal{A} \to \mathcal{B}$: we think of these as representations of $\mathcal{A}$ in $\mathcal{B}$. For example, if $\mathcal{A}$ has a unique object $*$ and $\mathcal{B} = \mathcal{S}ets$, the category $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ is equivalent to the category of sets acted on by the monoid $\mathrm{End}(*)$.

Given $a \in \mathcal{A}$, we have a functor $\mathrm{Hom}(a, -) : \rho_a : \mathcal{A} \to \mathcal{S}ets$ (the regular representation when $\mathcal{A}$ has a unique object).

We put $\mathcal{A}^\vee = \mathcal{H}om(\mathcal{A}^{\mathrm{opp}}, \mathcal{S}ets^{\mathrm{opp}})$. The functor

$$\mathcal{A} \to \mathcal{A}^\vee,\ M \mapsto \mathrm{Hom}(M, -)$$

is fully faithful (Yoneda's Lemma) and we identify $\mathcal{A}$ with a full subcategory of $\mathcal{A}^\vee$ through this embedding.

Assume $\mathcal{A}$ is enriched in abelian groups. The *additive closure* of $\mathcal{A}$ is the full additive subcategory $\mathcal{A}^a$ of the category of functors $\mathcal{A}^{\mathrm{opp}} \to \mathcal{A}b^{\mathrm{opp}}$ generated by objects of $\mathcal{A}$. Given $\mathcal{A}'$ an additive category, the restriction functor gives an equivalence from the category of additive functors $\mathcal{A}^a \to \mathcal{A}'$ to the category of functors enriched in abelian groups $\mathcal{A} \to \mathcal{A}'$.

Assume $\mathcal{A}$ is an additive category. We denote by $\mathcal{A}^i$ the idempotent completion of $\mathcal{A}$. Given $\mathcal{A}'$ an idempotent-complete additive category, restriction gives an equivalence from the category of additive functors $\mathcal{A}^i \to \mathcal{A}'$ to the category of additive functors $\mathcal{A} \to \mathcal{A}'$.

Let $M \in \mathcal{A}$ and let L be a right $\mathrm{End}(M)$-module. We denote by $L \otimes_{\mathrm{End}(M)} M$ the object of $\mathcal{C}^\vee$ defined by $\mathrm{Hom}_{\mathrm{End}(M)^{\mathrm{opp}}}(L, \mathrm{Hom}(M, -))$.

Given A a ring, the category of A-modules in $\mathcal{A}$ is the category of additive functors $A \to \mathcal{A}$, where A is the category with one object $*$ and with $\mathrm{End}(*) = A$. An object of that category is an object M of $\mathcal{A}$ endowed with a morphism of rings $A \to \mathrm{End}(M)$.

Given an A-module M in $\mathcal{A}$ and L a right A-module, we put $L\otimes_A M = (L\otimes_A \mathrm{End}(M))\otimes_{\mathrm{End}(M)} M$. For example, there is a canonical isomorphism $\mathbf{Z}^n \otimes_{\mathbf{Z}} M \xrightarrow{\sim} M^n$.

Let B be a commutative ring endowed with a morphism $B \to Z(\mathcal{A})$ and let A be a B-algebra. We denote by $\mathcal{A} \otimes_B A$ the additive category with same objects as $\mathcal{A}$ and $\mathrm{Hom}_{\mathcal{A}\otimes_B A}(M,N) = \mathrm{Hom}_{\mathcal{A}}(M,N) \otimes_B A$, where B acts via $Z(\mathcal{A})$. Let $\mathcal{A}'$ be an additive category endowed with a morphism $B \to Z(\mathcal{A}')$. We denote by $\mathcal{A} \otimes_B \mathcal{A}'$ the additive closure of the category with set of objects $\mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{A}')$ and $\mathrm{Hom}((M,M'),(N,N')) = \mathrm{Hom}_{\mathcal{A}}(M,N) \otimes_B \mathrm{Hom}_{\mathcal{A}'}(M',N')$. Given $\mathcal{A}''$ an additive category, we have an equivalence of categories

$$\mathcal{H}om_{\mathrm{add}}(\mathcal{A} \otimes_B \mathcal{A}', \mathcal{A}'') \xrightarrow{\sim} \mathcal{H}om_{\mathrm{add}}(\mathcal{A}, \mathcal{H}om_{\mathrm{add}}(\mathcal{A}', \mathcal{A}'')).$$

An equivalence relation $\sim$ on a category is a relation on arrows such that $f \sim f'$ implies $fg \sim f'g$ and $gf \sim gf'$ (whenever this makes sense). Recall that given $\mathcal{A}$ a category and $\sim$ a relation on arrows of $\mathcal{A}$, we have a quotient category $\mathcal{A}/{\sim}$ with same objects as $\mathcal{A}$. The quotient functor $\mathcal{A} \to \mathcal{A}/{\sim}$ induces a fully faithful functor $\mathcal{H}om(\mathcal{A}/{\sim}, \mathcal{B}) \to \mathcal{H}om(\mathcal{A}, \mathcal{B})$ for any category $\mathcal{B}$. A functor is in the image if and only if two equivalent arrows have the same image under the functor. The construction depends only on the equivalence relation on $\mathcal{A}$ generated by $\sim$.

Let k be a commutative ring and $\mathcal{A}$ a k-linear category. Given S a set of arrows of $\mathcal{A}$, let $\sim = \sim_S$ be the coarsest equivalence relation on $\mathcal{A}$ such that $f \sim 0$ for every $f \in S$ and $\{(f,g) \mid f \sim g\}$ is a k-submodule of $\mathrm{Hom}(a,a') \oplus \mathrm{Hom}(a,a')$. We denote by $\mathcal{A}/S = \mathcal{A}/{\sim}$ the quotient k-linear category: a k-linear functor $\mathcal{A} \to \mathcal{B}$ factors through $\mathcal{A}/S$, and then the factorization is unique up to unique isomorphism, if and only if it sends arrows in S to 0.

Given $\mathcal{A}$ a category, we denote by $k\mathcal{A}$ the k-linear category associated with $\mathcal{A}$: there is a canonical functor $\mathcal{A} \to k\mathcal{A}$ and given a k-linear category $\mathcal{B}$ and a functor $F : \mathcal{A} \to \mathcal{B}$, there is a k-linear functor $G : k\mathcal{A} \to \mathcal{B}$ unique up to unique isomorphism such that $F = G \cdot \mathrm{can}$.

Let $I = (I_0, I_1, s, t)$ be a quiver: this is the data of

- a set I_0 (vertices) and a set I_1 (arrows)
- maps $s, t : I_1 \to I_0$ (source and target).

We denote by $\mathcal{P} = \mathcal{P}(I)$ the set of paths in I, *i.e.*, sequences $(b_1, \ldots, b_n)$ of elements of I_1 such that $t(b_i) = s(b_{i-1})$ for $1 < i \leq n$. It comes with maps $s : \mathcal{P} \to I_0$, $(b_1, \ldots, b_n) \mapsto s(b_n)$ (source) and $t : \mathcal{P} \to I_0$, $(b_1, \ldots, b_n) \mapsto t(b_1)$ (target). We write $b_1 \cdots b_n$ for the element $(b_1, \ldots, b_n)$ of $\mathcal{P}$.

We denote by $\mathcal{C}(I)$ the category generated by I. Its set of objects is I_0 and $\mathrm{Hom}(i,j) = (s,t)^{-1}(i,j)$. Composition is concatenation of paths.

Let $\mathcal{A}$ be a category. The category of diagrams of type I in $\mathcal{A}$ is canonically isomorphic to the category of functors $\mathcal{C}(I) \to \mathcal{A}$ (the isomorphism is given by restricting the functor).

A *graded category* is a category endowed with a self-equivalence T. Given M an object with isomorphism class $[M]$, we put $v[M] = [T^{-1}(M)]$.

The 2-category of graded k-linear categories is equivalent to the 2-category of k-linear categories enriched in graded k-modules:

- Let $\mathcal{C}$ be a graded k-linear category. We define $\mathcal{D}$ as the category with objects those of $\mathcal{C}$ and with $\mathrm{Hom}_{\mathcal{D}}(V,W) = \bigoplus_i \mathrm{Hom}_{\mathcal{C}}(V, T^i W)$. The composition of the maps of $\mathcal{D}$ coming from maps $f : V \to T^i W$ and $g : W \to T^j X$ of $\mathcal{C}$ is the map coming from $T^i(g) \circ f : V \to T^{i+j} X$.

- Let $\mathcal{D}$ be a k-linear category enriched in graded k-modules. Define $\mathcal{C}$ as the category with objects families $\{V_i\}_{i\in\mathbf{Z}}$ with V_i an object of $\mathcal{D}$ and $V_i = 0$ for almost all i. We put $\mathrm{Hom}_{\mathcal{C}}(\{V_i\},\{W_i\}) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{D}}(V_i, W_j)_{j-i}$. We define $T(\{V_i\})_n = V_{n+1}$.

2.2.2. *Definitions.* Our main reference for basic definitions and results on 2-categories is [Gra] (cf also [Le] for the basic definitions).

Definition 2.1. *A 2-category $\mathfrak{A}$ is the data of*

- *a set $\mathfrak{A}_0$ of objects*
- *categories $\mathcal{H}om(a,a')$ for $a,a' \in \mathfrak{A}_0$*
- *functors $\mathcal{H}om(a_1,a_2)\times\mathcal{H}om(a_2,a_3)\to\mathcal{H}om(a_1,a_3)$, $(b_1,b_2)\mapsto b_2b_1$ for $a_1,a_2,a_3\in\mathfrak{A}$*
- *functors $I_a \in \mathcal{E}nd(a)$ for $a\in\mathfrak{A}$*
- *natural isomorphisms $(b_3b_2)b_1 \xrightarrow{\sim} b_3(b_2b_1)$ for $b_i\in\mathcal{H}om(a_i,a_{i+1})$ and $a_1,\ldots,a_4\in\mathfrak{A}$.*
- *natural isomorphisms $bI_a \xrightarrow{\sim} b$ for $b\in\mathcal{H}om(a,a')$ and $a,a'\in\mathfrak{A}$*
- *natural isomorphisms $I_ab \xrightarrow{\sim} b$ for $b\in\mathcal{H}om(a',a)$ and $a,a'\in\mathfrak{A}$*

such that the following diagrams commute

$$\begin{tikzcd}
 & \big((b_4b_3)b_2\big)b_1 \arrow[rr, "\mathrm{can}(b_4,b_3,b_2)\cdot b_1"] \arrow[dl, "\mathrm{can}(b_4b_3,b_2,b_1)"'] & & \big(b_4(b_3b_2)\big)b_1 \arrow[dr, "\mathrm{can}(b_4,b_3b_2,b_1)"] & \\
(b_4b_3)(b_2b_1) \arrow[drr, "\mathrm{can}(b_4,b_3,b_2b_1)"'] & & & & b_4\big((b_3b_2)b_1\big) \arrow[dll, "b_4\cdot\mathrm{can}(b_3,b_2,b_1)"] \\
 & & b_4\big(b_3(b_2b_1)\big) & &
\end{tikzcd}$$

$$\begin{tikzcd}
(b_2I_a)b_1 \arrow[rr, "\mathrm{can}(b_2,I_a,b_1)"] \arrow[dr, "\mathrm{can}(b_2)\cdot b_1"'] & & b_2(I_ab_1) \arrow[dl, "b_2\cdot\mathrm{can}(b_1)"] \\
 & b_2b_1 &
\end{tikzcd}$$

Note that 2-categories are called bicategories in [Gra]. A *strict* 2*-category* is a 2-category where the associativity and unit isomorphisms are identity maps: $(b_3b_2)b_1 = b_3(b_2b_1)$ and $bI_a = b$, $I_ab = b$ (called 2-category in [Gra]).

Let $\mathfrak{A}$ be a 2-category. Its 1-arrows (resp. 2-arrows) are the objects (resp. arrows) of the categories $\mathcal{H}om(a,a')$

Given $b : a\to a'$ and $b' : a'\to a''$ two 1-arrows, we denote by $b'b : a\to a''$ their composition. The composition of 2-arrows (viewed as arrows in a category $\mathcal{H}om(a,a')$) is denoted by $c'\circ c$. Given a'' an object of $\mathfrak{A}$, $b_1,b_2 : a\to a'$, $c : b_1\to b_2$ and $b'_1,b'_2 : a'\to a''$, $c' : b'_1\to b'_2$, we denote by $c'c : b'_1b_1\to b'_2b_2$ the "juxtaposition".

We say that a 1-arrow $b : a_1\to a_2$ is

- an *equivalence* if there is a 1-arrow $b' : a_2\to a_1$ and isomorphisms $I_{a_1}\xrightarrow{\sim} b'b$ and $bb'\xrightarrow{\sim} I_{a_2}$
- *fully faithful* if given any object a'', the functor $\mathcal{H}om(a'',b) : \mathcal{H}om(a'',a_1)\to\mathcal{H}om(a'',a_2)$ is fully faithful.

Note that these notions coincide with the usual notions for $\mathfrak{A} = \mathfrak{C}at$, $\mathfrak{A} = \mathfrak{A}dd$, $\mathfrak{A} = \mathfrak{A}b$ or $\mathfrak{A} = \mathfrak{T}ri$.

Given a 2-category $\mathfrak{A}$, we denote by $\mathfrak{A}_{\leq 1}$ the category with objects those of $\mathfrak{A}$ and with arrows the isomorphism classes of 1-arrows of $\mathfrak{A}$.

The *opposite* 2-category $\mathfrak{A}^{\mathrm{opp}}$ of $\mathfrak{A}$ has same set of objects as $\mathfrak{A}$ and $\mathcal{H}om_{\mathfrak{A}^{\mathrm{opp}}}(a,a') = \mathcal{H}om_{\mathfrak{A}}(a,a')^{\mathrm{opp}}$, while the rest of the structure is inherited from that of $\mathfrak{A}$.

The *reverse* 2-category $\mathfrak{A}^{\mathrm{rev}}$ of $\mathfrak{A}$ has same set of objects as $\mathfrak{A}$ and $\mathcal{H}om_{\mathfrak{A}^{\mathrm{opp}}}(a,a') = \mathcal{H}om_{\mathfrak{A}}(a',a)$. The composition

$$\mathcal{H}om_{\mathfrak{A}^{\mathrm{rev}}}(a_1,a_2) \times \mathcal{H}om_{\mathfrak{A}^{\mathrm{rev}}}(a_2,a_3) \to \mathcal{H}om_{\mathfrak{A}^{\mathrm{rev}}}(a_1,a_3)$$

is given by $(b_1,b_2) \mapsto b_1b_2$ (composition in $\mathfrak{A}$). The rest of the structure is inherited from that of $\mathfrak{A}$.

Definition 2.2. *A* 2-functor $R : \mathfrak{A} \to \mathfrak{B}$ *between* 2*-categories is the data of*

- *a map* $R : \mathrm{Ob}(\mathfrak{A}) \to \mathrm{Ob}(\mathfrak{B})$
- *functors* $R : \mathfrak{Hom}(a,a') \to \mathfrak{Hom}(R(a),R(a'))$ *for* $a,a' \in \mathfrak{A}$
- *natural isomorphisms* $R(b_2)R(b_1) \xrightarrow{\sim} R(b_2b_1)$ *for* b_1,b_2 1*-arrows of* $\mathfrak{A}$
- *invertible* 2*-arrows* $I_{R(a)} \xrightarrow{\sim} R(I_a)$ *for* $a \in \mathfrak{A}$

such that the following diagrams commute

$$\begin{tikzcd}
\big(R(b_3)R(b_2)\big)R(b_1) \arrow[r, "\mathrm{can}(b_3,b_2)\cdot R(b_1)"] \arrow[d, "\mathrm{can}(R(b_3),R(b_2),R(b_1))"'] & R(b_3b_2)R(b_1) \arrow[r, "\mathrm{can}(b_3b_2,b_1)"] & R\big((b_3b_2)b_1\big) \arrow[d, "R(\mathrm{can}(b_3,b_2,b_1))"] \\
R(b_3)\big(R(b_2)R(b_1)\big) \arrow[r, "R(b_3)\cdot\mathrm{can}(b_2,b_1)"'] & R(b_3)R(b_2b_1) \arrow[r, "\mathrm{can}(b_3,b_2b_1)"'] & R\big(b_3(b_2b_1)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
R(b)I_{R(a)} \arrow[r, "R(b)\cdot\mathrm{can}(a)"] \arrow[d, "\mathrm{can}(R(b))"'] & R(b)R(I_a) \arrow[d, "\mathrm{can}(b,I_a)"] \\
R(b) & R(bI_a) \arrow[l, "R(\mathrm{can}(b))"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{R(a')}R(b) \arrow[r, "\mathrm{can}(a')\cdot R(b)"] \arrow[d, "\mathrm{can}(R(b))"'] & R(I_{a'})R(b) \arrow[d, "\mathrm{can}(I_{a'},b)"] \\
R(b) & R(I_{a'}b) \arrow[l, "R(\mathrm{can}(b))"]
\end{tikzcd}$$

When the 2-arrows are identity maps $I_{R(a)} = R(I_a)$, we say that the 2-functor is *strict* (called strict pseudo-functor in [Gra]).

Definition 2.3. *A* morphism of 2-functors $\sigma : R \to R'$ *is the data of*

- 1*-arrows* $\sigma(a) : R(a) \to R'(a)$
- *natural isomorphisms* $R'(b)\sigma(a_1) \xrightarrow{\sim} \sigma(a_2)R(b)$ *for all* 1*-arrows* $b : a_1 \to a_2$

such that the following diagrams commute

$$\begin{tikzcd}
\big(R'(b_2)R'(b_1)\big)\sigma(a_1) \arrow[r, "\mathrm{can}(R'(b_2),R'(b_1),\sigma(a_1))"] \arrow[d, "\mathrm{can}(b_2,b_1)\cdot\sigma(a_1)"] & R'(b_2)\big(R'(b_1)\sigma(a_1)\big) \arrow[r, "R'(b_2)\cdot\mathrm{can}(b_1)"] & R'(b_2)\big(\sigma(a_2)R(b_1)\big) \arrow[d, "\mathrm{can}(R'(b_2),\sigma(a_2),R(b_1))^{-1}"'] \\
R'(b_2b_1)\sigma(a_1) \arrow[d, "\mathrm{can}(b_2b_1)"] & & \big(R'(b_2)\sigma(a_2)\big)R(b_1) \arrow[d, "\mathrm{can}(b_2)R(b_1)"'] \\
\sigma(a_3)R(b_2b_1) & \sigma(a_3)\big(R(b_2)R(b_1)\big) \arrow[l, "\sigma(a_3)\cdot\mathrm{can}(b_2,b_1)"] & \big(\sigma(a_3)R(b_2)\big)R(b_1) \arrow[l, "\mathrm{can}(\sigma(a_3),R(b_2),R(b_1))"]
\end{tikzcd}$$

$$\begin{tikzcd}
I_{R'(a)}\sigma(a) \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{can}\cdot\sigma(a)"'] & \sigma(a) \arrow[r, "\mathrm{can}^{-1}"] & \sigma(a)I_{R(a)} \arrow[d, "\sigma(a)\cdot\mathrm{can}"] \\
R'(I_a)\sigma(a) \arrow[rr, "\mathrm{can}"'] & & \sigma(a)R(I_a)
\end{tikzcd}$$

These are quasi-natural transformations with invertible 2-arrows in [Gra].

Definition 2.4. *A* morphism $\gamma : \sigma \to \tilde{\sigma}$, *where* $\sigma, \tilde{\sigma} : R \to R'$ *are morphisms of* 2*-functors, is the data of* 2*-arrows* $\gamma(a) : \sigma(a) \to \tilde{\sigma}(a)$ *for* $a \in \mathfrak{A}$ *such that the following diagrams commute*

$$\begin{tikzcd}
R'(b)\sigma(a_1) \arrow[r, "R'(b)\gamma(a_1)"] \arrow[d, "\mathrm{can}"', "\sim"] & R'(b)\tilde{\sigma}(a_1) \arrow[d, "\sim"', "\mathrm{can}"] \\
\sigma(a_2)R(b) \arrow[r, "\gamma(a_2)R'(b)"'] & \tilde{\sigma}(a_2)R(b)
\end{tikzcd}$$

These are called modifications in [Gra].

We denote by $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$ denotes the 2-category of 2-functors $\mathfrak{A} \to \mathfrak{B}$. When $\mathfrak{B}$ is a strict 2-category, then $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$ is strict as well.

Given a property of functors, we say that a 2-functor $F : \mathfrak{A} \to \mathfrak{B}$ has *locally* that property if the functors $\mathcal{H}om(a, a') \to \mathcal{H}om(F(a), F(a'))$ have the property for all a, a' objects of $\mathfrak{A}$.

A 2-functor $F : \mathfrak{A} \to \mathfrak{B}$ is a 2*-equivalence* if there is a 2-functor $G : \mathfrak{B} \to \mathfrak{A}$ and equivalences $\mathrm{id}_{\mathfrak{A}} \xrightarrow{\sim} GF$ and $FG \xrightarrow{\sim} \mathrm{id}_{\mathfrak{B}}$. This is equivalent to the requirement that F is locally an equivalence and every object of $\mathfrak{B}$ is equivalent to an object in the image of F.

Every 2-category is 2-equivalent to a strict 2-category, but there are 2-functors between strict 2-categories that are not equivalent to strict ones.

Given a an object of $\mathfrak{A}$, then $\mathcal{E}nd(a)$ is a monoidal category. Conversely, a monoidal category gives rise to a 2-category with a single object $*$, and the notion of monoidal functor coincides with that of 2-functor (*i.e.*, there is a $1, 2, 3$-fully faithful strict 3-functor from the 3-category of monoidal categories to that of 2-categories).

Let k be a commutative ring. A k*-linear* 2-category is a 2-category $\mathfrak{A}$ that is locally k-linear and such that juxtaposition is k-linear. Given $\mathfrak{A}$ and $\mathfrak{B}$ two k-linear 2-categories, we denote by $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$ the 2-category of k-linear 2-functors $\mathfrak{A} \to \mathfrak{B}$: this is the locally full sub-2-category of the category of 2-functors obtained by requiring the functors in the definition of 2-functors to be k-linear.

Given $\mathfrak{A}$ a 2-category, we denote by $k\mathfrak{A}$ the k-linear closure of $\mathfrak{A}$: its objects are those of $\mathfrak{A}$ and $\mathcal{H}om_{k\mathfrak{A}}(a, a') = k\mathcal{H}om_{\mathfrak{A}}(a, a')$.

Let $b : a \to a'$ be a 1-arrow. A *right adjoint* (or *right dual*) of b is a triple $(b^\vee, \varepsilon_b, \eta_b)$ where $b^\vee : a' \to a$ is a 1-arrow and $\varepsilon_b : bb^\vee \to \mathrm{id}_{a'}$ and $\eta_b : \mathrm{id}_a \to b^\vee b$ are 2-arrows such that

$$(\varepsilon_b b) \circ (b\eta_b) = \mathrm{id}_b \text{ and } (b^\vee \varepsilon_b) \circ (\eta_b b^\vee) = \mathrm{id}_{b^\vee}.$$

We also say that $(b, \varepsilon_b, \eta_b)$ is a left adjoint (or dual) of $b^\vee$ and that $(b, b^\vee, \varepsilon_b, \eta_b)$ (or simply $(b, b^\vee)$) is an adjoint quadruple (resp. an adjoint pair).

Given $b_1 : a \to a'$ a 1-arrow and $(b_1, b_1^\vee)$ an adjoint pair, we have a canonical isomorphism

$$\mathrm{Hom}(b, b_1) \xrightarrow{\sim} \mathrm{Hom}(b_1^\vee, b^\vee),\ f \mapsto f^\vee = (b^\vee \varepsilon_{b_1}) \circ (b^\vee f b_1^\vee) \circ (\eta_b b_1^\vee).$$

Assume now there are dual pairs $(b, b^\vee)$ and $(b^\vee, b)$. We have an automorphism

$$\mathrm{End}(b) \xrightarrow{\sim} \mathrm{End}(b),\ f \mapsto (f^\vee)^\vee. \tag{1}$$

2.2.3. *Generators and relations.* An *equivalence relation* $\sim$ on $\mathfrak{A}$ is the data for every a, a' objects, for every $b, b' : a \to a'$ of an equivalence relation on $\mathrm{Hom}(b, b')$ compatible with composition and juxtaposition, *i.e.*, if $c_1 \sim c_2$, then given a 2-arrow c, we have $c_1 \circ c \sim c_2 \circ c$, $c \circ c_1 \sim c \circ c_2$, $c_1 c \sim c_2 c$ and $cc_1 \sim cc_2$, whenever this makes sense. Given a relation $\sim$ on 2-arrows of $\mathcal{C}$, the equivalence relation generated by $\sim$ is the coarsest refinement of $\sim$ that is an equivalence relation.

Let $\mathfrak{A}$ be a 2-category and $\sim$ an equivalence relation. We denote by $\mathfrak{A}/\sim$ the 2-category with same objects as $\mathfrak{A}$ and with $\mathcal{H}om_{\mathfrak{A}/\sim}(a, a') = \mathcal{H}om_{\mathfrak{A}}(a, a')/\sim$ (so, $\mathfrak{A}/\sim$ has the same 1-arrows as $\mathfrak{A}$). The local quotient functors induce a strict quotient 2-functor $\mathfrak{A} \to \mathfrak{A}/\sim$. Given a 2-category $\mathfrak{B}$, the quotient strict 2-functor $\mathfrak{A} \to \mathfrak{A}/\sim$ induces a strict 2-functor $\mathfrak{Hom}(\mathfrak{A}/\sim, \mathfrak{B}) \to \mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$ that is locally an isomorphism. A 2-functor is in the image if and only if two equivalent 2-arrows have the same image.

Given S a set of 2-arrows of $\mathfrak{A}$, we denote by $\tilde{S}$ the smallest set of 2-arrows of $\mathfrak{A}$ closed under juxtaposition and composition and containing S and the invertible 2-arrows.

We denote by $\mathfrak{A}[S^{-1}]$ the 2-category with same objects as $\mathfrak{A}$ and with $\mathcal{H}om_{\mathfrak{A}[S^{-1}]}(a, a') = \mathcal{H}om_{\mathfrak{A}}(a, a')[S(a, a')^{-1}]$, where $S(a, a')$ are the 2-arrows of $\tilde{S}$ that are in $\mathcal{H}om_{\mathfrak{A}}(a, a')$ (so, $\mathfrak{A}[S^{-1}]$ has the same 1-arrows as $\mathfrak{A}$).

The canonical strict 2-functor $\mathfrak{A} \to \mathfrak{A}[S^{-1}]$ induces a strict 2-functor $\mathfrak{Hom}(\mathfrak{A}[S^{-1}], \mathfrak{B}) \to \mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$ that is locally an isomorphism. A 2-functor is in the image if and only if the image of any 2-arrow in S is invertible.

Assume $\mathfrak{A}$ is a k-linear 2-category. Let S be a set of 2-arrows of $\mathfrak{A}$. Given a, a' objects of $\mathfrak{A}$, we consider the equivalence relation $\sim_{S(a,a')}$ on $\mathcal{H}om(a, a')$. Let $\sim$ be the coarsest equivalence relation on $\mathfrak{A}$ that refines the relations $\sim_{S(a,a')}$. We put $\mathfrak{A}/S = \mathfrak{A}/\sim$.

A 2-*quiver* $I = (I_0, I_1, I_2, s, t, s_2, t_2)$ is the data of

- three sets I_0 (vertices), I_1 (1-arrows) and I_2 (2-arrows)
- maps $s, t : I_1 \to I_0$ (source and target)
- maps $s_2, t_2 : I_2 \to \mathcal{P} = \mathcal{P}(I_0, I_1, s, t)$ such that $s(s_2(c)) = s(t_2(c))$ and $t(s_2(c)) = t(t_2(c))$ for all $c \in I_2$.

Let I be a 2-quiver. Let $a, a' \in I_0$. We define a quiver $I(a, a') = (\tilde{I}_0, \tilde{I}_1, \tilde{s}, \tilde{t})$. We put $\tilde{I}_0 = (s, t)^{-1}(a, a')$, the set of paths from a to a'. The set $\tilde{I}_1$ is given by triples (b, c, b') where $b, b' \in \mathcal{P}$, $c \in I_2$ satisfy $t(b') = s(s_2(c))$, $t(s_2(c)) = s(b)$, $s(b') = a$, $t(b) = a'$. We put $\tilde{s}(b, c, b') = bs_2(c)b'$ and $\tilde{t}(b, c, b') = bt_2(c)b'$. We introduce a relation $\sim$ on $\mathcal{P}(I(a, a'))$ by

$$(b_1 t_2(c_1) b_2, c_2, b_3)(b_1, c_1, b_2 s_2(c_2) b_3) \sim (b_1, c_1, b_2 t_2(c_2) b_3)(b_1 s_2(c_1) b_2, c_2, b_3)$$

(whenever this makes sense).

The strict 2-category $\mathfrak{C}(I)$ generated by I is defined as follows. Its set of objects is I_0. We put $\mathcal{H}om(a, a') = \mathcal{C}(I(a, a'))/\sim$. Composition of 1-arrows is concatenation of paths. Juxtaposition is given by

$$(b_1, c_1, b_1')(b_2, c_2, b_2') = (b_1, c_1, b_1' b_2 t_2(c_2) b_2') \circ (b_1 s_2(c_1) b_1' b_2, c_2, b_2').$$

Note that the category $\mathfrak{C}(I)_{\le 1}$ is $\mathcal{C}(I_0, I_1, s, t)$.

Let $\mathfrak{B}$ be a strict 2-category. An I-*diagram* D in $\mathfrak{B}$ is the data of

- an object a_i of $\mathfrak{B}$ for any $i \in I_0$
- a 1-arrow $b_j : a_{s(j)} \to a_{t(j)}$ for any $j \in I_1$
- a 2-arrow $c_k : b_{s_2(k)} \to b_{t_2(k)}$ for any $k \in I_2$

where given $p = (p_1, \dots, p_n) \in \mathcal{P}$, we put $b_p = b_{p_1} \cdots b_{p_n}$.

$$\begin{tikzcd}
a_{s(s_2(k))} \arrow[r, bend left, "b_{s_2(k)}"{name=U}] \arrow[r, bend right, "b_{t_2(k)}"'{name=D}] & a_{t(s_2(k))} \arrow[from=U, to=D, Rightarrow, "c_k"]
\end{tikzcd}$$

The data of b_j's and c_k's is the same as the data, for $i, i' \in I_0$, of an $I(i, i')$-diagram in $\mathcal{H}om(a_i, a_{i'})$.

A morphism $\sigma : D \to D'$ is the data of

- 1-arrows $\sigma_i : a_i \to a'_i$ for $i \in I_0$
- invertible 2-arrows $\sigma_j : b'_j \sigma_{s(j)} \xrightarrow{\sim} \sigma_{t(j)} b_j$ for every $j \in I_1$

$$\begin{tikzcd}
 & a'_{s(j)} \arrow[dr, "b'_j"] & \\
a_{s(j)} \arrow[ur, "\sigma_{s(j)}"] \arrow[dr, "b_j"'] \arrow[rr, phantom, "\sim \Downarrow \sigma_j"] & & a'_{t(j)} \\
 & a_{t(j)} \arrow[ur, "\sigma_{t(j)}"'] &
\end{tikzcd}$$

such that for every $k \in I_2$ with $s_2(k) = (j_1, \dots, j_n)$ and $t_2(k) = (\bar{j}_1, \dots, \bar{j}_{\bar{n}})$, the following 2-arrows $b'_{s_2(k)} \sigma_{s(j_n)} \to \sigma_{t(j_1)} b_{t_2(k)}$ are equal:

$$\begin{tikzcd}
\bullet \arrow[r, "b_{j_n}"] \arrow[d, "\sigma_{s(j_n)}"'] \arrow[rrr, bend left=50, "b_{\bar{j}_1} \cdots b_{\bar{j}_{\bar{n}}}"] \arrow[dr, phantom, "\sigma_{j_n} \Nearrow"] & \bullet \arrow[r, dash, dotted] \arrow[d, "\sigma_{t(j_n)}"] \arrow[r, phantom, shift left=4, "\Uparrow c'_k"] & \bullet \arrow[r, "b_{j_1}"] \arrow[d, "\sigma_{t(j_2)}"'] \arrow[dr, phantom, "\sigma_{j_1} \Nearrow"] & \bullet \arrow[d, "\sigma_{t(j_1)}"] \\
\bullet \arrow[r, "b'_{j_n}"'] & \bullet \arrow[r, dash, dotted] & \bullet \arrow[r, "b'_{j_1}"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "b_{\bar{j}_{\bar{n}}}"] \arrow[d, "\sigma_{s(\bar{j}_{\bar{n}})}"'] \arrow[dr, phantom, "\sigma_{\bar{j}_{\bar{n}}} \Nearrow"] & \bullet \arrow[r, dash, dotted] \arrow[d, "\sigma_{t(\bar{j}_{\bar{n}})}"] & \bullet \arrow[r, "b_{\bar{j}_1}"] \arrow[d, "\sigma_{t(\bar{j}_2)}"'] \arrow[dr, phantom, "\sigma_{\bar{j}_1} \Nearrow"] & \bullet \arrow[d, "\sigma_{t(\bar{j}_1)}"] \\
\bullet \arrow[r, "b'_{\bar{j}_{\bar{n}}}"'] \arrow[rrr, bend right=50, "b'_{j_1} \cdots b'_{j_n}"'] & \bullet \arrow[r, dash, dotted] \arrow[r, phantom, shift right=4, "\Uparrow c_k"] & \bullet \arrow[r, "b'_{\bar{j}_1}"'] & \bullet
\end{tikzcd}$$

A morphism $\gamma : \sigma \to \tilde{\sigma}$ is the data of 2-arrows $\gamma_i : \sigma_i \to \tilde{\sigma}_i$ for $i \in I_0$ such that for every $j \in I_1$, we have $(\gamma_{t(j)} b_j) \circ \sigma_j = \tilde{\sigma}_j \circ (b'_j \gamma_{s(j)})$, *i.e.*, the following diagram of 2-arrows is commutative:

$$\sigma_j$$

$$\begin{tikzcd}
\bullet \arrow[r, "b_j"] & \bullet \arrow[r, bend left, "\sigma_{t(j)}"] \arrow[r, bend right, "\tilde{\sigma}_{t(j)}"'] \arrow[r, phantom, "\Downarrow\gamma_{t(j)}"] & \bullet & \bullet \arrow[r, bend left, "\sigma_{s(j)}"] \arrow[r, bend right, "\tilde{\sigma}_{s(j)}"'] \arrow[r, phantom, "\Downarrow\gamma_{s(j)}"] & \bullet \arrow[r, "b'_j"] & \bullet
\end{tikzcd}$$

$$\tilde{\sigma}_j$$

This gives rise to a strict 2-category $\mathfrak{Hom}(I, \mathfrak{B})$ of I-diagrams in $\mathfrak{B}$.

Restriction gives a strict 2-functor $H : \mathfrak{Hom}(\mathfrak{C}(I), \mathfrak{B}) \to \mathfrak{Hom}(I, \mathfrak{B})$. It is locally an isomorphism and it is surjective on objects, so it is a 2-equivalence.

2.2.4. 2-*Representations.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be two 2-categories. We will consider 2-representations of $\mathfrak{A}$ in $\mathfrak{B}$, *i.e.*, 2-functors $R : \mathfrak{A} \to \mathfrak{B}$. We put $\mathfrak{A}\text{-Mod}(\mathfrak{B}) = \mathfrak{Hom}(\mathfrak{A}, \mathfrak{B})$, a 2-category. Given $R : \mathfrak{A} \to \mathfrak{B}$, a *sub-2-representation* is a 2-functor $R' : \mathfrak{A} \to \mathfrak{B}$ equiped with a fully faithful morphism $R' \to R$.

Let S be a collection of objects of $\mathfrak{B}$. An *action* of $\mathfrak{A}$ on S is a representation of $\mathfrak{A}$ in $\mathfrak{B}$ with image contained in S. Note that if $\mathfrak{A}$ has only one object and is viewed as a monoidal category $\mathcal{A}$ and $S = \{\mathcal{C}\}$, we recover the usual notion of an action of $\mathcal{A}$ on $\mathcal{C}$.

Let $a \in \mathcal{A}$. We define $\mathcal{H}om(a, -) : \mathfrak{A} \to \mathfrak{C}at$ by $a' \mapsto \mathcal{H}om(a, a')$. The functor $\mathcal{H}om(a', a'') \to \mathcal{H}om(\mathcal{H}om(a, a'), \mathcal{H}om(a, a''))$ is given by juxtaposition. The associativity and unit maps of $\mathfrak{A}$ provide the required 2-arrows.

Let $R : \mathfrak{A} \to \mathfrak{C}at$ be a 2-functor. Given a an object of $\mathfrak{A}$, there is an equivalence of categories from $R(a)$ to the category of morphisms $\mathcal{H}om(a, -) \to R$:

- Given M an object of the category $R(a)$, we define a morphism $\sigma : \mathcal{H}om(a, -) \to R$. The functor $\mathcal{H}om(a, a') \to R(a')$ is $b \mapsto R(b)(M)$. The required natural isomorphisms come from the natural isomorphisms $R(b)R(f) \xrightarrow{\sim} R(bf)$.
- Conversely, given $\sigma : \mathcal{H}om(a, -) \to R$, we put $M = \sigma(I_a)$.

Assume from now on that our 2-categories are k-linear.

Let $b : a \to a'$ be a 1-arrow. A *cokernel* of b is the data of an object $\mathcal{C}oker(b)$ and of a 1-arrow $b' : a' \to \mathcal{C}oker(b)$ such that for any object a'', the functor $\mathcal{H}om(b', a'') : \mathcal{H}om(\mathcal{C}oker(b), a'') \to \mathcal{H}om(a', a'')$ is fully faithful with image equivalent to the full subcategory of 1-arrows $b'' : a' \to a''$ such that $b''b = 0$. When a cokernel of b exists, it is unique up to an equivalence unique up to a unique isomorphism.

We say that $\mathfrak{A}$ *admits cokernels* if all 1-arrows admit cokernels. This is the case for the 2-category of k-linear categories, of abelian categories or of triangulated categories.

We define *kernels* as cokernels taken in $\mathfrak{B}^{\mathrm{rev}}$.

Assume $\mathfrak{B}$ admits kernel and cokernels.

Let $b : a \to a'$ be a fully faithful 1-arrow. We say that it is *thick* if b is a kernel of $a \to \mathcal{C}oker(b)$.

When $\mathfrak{B} \subset \mathfrak{Lin}_k$, the notion of thickness corresponds to

- $\mathfrak{Lin}_k$ or $\mathfrak{Tri}$: a is closed under direct summands
- $\mathfrak{Ab}$: a is closed under extensions, subobjects and quotients

Lemma 2.5. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two additive categories, $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ two additive functors and $f : F \to G$. Let $\mathcal{C}'_1$ and $\mathcal{C}'_2$ be thick subcategories of $\mathcal{C}_1$ and $\mathcal{C}_2$. Assume*

- *F, G admit right adjoints $F^\vee, G^\vee$*
- *F and G send an object of $\mathcal{C}'_1$ to an object of $\mathcal{C}'_2$ and*
- *$F^\vee$ and $G^\vee$ send an object of $\mathcal{C}'_2$ to an object of $\mathcal{C}'_1$.*

Denote by $\bar{F}, \bar{G} : \mathcal{C}_1/\mathcal{C}'_1 \to \mathcal{C}_2/\mathcal{C}'_2$ the induced functors and $\bar{f} : \bar{F} \to \bar{G}$ induced by f. If $f_{|\mathcal{C}'_1}$ and $\bar{f}$ are isomorphisms, then f is an isomorphism.

Proof. It is enough to prove the lemma for the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ replaced by idempotent completions and we will assume now that these categories are idempotent-complete.

Let $M \in \mathcal{C}_1$ and $N \in \mathcal{C}_2$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^{\mathcal{C}'_2}(F(M), N) \arrow[r, "\mathrm{can}"] & \mathrm{Hom}_{\mathcal{C}_2}(F(M), N) \arrow[r, "\mathrm{can}"] & \mathrm{Hom}_{\mathcal{C}_2/\mathcal{C}'_2}(\bar{F}(M), N) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}^{\mathcal{C}'_2}(G(M), N) \arrow[r, "\mathrm{can}"] \arrow[u, "\mathrm{Hom}^{\mathcal{C}'_2}(f,N)"] & \mathrm{Hom}_{\mathcal{C}_2}(G(M), N) \arrow[r, "\mathrm{can}"] \arrow[u, "\mathrm{Hom}(f,N)"] & \mathrm{Hom}_{\mathcal{C}_2/\mathcal{C}'_2}(\bar{G}(M), N) \arrow[r] \arrow[u, "\mathrm{Hom}(\bar{f},N)"] & 0
\end{tikzcd}$$

where $\mathrm{Hom}^{\mathcal{C}'_2}$ denotes the subgroup of maps factoring through an object of $\mathcal{C}'_2$. By assumption, $\mathrm{Hom}(\bar{f}, N)$ is an isomorphism, so it is enough to show that

$$\mathrm{Hom}^{\mathcal{C}'_2}(f, N) : \mathrm{Hom}^{\mathcal{C}'_2}(G(M), N) \to \mathrm{Hom}^{\mathcal{C}'_2}(F(M), N)$$

is an isomorphism. It is so when $M \in \mathcal{C}'_1$. Consider $f^\vee : G^\vee \to F^\vee$. Since $f_{|\mathcal{C}'_1}$ is an isomorphism, it follows that $f^\vee_{|\mathcal{C}'_2}$ is an isomorphism. In particular,

$$\mathrm{Hom}(M, f^\vee) : \mathrm{Hom}^{\mathcal{C}'_1}(M, G^\vee(N)) \to \mathrm{Hom}^{\mathcal{C}'_1}(M, F^\vee(N))$$

is an isomorphism when $N \in \mathcal{C}'_2$, hence $\mathrm{Hom}^{\mathcal{C}'_2}(f, N)$ is an isomorphism when $N \in \mathcal{C}'_2$ and consequently $\mathrm{Hom}(f, N)$ is an isomorphism when $N \in \mathcal{C}'_2$. It follows also that the map $\mathrm{Hom}^{\mathcal{C}'_2}(f, N)$ is surjective for all M, N, hence $\mathrm{Hom}(f, N)$ is surjective for all M, N. Taking $N = F(M)$ shows that $f(M)$ has a left inverse ϕ. Let $N' = \operatorname{coker} \phi$. This is an object of $\mathcal{C}'_2$ since $\bar{f}(M)$ is an isomorphism. The canonical map $F(M) \to N'$ lifts through $f(M)$ to a map $G(M) \to N'$, hence $N' = 0$ and $F(M)$ is an isomorphism. ☐

Let $R, R' : \mathfrak{A} \to \mathfrak{B}$ be two 2-functors and $\sigma : R' \to R$. Assume σ is locally fully faithful. We define $R'' = \mathcal{C}oker\sigma : \mathfrak{A} \to \mathfrak{B}$ (denoted also by R/R' when there is no ambiguity) by $R'' : a \mapsto \mathcal{C}oker\sigma(a)$. The composition $\mathcal{H}om(a, a') \xrightarrow{R(a,a')} \mathcal{H}om(R(a), R(a')) \to \mathcal{H}om(R(a), \mathcal{C}oker\sigma(a'))$ factors uniquely through $\mathcal{H}om(\mathcal{C}oker\sigma(a), \mathcal{C}oker\sigma(a'))$ and this defines a functor $\mathcal{H}om(a, a') \to$

$\mathcal{H}om(\mathcal{C}oker\sigma(a), \mathcal{C}oker\sigma(a'))$. The constraints are deduced by taking quotients.

$$\begin{tikzcd}
 & & \mathcal{H}om(\mathcal{C}oker\sigma(a), \mathcal{C}oker\sigma(a')) \arrow[d] \\
\mathcal{H}om(a,a') \arrow[r] \arrow[d] \arrow[urr, dashed] \arrow[drr, bend right=60, "0"'] & \mathcal{H}om(R(a),R(a')) \arrow[r] \arrow[d] & \mathcal{H}om(R(a),\mathcal{C}oker\sigma(a')) \arrow[d] \\
\mathcal{H}om(R'(a),R'(a')) \arrow[r] \arrow[ur, Rightarrow, "\sim"] & \mathcal{H}om(R'(a),R(a')) \arrow[r] & \mathcal{H}om(R'(a),\mathcal{C}oker\sigma(a'))
\end{tikzcd}$$

We have a Grothendieck group functor $K_0 : \mathfrak{Tri}_{\le 1} \to \mathcal{A}b$. When $\mathfrak{B}$ is endowed with a canonical 2-functor to the 2-category of triangulated categories, we will still denote by K_0 the composite functor $\mathfrak{B}_{\le 1} \to \mathrm{Ab}$. For example, $\mathfrak{B}$ is the category of exact categories or of dg-categories and we consider the derived category 2-functor. Viewing additive categories as exact categories for the split structure provides another example (this is the homotopy category functor). This gives a "decategorification" functor $\mathfrak{A}\text{-Mod}(\mathfrak{B})_{\le 1} \to \mathcal{H}om(\mathfrak{A}_{\le 1}, \mathcal{A}b)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be k-linear 2-categories. Assume $\mathfrak{B}$ is locally idempotent-complete. Let $\mathfrak{A}^i$ be the idempotent completion of $\mathfrak{A}$. The canonical strict 2-functor $\mathfrak{A} \to \mathfrak{A}^i$ induces a 2-equivalence $\mathfrak{A}^i\text{-Mod}(\mathfrak{B}) \xrightarrow{\sim} \mathfrak{A}\text{-Mod}(\mathfrak{B})$.

2.3. **Symmetric algebras.** The theory of symmetric or Frobenius algebras is classical. We need here a version over a non-commutative base algebra and we study transitivity properties.

2.3.1. *Frobenius forms.* Let B be a k-algebra and A a B-algebra. We denote by $m : A \otimes_B A \to A$ the multiplication map.

The canonical isomorphism of (A,B)-bimodules $\mathrm{Hom}_B(A,B) \xrightarrow{\sim} \mathrm{Hom}_A(A, \mathrm{Hom}_B(A,B))$ restricts to an isomorphism

$$t \mapsto \hat{t} : \mathrm{Hom}_{B,B}(A,B) \xrightarrow{\sim} \mathrm{Hom}_{A,B}(A, \mathrm{Hom}_B(A,B)).$$

Let us describe this explicitely. Given $t : A \to B$ a morphism of (B,B)-bimodules, we have the morphism of (A,B)-bimodules

$$\begin{aligned} \hat{t} : A &\to \mathrm{Hom}_B(A,B) \\ a &\mapsto (a' \mapsto t(a'a)). \end{aligned}$$

Conversely, given $f : A \to \mathrm{Hom}_B(A,B)$ a morphism of (A,B)-bimodules, then $f(1) : A \to B$ is a morphism of (B,B)-bimodules and we have $f = \widehat{f(1)}$.

Definition 2.6. *Let $t : A \to B$ be a morphism of (B,B)-bimodules. We say that t is a Frobenius form if A is a projective B-module of finite type and $\hat{t} : A \to \mathrm{Hom}_B(A,B)$ is an isomorphism.*

Let $t : A \to B$ be a Frobenius form. It defines an automorphism of $Z(B)$-algebras, the Nakayama automorphism:

$$\gamma_t : A^B \xrightarrow{\sim} A^B,\ a \mapsto \hat{t}^{-1}\left(a' \mapsto t(aa')\right).$$

We have
$$t(aa') = t(a'\gamma_t(a)) \text{ for all } a \in A^B \text{ and } a' \in A.$$
This makes $\hat{t}$ into an isomorphism of $(A, B \otimes_{Z(B)} A^B)$-modules
$$\hat{t} : A_{1\otimes\gamma_t} \xrightarrow{\sim} \mathrm{Hom}_B(A, B).$$

We say that t is *symmetric* if $\gamma_t = \mathrm{id}_{A^B}$.

Remark 2.7. Note that if $t(aa') = t(a'a)$ for all $a, a' \in A$, then $A^B = A$.

Given t and t' two Frobenius forms, there is a unique element $z \in (A^B)^\times$ such that $t'(a) = t(az)$ for all $a \in A$. If in addition t and t' are symmetric, then $z \in Z(A^B)^\times$.

2.3.2. *Adjunction* $(\mathrm{Res}, \mathrm{Ind})$. Let B be a k-algebra and A a B-algebra.

The data of an adjunction $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$ is the same as the data of an isomorphism $A \otimes_B - \xrightarrow{\sim} \mathrm{Hom}_B(A, -)$ of functors B-Mod $\to$ A-Mod.

Assume there is such an adjunction. The functor $\mathrm{Hom}_B(A, -)$ is right exact, hence A is projective as a B-module. The functor $\mathrm{Hom}_B(A, -)$ commutes with direct sums, hence A is a finitely generated projective B-module.

Assume now A is a finitely generated projective B-module. We have a canonical isomorphism
$$\mathrm{Hom}_B(A, B) \otimes_B - \xrightarrow{\sim} \mathrm{Hom}_B(A, -).$$
So, the data of an adjunction $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$ is the same as the data of an isomorphism $f : A \xrightarrow{\sim} \mathrm{Hom}_B(A, B)$ of (A, B)-bimodules. Given f, let $t = f(1) : A \to B$. This is the morphism of (B, B)-bimodules corresponding to the counit $\varepsilon : \mathrm{Res}_B^A \mathrm{Ind}_B^A \to \mathrm{id}_B$. On the other hand, we have $f = \hat{t}$. Summarizing, we have the following Proposition.

Proposition 2.8. *Let B be an algebra and A a B-algebra. We have inverse bijections between the set of Frobenius forms and the set of adjunctions* $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$*:*
$$\begin{aligned} t &\mapsto \text{ adjunction defined by } \hat{t} \\ \text{counit} &\leftarrow\!\shortmid \text{ adjunction} \end{aligned}$$

Assume we have a Frobenius form $t : A \to B$. The unit of adjunction of the pair $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$ corresponds to a morphism of (A, A)-bimodules $A \to A \otimes_B A$. The image of 1 under this morphism is the *Casimir element* $\pi = \pi_B^A \in (A \otimes_B A)^A$. It satisfies
$$(t \otimes 1)(\pi) = (1 \otimes t)(\pi) = 1 \in A. \tag{2}$$
Conversely, given an element $\pi \in (A \otimes_B A)^A$, there exists at most one $t \in \mathrm{Hom}_{B,B}(A, B)$ satisfying (2), and such a morphism is a Frobenius form.

Note that right multiplication induces an isomorphism $A^B \xrightarrow{\sim} \mathrm{End}(\mathrm{Ind}_B^A)$ and the automorphism (1) is the Nakayama automorphism γ_t.

Remark 2.9. We developed the theory for left modules, but this is the same as the theory for right modules. Namely, let $t : A \to B$ be a Frobenius form. Since A is finitely generated and projective as a B-module, it follows that $\mathrm{Hom}_B(A, B)$ is a finitely generated projective right B-module, hence A is a finitely generated projective right B-module. Consider the composition
$$\check{t} : A \xrightarrow{a \mapsto (\zeta \mapsto \zeta(a))} \mathrm{Hom}_{B^{\mathrm{opp}}}(\mathrm{Hom}_B(A, B), B) \xrightarrow{\mathrm{Hom}_{B^{\mathrm{opp}}}(\hat{t}, B)} \mathrm{Hom}_{B^{\mathrm{opp}}}(A, B),\ a \mapsto (a' \mapsto t(aa')).$$
The first map is an isomorphism since A is finitely generated and projective as a B-module. It follows that $\check{t}$ is an isomorphism.

2.3.3. *Transitivity.* Let C be an algebra, B a C-algebra and A a B-algebra. We assume that A (resp. B) is a finitely generated projective B-module (resp. C-module).

Given $t \in \mathrm{Hom}_{B,B}(A,B)$, $t' \in \mathrm{Hom}_{C,C}(B,C)$ and $t'' = t' \circ t \in \mathrm{Hom}_{C,C}(A,C)$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\hat{t}''"] \arrow[d, "\hat{t}"'] & & \mathrm{Hom}_C(A,C) \arrow[d, "\mathrm{can}", "\sim"'] \\ \mathrm{Hom}_B(A,B) \arrow[urr, "\mathrm{Hom}(A,t')"] \arrow[rr, "\mathrm{Hom}(A,\hat{t}')"'] & & \mathrm{Hom}_B(A, \mathrm{Hom}_C(B,C)) \end{tikzcd}$$

The units of adjunction are given by composition:

$$A \xrightarrow{1 \mapsto \pi_B^A} A \otimes_B A \xrightarrow{1\otimes 1 \mapsto 1\pi_C^B 1} A \otimes_C A,\ 1 \mapsto \pi_C^A.$$

Lemma 2.10. *If $t \in \mathrm{Hom}_{B,B}(A,B)$ and $t' \in \mathrm{Hom}_{C,C}(B,C)$ are Frobenius forms, then $t' \circ t : A \to C$ is a Frobenius form.*

Lemma 2.11. *Let $t' \in \mathrm{Hom}_{C,C}(B,C)$ and $t'' \in \mathrm{Hom}_{C,C}(A,C)$ be Frobenius forms. There is a unique $t \in \mathrm{Hom}_B(A,B)$ such that $t'' = t' \circ t$. It is a Frobenius form and it is given by $t = \mathrm{Hom}_B(A,\hat{t}')^{-1}(\hat{t}''(1)) \in \mathrm{Hom}_{B,B}(A,B)$.*

Let $t'' \in \mathrm{Hom}_{C,C}(A,C)$ and $\zeta \in A^C$. Define $t' \in \mathrm{Hom}_{C,C}(B,C)$ by $t'(b) = t''(b\zeta)$. If t'' is a Frobenius morphism and the pairing

$$B \times B \to C,\ (b,b') \mapsto t''(bb'\zeta)$$

is perfect, then t' is a Frobenius form.

Assume now t, t' and t'' are given and let $\zeta \in A^C$. Then,

$$t(\zeta) = 1 \Leftrightarrow \forall b \in B,\ t'(bt(\zeta)) = t'(b) \Leftrightarrow \forall b \in B,\ t''(b\zeta) = t'(b).$$

Note that ζ is determined by t' up to adding an element $\xi \in A^C$ such that $t''(B\xi) = 0$. The next lemma shows that under certain conditions on A, the form t' is always obtained from such a ζ.

Lemma 2.12. *Assume B is a quotient of A as a (B,C)-bimodule (this is the case if A is a progenerator for B and $C \subset Z(A)$). Let $t \in \mathrm{Hom}_{B,B}(A,B)$ and $t'' \in \mathrm{Hom}_{C,C}(A,C)$ be Frobenius forms. There is a unique $t' \in \mathrm{Hom}_{C,C}(B,C)$ such that $t'' = t' \circ t$. It is a Frobenius form.*

Proof. Since A is a progenerator for B, the morphism $\hat{t}'$ is determined by $\mathrm{Hom}_B(A,\hat{t}')$. The unicity of t' follows.

Assume A is a progenerator for B and C is central in A. Since A is a progenerator for B, there exists an integer n and a surjection of B-modules $f : A^n \to B$. Let $m \in f^{-1}(1)$ and consider the morphism $A \to A^n$, $a \mapsto am$. The composition $g : A \to A^n \to B$ is a morphism of B-modules with $g(1) = 1$. Since C is central, g is a morphism of (B,C)-bimodules.

Assume now there is a surjective morphism of (B,C)-bimodules $h : A \to B$. Then, $h(1) \in Z(C)^\times$. let $g : A \to B$, $a \mapsto ah(1)^{-1}$. This is a morphism of (B,C)-bimodules with $g(1) = 1$.

Let $\zeta = \hat{t}^{-1}(g)$. We have $t(\zeta) = 1$ and we define t' by $t'(b) = t''(b\zeta)$. We have $t'' = t' \circ t$, the morphism $\mathrm{Hom}_B(A,\hat{t}')$ is invertible and since A is a progenerator for B, it follows that $\hat{t}'$ is an isomorphism. $\square$

2.3.4. *Bases.* Let B be an algebra, A a B-algebra and assume A is free of finite rank as a B-module. Let $\mathcal{B}$ be a basis of A as a left B-module: $A = \bigoplus_{v\in\mathcal{B}} Bv$.

Let $t \in \mathrm{Hom}_{B,B}(A,B)$. Then, t is a Frobenius form if and only if there exists a *dual basis* $\mathcal{B}^\vee = \{v^\vee\}_{v\in\mathcal{B}}$: *i.e.*, $\mathcal{B}^\vee$ satisfies $t(v'v^\vee) = \delta_{vv'}$ for $v, v' \in \mathcal{B}$.

Assume t is a Frobenius form. Then, $\mathcal{B}^\vee$ exists and is unique. It is a basis of A as a right B-module. We have

$$\hat{t}(v^\vee) = (\mathcal{B} \ni v' \mapsto \delta_{v,v'}) \text{ for } v \in \mathcal{B}.$$

Given $a \in A$, we have

$$a = \sum_{v\in\mathcal{B}} t(av^\vee)v = \sum_{v\in\mathcal{B}} v^\vee t(va).$$

Given $a \in A^B$, we have

$$\gamma_t(a) = \sum_{v\in\mathcal{B}} v^\vee t(av).$$

The unit of the adjoint pair $(\mathrm{Res}_B^A, \mathrm{Ind}_B^A)$ is given by the morphism of (A,A)-bimodules

$$A \to A \otimes_B A,\ 1 \mapsto \pi_B^A = \sum_{v\in\mathcal{B}} v^\vee \otimes v.$$

Consider now C an algebra and a C-algebra structure on B such that B is free of finite rank as a C-module. Let $\mathcal{B}'$ be a basis of B as a C-module. Then, $\mathcal{B}'' = \mathcal{B}'\mathcal{B} = \{v'v\}_{v\in\mathcal{B},v'\in\mathcal{B}'}$ is a basis of A as a C-module.

Let $t' : B \to C$ be a Frobenius form. The dual basis to $\mathcal{B}''$ for the Frobenius form $t'' = t' \circ t : A \to C$ is $\mathcal{B}''^\vee = \{v^\vee v'^\vee\}_{v\in\mathcal{B},v'\in\mathcal{B}'}$. Given $a \in A$, we have

$$t(a) = \sum_{v'\in\mathcal{B}'} t''(av'^\vee)v' = \sum_{v'\in\mathcal{B}'} v'^\vee t''(v'a).$$

Given $v \in \mathcal{B}$, we have

$$v^\vee = \sum_{v'\in\mathcal{B}'} (v'v)^\vee t'(v').$$

2.3.5. *Ramification.* Let A be a B-algebra endowed with a Frobenius form t and assume $A^B = A$.

The following statements are equivalent:

(a) A is a projective $(A \otimes_B A^{\mathrm{opp}})$-module
(b) there exists $a \in A$ such that $m((1 \otimes a \otimes 1 \otimes 1)\pi) = 1$
(c) there exists $a \in A$ such that $m((1 \otimes 1 \otimes a \otimes 1)\pi) = 1$

where $A \otimes_B A$ is viewed as a module over $((A \otimes A^{\mathrm{opp}}) \otimes_B (A \otimes A^{\mathrm{opp}}))$.

When A is commutative, the statements (a)-(c) above are equivalent to the following two statements

(d) A is étale over B
(e) $m(\pi) \in A^\times$.

3. Hecke algebras

3.1. **Classical Hecke algebras.** We recall in this section the various versions of affine Hecke algebras and the isomorphisms between them after suitable localizations. We consider only the case of GL_n: in this case, the inclusion $\mathbf{G}_m^n \hookrightarrow \mathbf{G}_a^n$ gives an algebraic $\mathfrak{S}_n$-equivariant map that makes it possible to avoid completions. In general, one needs to use the expotential map from the Lie algebra of a torus to the torus. All constructions and results in this section extend to arbitrary Weyl groups.

3.1.1. *BGG-Demazure operators.* Given $1 \le i \le n$, we put $s_i = (i, i+1) \in \mathfrak{S}_n$. We define an endomorphism of abelian groups $\partial_i \in \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}[X_1, \ldots, X_n])$ by

$$\partial_i(P) = \frac{P - s_i(P)}{X_{i+1} - X_i}.$$

The formula defines endomorphisms of various localizations, for example $\mathbf{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

Given $w = s_{i_1} \cdots s_{i_r}$, we put

$$\partial_w = \partial_{i_1} \cdots \partial_{i_r}.$$

This is independent of the choice of the reduced decomposition.

The $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-linear morphism $\partial_{w[1,n]}$ takes values in $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$. It is a symmetrizing form for the $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-algebra $\mathbf{Z}[X_1, \ldots, X_n]$. We view $\mathbf{Z}[X_1, \ldots, X_n]$ as a graded algebra with $\deg(X_i) = 2$. Then, $\partial_{w[1,n]}$ is homogeneous of degree $-n(n-1)$.

Lemma 3.1. *Denote by π the Casimir element for $\partial_{w[1,n]}$. Then $m(\pi) = \prod_{1 \le j < i \le n}(X_i - X_j)$.*

Proof. The algebra $\mathbf{Z}[X_1, \ldots, X_n]$ is étale over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ outside $m(\pi) = 0$. So, $\prod_{1 \le j < i \le n}(X_i - X_j) \mid m(\pi)$ (cf §2.3.5). Since $m(\pi)$ is homogeneous of degree $n(n-1)$, it follows that there is $a \in \mathbf{Z}$ such that $m(\pi) = a \prod_{1 \le j < i \le n}(X_i - X_j)$. On the other hand, $\partial_{w[1,n]}(m(\pi)) = n! = \partial_{w[1,n]}\left(\prod_{1 \le j < i \le n}(X_i - X_j)\right)$ and the lemma follows. $\square$

Let $A = \mathbf{Z}[X_1, \ldots, X_n] \rtimes \mathfrak{S}_n$. This algebra has a Frobenius form over $\mathbf{Z}[X_1, \ldots, X_n]$ given by

$$Pw \mapsto P\delta_{w \cdot w[1,n]} \text{ for } P \in \mathbf{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n.$$

By composition, we obtain a Frobenius form t for A over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by

$$t(Pw) = \partial_{w[1,n]}(P)\delta_{w \cdot w[1,n]} \text{ for } P \in \mathbf{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n.$$

The corresponding Nakayama automorphism of A is the involution

$$X_i \mapsto X_{n-i+1}, \ s_i \mapsto -s_{n-i}.$$

3.1.2. *Degenerate affine Hecke algebras.* Let $\bar{H}_n$ be the *degenerate affine Hecke algebra* of GL_n: $\bar{H}_n = \mathbf{Z}[X_1, \ldots, X_n] \otimes \mathbf{Z}\mathfrak{S}_n$ where $\mathbf{Z}[X_1, \ldots, X_n]$ and $\mathbf{Z}\mathfrak{S}_n$ are subalgebras and

$$T_i X_j = X_j T_i \text{ if } j - i \neq 0, 1 \text{ and } T_i X_{i+1} - X_i T_i = 1.$$

We denote here by $T_1, \ldots, T_{n-1}$ the Coxeter generators for $\mathfrak{S}_n$ and we write T_w for the element w of $\mathfrak{S}_n$.

Given $P \in \mathbf{Z}[X_1, \ldots, X_n]$, we have $T_i P - s_i(P) T_i = \partial_i(P)$.

We have a faithful representation on $\mathbf{Z}[X_1, \ldots, X_n] = \bar{H}_n \otimes_{\mathbf{Z}\mathfrak{S}_n} \mathbf{Z}$ where

$$T_i(P) = s_i(P) + \partial_i(P).$$

Here, $\mathbf{Z}$ is the trivial representation of $\mathfrak{S}_n$.

The algebra $\bar{H}_n$ has a Frobenius form over $\mathbf{Z}[X_1, \ldots, X_n]$ given by

$$PT_w \mapsto P\partial_{w,w[1,n]} \text{ for } P \in \mathbf{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n. \tag{3}$$

By composition, we obtain a Frobenius form t for $\bar{H}_n$ over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by

$$t(PT_w) = \partial_{w[1,n]}(P)\delta_{w\cdot w[1,n]} \text{ for } P \in \mathbf{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n. \tag{4}$$

The corresponding Nakayama automorphism of $\bar{H}_n$ is the involution

$$X_i \mapsto X_{n-i+1},\ T_i \mapsto -T_{n-i}.$$

3.1.3. *Finite Hecke algebras.* Let $R = \mathbf{Z}[q^{\pm 1}]$. Let H_n^f be the *Hecke algebra* of GL_n: this is the R-algebra generated by $T_1, \ldots, T_{n-1}$, with relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\quad T_iT_j = T_jT_i \text{ if } |i-j| > 1 \text{ and } (T_i - q)(T_i + 1) = 0.$$

Given $w = s_{i_1} \cdots s_{i_r}$ a reduced decomposition of an element $w \in \mathfrak{S}_n$, we put $T_w = T_{i_1} \cdots T_{i_r}$. Let t_f be the R-linear form on H_n^f defined by $t_f(T_w) = \delta_{w\cdot w[1,n]}$. This is a Frobenius form, with Nakayama automorphism the involution given by $T_i \mapsto T_{n-i}$.

Remark 3.2. The algebra H_n^f is actually symmetric, via the classical form given by $T_w \mapsto \delta_{1,w}$. In other terms, the Nakayama automorphism is inner: it is conjugation by $T_{w[1,n]}$. On the other hand, the Hecke algebra is not symmetric over $\mathbf{Z}[q]$ and the classical form induces a degenerate pairing, while the form t_f above is still a Frobenius form over $\mathbf{Z}[q]$ (cf §3.1.5).

3.1.4. *Affine Hecke algebras.* Let H_n be the *affine Hecke algebra* of GL_n: $H_n = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes H^f$ where $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and H^f are subalgebras and

$$T_iX_j = X_jT_i \text{ if } j - i \neq 0, 1 \text{ and } T_iX_{i+1} - X_iT_i = (q-1)X_{i+1}.$$

Given $P \in \mathbf{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, we have $T_iP - s_i(P)T_i = (q-1)X_{i+1}\partial_i(P)$.

We have a faithful representation on $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = H_n \otimes_{H_n^f} R$, where

$$T_i(P) = qs_i(P) + (q-1)X_{i+1}\partial_i(P).$$

Here R denotes the one-dimensional representation of H_n^f on which T_i acts by q.

The algebra H_n has a Frobenius form over $\mathbf{Z}[X_1, \ldots, X_n]$ given by (3) and a Frobenius form t over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by (4). The corresponding Nakayama automorphism of H_n is the involution

$$X_i \mapsto X_{n-i+1},\ T_i \mapsto -qT_{n-i}^{-1}.$$

3.1.5. *Nil Hecke algebras.* Let ${}^0H_n^f$ be the *nil Hecke algebra* of GL_n: this is the $\mathbf{Z}$-algebra generated by $T_1, \ldots, T_{n-1}$, with relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\quad T_iT_j = T_jT_i \text{ if } |i-j| > 1 \text{ and } T_i^2 = 0.$$

Let t_0 be the linear form on ${}^0H_n^f$ defined by $t_0(T_w) = \delta_{w\cdot w[1,n]}$. This is a Frobenius form, with Nakayama automorphism given by $T_i \mapsto T_{n-i}$.

Given $w = s_{i_1} \cdots s_{i_r}$ a reduced decomposition of an element $w \in \mathfrak{S}_n$, we put $T_w = T_{i_1} \cdots T_{i_r}$. The nil Hecke algebra ${}^0H_n$ is a graded algebra with $\deg T_i = -2$ and t_0 is homogeneous of degree $n(n-1)$.

Lemma 3.3. *Let $f : M \to N$ be a morphism of relatively $\mathbf{Z}$-projective ${}^0H_n^f$-modules. If $T_{w[1,n]}f : T_{w[1,n]}M \to T_{[1,n]}N$ is an isomorphism, then f is an isomorphism.*

Proof. The annihilator of $T_{w[1,n]}$ on a relatively $\mathbf{Z}$-projective module L is $({}^0H_n^f)_{\le -2}L$. Nakayama's Lemma shows that under the assumption of the lemma, the morphism f is surjective. On the other hand, $\ker f$ is a direct summand of M, hence $\ker f$ is relatively $\mathbf{Z}$-projective. Since $T_{w[1,n]}\ker f = 0$, it follows that $\ker f = 0$. $\square$

Let A be an algebra. We denote by $A \wr {}^0H_n^f$ the algebra whose underlying abelian group is $A^{\otimes n} \otimes {}^0H_n^f$, where $A^{\otimes n}$ and ${}^0H_n^f$ are subalgebras and where $(a_1 \otimes \cdots \otimes a_n)T_i = T_i(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes a_i \otimes a_{i+2} \otimes \cdots \otimes a_n)$.

3.1.6. *Nil affine Hecke algebras.* Let ${}^0H_n$ be the *nil affine Hecke algebra* of GL_n: ${}^0H_n = \mathbf{Z}[X_1, \ldots, X_n] \otimes {}^0H_n^f$, where $\mathbf{Z}[X_1, \ldots, X_n]$ and ${}^0H_n^f$ are subalgebras and

$$T_iX_j = X_jT_i \text{ if } j - i \ne 0, 1,\ \ T_iX_{i+1} - X_iT_i = 1 \text{ and } T_iX_i - X_{i+1}T_i = -1.$$

Given $P \in \mathbf{Z}[X_1, \ldots, X_n]$, we have $T_iP - s_i(P)T_i = PT_i - T_is_i(P) = \partial_i(P)$.

We have a faithful representation on $\mathbf{Z}[X_1, \ldots, X_n] = {}^0H_n \otimes_{{}^0H_n^f} \mathbf{Z}$ where

$$T_i(P) = \partial_i(P).$$

Let $b_n = T_{w[1,n]}X_1^{n-1}X_2^{n-2}\cdots X_{n-1}$. By induction on n, one sees that $\partial_{w[1,n]}(X_1^{n-1}X_2^{n-2}\cdots X_{n-1}) = 1$, hence $b_n^2 = b_n$. We have an isomorphism of ${}^0H_n$-modules

$$\mathbf{Z}[X_1, \ldots, X_n] \xrightarrow{\sim} {}^0H_nb_n,\ \ P \mapsto Pb_n.$$

Since $\{\partial_w(X_1^{n-1}\cdots X_{n-1})_{w\in\mathfrak{S}_n}$ is a basis of $\mathbf{Z}[X_1, \ldots, X_n]$ over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$, it follows that the multiplication map gives an isomorphism of $({}^0H_n^f, \mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n})$-bimodules

$${}^0H_n^f \otimes (\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}X_1^{n-1}\cdots X_{n-1}b_n) \xrightarrow{\sim} {}^0H_nb_n.$$

Proposition 3.4. *The action of ${}^0H_n$ on $\mathbf{Z}[X_1, \ldots, X_n]$ induces an isomorphism*

$${}^0H_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}}(\mathbf{Z}[X_1, \ldots, X_n]).$$

Since $\mathbf{Z}[X_1, \ldots, X_n]$ is a free $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-module of rank $n!$, the algebra ${}^0H_n$ is isomorphic to a $(n! \times n!)$-matrix algebra over $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$.

The restriction to ${}^0H_n^f$ of any ${}^0H_n$-module is relatively $\mathbf{Z}$-projective.

Proof. Since $\mathbf{Z}[X_1, \ldots, X_n]$ is a finitely generated projective ${}^0H_n$-module, the canonical map ${}^0H_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}}(\mathbf{Z}[X_1, \ldots, X_n])$ splits as a morphism of $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-modules. The first two assertions of the proposition follow from the fact that ${}^0H_n$ is a free $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-module of rank $(n!)^2$.

The $({}^0H_n^f, \mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-module $\mathbf{Z}[X_1, \ldots, X_n]$ is a direct summand of ${}^0H_n$. So, given M an $\mathbf{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-module, then $\mathbf{Z}[X_1, \ldots, X_n] \otimes_{\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}} M$ is a direct summand of ${}^0H_n^f \otimes_{\mathbf{Z}} (\mathbf{Z}[X_1, \ldots, X_n] \otimes_{\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}} M)$ as an ${}^0H_n^f$-module. So, given N an ${}^0H_n$-module, then N is a direct summand of ${}^0H_n^f \otimes_{\mathbf{Z}} N$ as an ${}^0H_n^f$-module and the proposition is proven. $\square$

Lemma 3.3 joined with Proposition 3.4 gives a useful criterion to check that a morphism of ${}^0H_n$-modules is invertible. Note also that the proposition shows that ${}^0H_n$ is projective as a $({}^0H_n^f, {}^0H_n)$-bimodule.

The algebra ${}^0H_n$ has a Frobenius form over $\mathbf{Z}[X_1,\ldots,X_n]$ given by (3) and a Frobenius form t over $\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}$ given by (4). The corresponding Nakayama automorphism of ${}^0H_n$ is the involution

$$X_i \mapsto X_{n-i+1},\ T_i \mapsto -T_{n-i}.$$

A special feature of the nil affine Hecke algebra, compared to the affine Hecke algebra and the degenerate affine Hecke algebra, is that the Nakayama automorphism γ is inner, hence the nil affine Hecke algebra is actually symmetric over $\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}$. Indeed, when viewed as a subalgebra of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}[X_1,\ldots,X_n])$, then ${}^0H_n$ contains $\mathfrak{S}_n$. The injection of $\mathfrak{S}_n$ in ${}^0H_n$ is given by $s_i \mapsto (X_i - X_{i+1})T_i + 1$ (cf also §3.1.7). We have

$$w[1,n]aw[1,n] = \gamma(a) \text{ for all } a \in {}^0H_n.$$

It follows that the linear form t' given by $t'(a) = t(aw[1,n])$ is a symmetrizing form for ${}^0H_n$ over $\mathbf{Z}[X_1,\ldots,X_n]^{\mathfrak{S}_n}$.

The nil affine Hecke algebra ${}^0H_n$ is a graded algebra with $\deg X_i = 2$ and $\deg T_i = -2$ and t is homogeneous of degree 0. The nil affine Hecke algebra has also a bifiltration given by

$$F^{\le(i,j)}\left({}^0H_n\right) = \mathbf{Z}[X_1,\ldots,X_n]_{\le i} \otimes \left({}^0H_n^f\right)_{\ge -j}.$$

Note that $t(F^{<(n(n-1),n(n-1))}) = 0$.

3.1.7. *Isomorphisms.* The polynomial representations above induce isomorphisms with the semi-direct product of the algebra of polynomials with $\mathfrak{S}_n$, after a suitable localization.

Let $R' = \mathbf{Z}[X_1,\ldots,X_n,(X_i-X_j)^{-1},(X_i-X_j-1)^{-1}]_{i\neq j}$. We put $s_1 = (1,2),\ldots,s_{n-1} = (n-1,n) \in \mathfrak{S}_n$. We have an isomorphism of R'-algebras

$$R' \rtimes \mathfrak{S}_n \xrightarrow{\sim} R' \otimes_{\mathbf{Z}[X_1,\ldots,X_n]} \bar{H}_n,\ s_i \mapsto \frac{X_i - X_{i+1}}{X_i - X_{i+1}+1}(T_i - 1) + 1 = (T_i+1)\frac{X_i - X_{i+1}}{X_i - X_{i+1} - 1} - 1$$

Let $R'_q = R[X_1^{\pm1},\ldots,X_n^{\pm1},(X_i-X_j)^{-1},(qX_i-X_j)^{-1}]_{i\neq j}$. We have an isomorphism of R'_q-algebras

$$R'_q \rtimes \mathfrak{S}_n \xrightarrow{\sim} R'_q \otimes_{R[X_1^{\pm1},\ldots,X_n^{\pm1}]} H_n,\ s_i \mapsto \frac{X_i - X_{i+1}}{qX_i - X_{i+1}}(T_i - q) + 1 = (T_i+1)\frac{X_i - X_{i+1}}{X_i - qX_{i+1}} - 1$$

Let ${}^0R' = \mathbf{Z}[X_1,\ldots,X_n,(X_i-X_j)^{-1}]_{i\neq j}$. We have an isomorphism of ${}^0R'$-algebras

$${}^0R' \rtimes \mathfrak{S}_n \xrightarrow{\sim} {}^0R' \otimes_{\mathbf{Z}[X_1,\ldots,X_n]} {}^0H_n,\ s_i \mapsto (X_i - X_{i+1})T_i + 1 = T_i(X_{i+1} - X_i) - 1$$

Let us finally note that the functor

$$M \mapsto M^{\mathfrak{S}_n} : ({}^0R' \rtimes \mathfrak{S}_n)\text{-mod} \to ({}^0R')^{\mathfrak{S}_n}\text{-mod}$$

is an equivalence of categories.

3.2. **Nil Hecke algebras associated with hermitian matrices.** In this section, we introduce a flat family of algebras presented by quiver and relations. To a non-symmetric Cartan datum and a choice of orientation of the underlying quiver, we associate a member of that family.

3.2.1. *Definition.* Let I be a set, k a commutative ring and $Q = (Q_{i,j})_{i,j\in I}$ a matrix in $k[u,v]$ with $Q_{ii} = 0$ for all $i \in I$.

Let n be a positive integer and $L = I^n$. We define a (possibly non-unitary) k-algebra $H_n(Q)$ by generators and relations. It is generated by elements 1_ν, $x_{i,\nu}$ for $i \in \{1,\ldots,n\}$ and $\tau_{i,\nu}$ for $i \in \{1,\ldots,n-1\}$ and $\nu \in L$ and the relations are

- $1_\nu 1_{\nu'} = 1_{\nu'} 1_\nu = \delta_{\nu,\nu'}$
- $\tau_{i,\nu} 1_{\nu'} = 1_{s_i(\nu')}\tau_{i,\nu} = \delta_{\nu,\nu'}\tau_{i,\nu}$
- $x_{a,\nu} 1_{\nu'} = 1_{\nu'} x_{a,\nu} = \delta_{\nu,\nu'} x_{a,\nu}$
- $x_{a,\nu} x_{b,\nu} = x_{b,\nu} x_{a,\nu}$
- $\tau_{i,s_i(\nu)}\tau_{i,\nu} = Q_{\nu_i,\nu_{i+1}}(x_{i,\nu}, x_{i+1,\nu})$
- $\tau_{i,s_j(\nu)}\tau_{j,\nu} = \tau_{j,s_i(\nu)}\tau_{i,\nu}$ if $|i-j| > 1$
- $\tau_{i+1,s_i s_{i+1}(\nu)}\tau_{i,s_{i+1}(\nu)}\tau_{i+1,\nu} - \tau_{i,s_{i+1}s_i(\nu)}\tau_{i+1,s_i(\nu)}\tau_{i,\nu} =$

$$\begin{cases} (x_{i+2,\nu} - x_{i,\nu})^{-1}\left(Q_{\nu_i,\nu_{i+1}}(x_{i+2,\nu}, x_{i+1,\nu}) - Q_{\nu_i,\nu_{i+1}}(x_{i,\nu}, x_{i+1,\nu})\right) & \text{if } \nu_i = \nu_{i+2} \\ 0 & \text{otherwise} \end{cases}$$

- $\tau_{i,\nu} x_{a,\nu} - x_{s_i(a),s_i(\nu)}\tau_{i,\nu} = \begin{cases} -1_\nu & \text{if } a = i \text{ and } \nu_i = \nu_{i+1} \\ 1_\nu & \text{if } a = i+1 \text{ and } \nu_i = \nu_{i+1} \\ 0 & \text{otherwise.} \end{cases}$

for $\nu, \nu' \in I$, $1 \le i, j \le n-1$ and $1 \le a, b \le n$, where $x_l = x_{l,\nu}$.

Remark 3.5. Note that when Γ is finite, then $H_n(\Gamma)$ has a unit $1 = \sum_{\nu\in L} 1_\nu$.

Remark 3.6. It is actually more natural to view $H_n(\Gamma)$ as a category $\mathcal{H}_n(\Gamma)$ with set of objects L and with Hom-spaces generated by

$$x_{a,\nu} \in \mathrm{End}(\nu) \text{ for } 1 \le a \le n$$

$$\tau_{i,\nu} \in \mathrm{Hom}(\nu, s_i(\nu)) \text{ for } 1 \le i \le n-1$$

with the relations above.

Given $a \in 1_\nu H_n(Q) 1_{\nu'}$, we will sometimes write $x_i a$ for $x_{i,\nu} a$ and $a x_i$ for $a x_{i,\nu'}$ and proceed similarly for τ_i.

Consider the (possibly non-unitary) algebra $R_n = \left(k^{(I)}[x]\right)^{\otimes n} = (k[x_1,\ldots,x_n] \otimes (k^{(I)})^{\otimes n})$. We denote by 1_s the idempotent corresponding to the s-th factor of $k^{(I)}$ and we put $1_\nu = 1_{\nu_1} \otimes \cdots \otimes 1_{\nu_n}$ for $\nu \in L$.

There is a morphism of algebras $R_n \to H_n(Q)$, $x_i 1_\nu \mapsto x_{i,\nu}$. It restricts to a morphism $R_n^{\mathfrak{S}_n} \to Z(H_n(Q))$. Note that $R_1 = H_1(Q)$ and we put $H_0(Q) = k$.

Let J be a set of finite sequences of elements of $\{1,\ldots,n-1\}$ such that $\{s_{i_1}\cdots s_{i_r}\}_{(i_1,\ldots,i_r)\in J}$ is a set of minimal length representatives of elements of $\mathfrak{S}_n$. Then,

$$S = \{\tau_{i_1,s_{i_2}\cdots s_{i_r}(\nu)} \cdots \tau_{i_r,\nu} x_{1,\nu}^{a_1} \cdots x_{n,\nu}^{a_n}\}_{(i_1,\ldots,i_r)\in J,(a_1,\ldots,a_n)\in \mathbf{Z}_{\ge 0}^n,\nu\in L}$$

generates $H_n(Q)$ as a k-module.

The algebra $H_n(Q)$ is filtered with 1_ν and $x_{i,\nu}$ in degree 0 and $\tau_{i,\nu}$ in degree 1. We have a surjective algebra morphism

$$k^{(I)}[x] \wr {}^0H_n^f \to \mathrm{gr} H_n(Q).$$

The algebra is said to satisfy the PBW (Poincaré-Birkhoff-Witt) property if that morphism is an isomorphism.

Theorem 3.7. *Assume $n \geq 2$. The following assertions are equivalent*
- *$H_n(Q)$ satisfies PBW*
- *$H_n(Q)$ is a free k-module with basis S*
- *$Q_{ij}(u,v) = Q_{ji}(v,u)$ for all $i,j \in I$.*

Proof. The first two assertions are equivalent, thanks to the generating family S described above.

Let $\nu \in L$ with $\nu_i \neq \nu_{i+1}$. We have

$Q_{\nu_{i+1},\nu_i}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)})\tau_{i,\nu} = \tau_{i,\nu}\tau_{i,s_i(\nu)}\tau_{i,\nu} = \tau_{i,\nu}Q_{\nu_i,\nu_{i+1}}(x_{i,\nu}, x_{i+1,\nu}) = Q_{\nu_i,\nu_{i+1}}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)})\tau_{i,\nu}.$

It follows that

$$(Q_{\nu_{i+1},\nu_i}(x_{i+1,s_i(\nu)}, x_{i,s_i(\nu)}) - Q_{\nu_i,\nu_{i+1}}(x_{i+1,s_i(\nu)}, x_{i,s_i(\nu)}))\tau_{i,\nu} = 0.$$

Assume S is a basis of $H_n(Q)$. We have $Q_{\nu_{i+1},\nu_i}(x_{i+1,s_i(\nu)}, x_{i,s_i(\nu)}) - Q_{\nu_i,\nu_{i+1}}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)}) = 0$. Consequently, $Q_{ij}(u,v) = Q_{ji}(v,u)$ for all $i,j \in I$.

Assume $Q_{ij}(u,v) = Q_{ji}(v,u)$ for all $i,j \in I$. Choose an ordering of pairs of distinct elements of I. Given $i < j$, put $P_{ij} = Q_{ij}$ and $P_{ji} = 1$. The theorem follows now from Proposition 3.12 below. $\square$

So, the algebras $H_n(Q)$ form a flat family of algebras over the space $k[u,v]^{\mathcal{P}_2(I)}$, where $\mathcal{P}_2(I)$ is the set of 2-element subsets of I. Denote by $Q \mapsto \bar{Q}$ the automorphism given by $\bar{Q}_{ij}(u,v) = Q_{ji}(v,u)$.

We identify this space with the space of matrices with vanishing diagonal and hermitian with respect to the automorphism of $k[u,v]$ swapping u and v, *i.e.*, such that $\bar{Q} = Q$.

Corollary 3.8. *Assume Q is hermitian. Let I' be a subset of I and $Q' = (Q_{i,j})_{i,j \in I'}$. Then, the canonical map $H_n(Q') \to H_n(Q)$ is injective and induces isomorphisms $1_\nu H_n(Q')1_{\nu'} \xrightarrow{\sim} 1_\nu H_n(Q)1_{\nu'}$ for $\nu, \nu' \in (I')^n$.*

From Proposition 3.12 below, we obtain a description of the center of $H_n(Q)$.

Proposition 3.9. *Assume Q is hermitian. Then, we have $Z(H_n(Q)) = R_n^{\mathfrak{S}_n}$.*

When $|I| = 1$, then $H_n(Q)$ is the nil affine Hecke algebra ${}^0H_n$ associated with GL_n.

Given $0 \leq i \leq n$, we have an injective morphism of R_n-algebras

$$H_i(Q) \otimes H_{n-i}(Q) \to H_n(Q)$$

given by $1_\nu \otimes 1_{\nu'} \mapsto 1_{\nu \cup \nu'}$, $x_{j,\nu} \otimes 1_{\nu'} \mapsto x_{j,\nu\cup\nu'}$, $1_\nu \otimes x_{j,\nu'} \mapsto x_{i+j,\nu\cup\nu'}$, etc.

Remark 3.10. The algebra Khovanov and Lauda [KhoLau] associate to a symmetrizable Cartan matrix (a_{ij}) corresponds to $Q_{ij}(u,v) = u^{-a_{ij}} + v^{-a_{ji}}$ for $i \neq j$.

Let us describe some isomorphisms between $H_n(Q)$'s.

Let $\{a_i\}_{i \in I}$ in k and $\{\beta_{ij}\}_{i,j \in I}$ in $k^\times$. Let $Q'_{ij}(u,v) = \beta_{ij}\beta_{ji}Q_{ij}(\beta_{jj}u + a_j, \beta_{ii}v + a_i)$. We have an isomorphism

$$H_n(Q') \xrightarrow{\sim} H_n(Q),\ 1_\nu \mapsto 1_\nu,\ x_{i,\nu} \mapsto \beta^{-1}_{\nu_i,\nu_i}(x_{i,\nu} - a_{\nu_i}),\ \tau_{i,\nu} \mapsto \beta_{\nu_i,\nu_{i+1}}\tau_{i,\nu}.$$

Put $\Delta I = \{(i,i)|i \in I\} \subset I \times I$. The construction above provides an action of the quotient of $(\mathbf{G}_m)^{(I\times I)-\Delta I}$ by $\{\beta_{i,j}|\beta_{ij}\beta_{ji} = 1\ \forall i \neq j\}$ on $H_n(Q)$.

Assume Q is hermitian. Given $\nu \in I^n$, we define $\bar{\nu} \in I^n$ by $\bar{\nu}_i = \nu_{n-i+1}$. There is an involution of $H_n(Q)$

$$H_n(Q) \xrightarrow{\sim} H_n(Q),\ 1_\nu \mapsto 1_{\bar{\nu}},\ x_{i,\nu} \mapsto x_{n-i+1,\bar{\nu}},\ \tau_{i,\nu} \mapsto -\tau_{n-i,\bar{\nu}}.$$

Let us finally construct a duality. There is an isomorphism

$$H_n(Q) \xrightarrow{\sim} H_n(Q)^{\mathrm{opp}},\ 1_\nu \mapsto 1_\nu,\ x_{i,\nu} \mapsto x_{i,\nu},\ \tau_{i,\nu} \mapsto \tau_{i,s_i(\nu)}.$$

Remark 3.11. One can also work with a matrix Q with values in $k(u,v)$ and define $H_n(Q)$ by adding inverses of the relevant polynomials in $x_{i,\nu}$'s.

3.2.2. *Polynomial realization.* Let $P = (P_{ij})_{i,j\in I}$ be a matrix in $k[u,v]$ with $P_{ii} = 0$ for all $i \in I$ and let $Q_{i,j}(u,v) = P_{i,j}(u,v)P_{j,i}(v,u)$.

Consider the (possibly non-unitary) k-algebra $A_n(I) = k^{(I)}[x] \wr \mathfrak{S}_n$.

The following Proposition provides a faithful representation of $H_n(Q)$ on the space R_n. It also shows that, after localization, the algebra $H_n(Q)$ depends only on the cardinality of I (assuming non-vanishing of Q_{ij} for $i \neq j$).

Proposition 3.12. *Let $\mathcal{O}' = \bigoplus_{\nu\in L} k[x_1,\ldots,x_n][\{(x_i-x_j)^{-1}\}_{i\neq j,\nu_i=\nu_j}]1_\nu$. We have an injective morphism of k-algebras*

$$H_n(Q) \to \mathcal{O}' \otimes_{\mathbf{Z}^{(I)}[x]^{\otimes n}} A_n(I)$$
$$1_\nu \mapsto 1_\nu,\ x_{i,\nu} \mapsto x_i 1_\nu,$$
$$\tau_{i,\nu} \mapsto \begin{cases} (x_i - x_{i+1})^{-1}(s_i 1_\nu - 1_\nu) & \text{if } \nu_i = \nu_{i+1} \\ P_{\nu_i,\nu_{i+1}}(x_{i+1},x_i)s_i 1_\nu & \text{otherwise} \end{cases}$$

for $1 \le i \le n$ and $\nu \in L$. It defines a faithful representation of $H_n(Q)$ on $R_n = \bigoplus_{\nu\in L} k[x_1,\ldots,x_n]1_\nu$.

Assume $P_{i,j} \neq 0$ for all $i \neq j$. Let

$$\mathcal{O} = \bigoplus_{\nu\in L} k[x_1,\ldots,x_n][\{P_{\nu_i,\nu_j}(x_i,x_j)^{-1}\}_{\nu_i\neq\nu_j}, \{(x_i-x_j)^{-1}\}_{i\neq j,\nu_i=\nu_j}]1_\nu.$$

The morphism above induces an isomorphism $\mathcal{O} \otimes_{k^{(I)}[x]^{\otimes n}} H_n(Q) \xrightarrow{\sim} \mathcal{O} \otimes_{k^{(I)}[x]^{\otimes n}} A_n(I)$.

Proof. Let $\tau'_{i,\nu} = \begin{cases} (x_i - x_{i+1})^{-1}(s_i 1_\nu - 1_\nu) & \text{if } \nu_i = \nu_{i+1} \\ P_{\nu_i,\nu_{i+1}}(x_{i+1},x_i)s_i 1_\nu & \text{otherwise.} \end{cases}$

Let us check that the defining relations of $H_n(Q)$ hold with $\tau_{i,\nu}$ replaced by $\tau'_{i,\nu}$. We will not write the idempotents 1_ν to make the calculations more easily readable.

We have

$$\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu} =$$
$$\begin{cases} (x_i - x_{i+1})^{-1}\left((x_i - x_{i+2})^{-1}(s_i s_{i+1} - s_i) - (x_{i+1} - x_{i+2})^{-1}(s_{i+1} - 1)\right) & \text{if } \nu_i = \nu_{i+1} = \nu_{i+2} \\ P_{\nu,\nu_{i+1}}(x_{i+1},x_i)(x_i - x_{i+2})^{-1}(s_i s_{i+1} - s_i) & \text{if } \nu_{i+1} = \nu_{i+2} \neq \nu_i \\ (x_i - x_{i+1})^{-1}\left(P_{\nu_{i+1},\nu_{i+2}}(x_{i+2},x_i)s_i s_{i+1} - P_{\nu_{i+1},\nu_{i+2}}(x_{i+2},x_{i+1})s_{i+1}\right) & \text{if } \nu_i = \nu_{i+2} \neq \nu_{i+1} \\ P_{\nu_i,\nu_{i+2}}(x_{i+1},x_i)P_{\nu_{i+1},\nu_{i+2}}(x_{i+2},x_i)s_i s_{i+1} & \text{if } \nu_{i+2} \notin \{\nu_i,\nu_{i+1}\}. \end{cases}$$

Assume $\nu_i = \nu_{i+1} = \nu_{i+2}$. We have

$$\tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ = (x_{i+1}-x_{i+2})^{-1}(x_i-x_{i+2})^{-1}(x_i-x_{i+1})^{-1}(s_{i+1}s_is_{i+1}-s_{i+1}s_i-s_is_{i+1}+s_i+s_{i+1}-1) \\ = \tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu}$$

Assume $\nu_i = \nu_{i+1} \neq \nu_{i+2}$. We have

$$\tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ = (x_{i+1}-x_{i+2})^{-1}P_{\nu_i,\nu_{i+2}}(x_{i+1},x_i)P_{\nu_i,\nu_{i+2}}(x_{i+2},x_i)(s_{i+1}s_is_{i+1}-s_is_{i+1}) \\ = \tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu}$$

Assume $\nu_{i+1} = \nu_{i+2} \neq \nu_i$. We have

$$\tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ = (x_i,x_{i+1})^{-1}P_{\nu_i,\nu_{i+1}}(x_{i+2},x_i)P_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})(s_{i+1}s_is_{i+1}-s_{i+1}s_i) \\ = \tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu}$$

Assume ν_i, ν_{i+1} and ν_{i+2} are distinct. We have

$$\tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ = P_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})P_{\nu_i,\nu_{i+2}}(x_{i+2},x_i)P_{\nu_{i+1},\nu_{i+2}}(x_{i+1},x_i)s_{i+1}s_is_{i+1} \\ = \tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu}$$

Assume finally $\nu_i = \nu_{i+2} \neq \nu_{i+1}$. We have

$$\tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ = (x_i-x_{i+2})^{-1}P_{\nu_{i+1},\nu_i}(x_{i+1},x_i)\left(P_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})s_is_{i+1}s_i - P_{\nu_i,\nu_{i+1}}(x_i,x_{i+1})\right)$$

and

$$\tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu} = \\ (x_i-x_{i+2})^{-1}P_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})\left(P_{\nu_{i+1},\nu_i}(x_{i+1},x_i)s_{i+1}s_is_{i+1} - P_{\nu_{i+1},\nu_i}(x_{i+1},x_{i+2})\right)$$

hence

$$\tau'_{i+1,s_is_{i+1}(\nu)}\tau'_{i,s_{i+1}(\nu)}\tau'_{i+1,\nu} - \tau'_{i,s_{i+1}s_i(\nu)}\tau'_{i+1,s_i(\nu)}\tau'_{i,\nu} = \\ (x_i-x_{i+2})^{-1}\left(P_{\nu_{i+1},\nu_i}(x_{i+1},x_i)P_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - P_{\nu_{i+1},\nu_i}(x_{i+1},x_{i+2})P_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})\right).$$

The other relations are immediate to check.

Let B be the k-subalgebra of $\mathcal{O}\otimes_{k^{(I)}[x]^{\otimes n}} A_n(I)$ image of the morphism. We have $\mathcal{O}\otimes_{k^{(I)}[x]^{\otimes n}} B = \mathcal{O}\otimes_{k^{(I)}[x]^{\otimes n}} A_n(I)$. The image of S is a basis of $\mathcal{O}\otimes_{k^{(I)}[x]^{\otimes n}} A_n(I)$ over k. It follows that the canonical map $H_n(Q)\to B$ is an isomorphism and that S is a basis of $H_n(Q)$ over k. $\square$

Remark 3.13. Consider the case of a matrix P with non-vanishing diagonal entries which we assume to be symmetric polynomials, and define Q as before, so that its diagonal coefficients are not all 0. The algebra $H_n(Q)$ can be defined as before and Proposition 3.12 extends to this setting, [1] where we need to add $P_{\nu_i,\nu_i}(x_i,x_{i+1})$ to the image of $\tau_{i,\nu}$ when $\nu_i=\nu_{i+1}$. This shows that

[1] check

the algebra $H_n(Q)$ satisfies PBW when $Q_{i,i}$ is a square of a symmetric polynomial. Suitable flat base change should allow to conclude it holds more generall when $Q_{i,i}$ is a symmetric polynomial, via an extension of the theory to the case where the algebra $k[x_1,\ldots,x_n]$ is replaced by a suitable algebraic extension. On the other hand, since $\tau_{i,\nu}^3=\tau_{i,\nu}Q_{\nu_i,\nu_i}(x_i,x_{i+1})=Q_{\nu_i,\nu_i}(x_i,x_{i+1})\tau_{i,\nu}$ when $\nu_i=\nu_{i+1}$, the PBW property implies that Q_{ν_i,ν_i} is a symmetric polynomial. When $|I|=1$ and $Q=1$, we obtain the degenerate affine Hecke algebras. [2].

3.2.3. *Cartan matrices.* Let $C=(a_{ij})$ be a Cartan matrix, *i.e.*,

- $a_{ii}=2$,
- $a_{ij}\in\mathbf{Z}_{\le 0}$ for $i\neq j$ and
- $a_{ij}=0$ if and only if $a_{ji}=0$.

We put $m_{ij}=-a_{ij}$. Let $\{t_{i,j,r,s}\}$ be a family of indeterminates with $i\neq j\in I$, $0\le r<m_{ij}$ and $0\le s<m_{ji}$ and such that $t_{j,i,s,r}=t_{i,j,r,s}$. Let $\{t_{ij}\}_{i\neq j}$ be a family of indeterminates with $t_{ij}=t_{ji}$ if $a_{ij}=0$.

Let $\mathbf{k}=\mathbf{k}_C=\mathbf{Z}[\{t_{i,j,r,s}\}\cup\{t_{ij}^{\pm 1}\}]$. Let $Q_{ii}=0$, $Q_{ij}=t_{ij}$ if $a_{ij}=0$ and

$$Q_{ij}=t_{ij}u^{m_{ij}}+\sum_{0\le r<m_{ij},0\le s<m_{ji}}t_{i,j,r,s}u^rv^s+t_{ji}v^{m_{ji}}\text{ for } i\neq j \text{ and } a_{ij}\neq 0.$$

We put $\tilde{H}_n(C)=H_n(Q)$. This is a $\mathbf{k}$-algebra, free as a $\mathbf{k}$-module.

Consider $s\neq t\in I$ and assume $n=m_{st}+2$. Let $\nu=(t,s,\ldots,s)\in I^n$. Given $0\le i\le n-1$, let $c_i=s_i\cdots s_1$: we have $c_i(\nu)=(s,\ldots,s,t,s,\ldots,s)$, where t is in the $(i+1)$-th position. The canonical isomorphisms ${}^0H_i\xrightarrow{\sim}1_{(s,\ldots,s)}H_i(Q)1_{(s,\ldots,s)}$ and ${}^0H_{n-i-1}\xrightarrow{\sim}1_{(s,\ldots,s)}H_{n-i-1}(Q)1_{(s,\ldots,s)}$ give rise to a morphism of unitary algebras

$${}^0H_i\otimes{}^0H_{n-1-i}\to 1_{c_i(\nu)}H_n(Q)1_{c_i(\nu)}.$$

We denote by e_{i+1} the image of $b_i\otimes b_{n-1-i}$ (cf §3.1.6).

The following Lemma generalizes a result of Khovanov and Lauda [KhoLau, Corollary 7].

Lemma 3.14. *Let $P^i=H_n(Q)e_{i+1}$. Define $\alpha_{i,i+1}=e_{i+1}\tau_{n-1}\cdots\tau_{i+2}\tau_{i+1}e_{i+2}$ and $\alpha_{i+1,i}=e_{i+2}\tau_1\tau_2\cdots\tau_{i+1}e_{i+1}$. We have a complex of projective $H_n(Q)$-modules*

$$\begin{tikzcd} 0 \arrow[r] & P^0 \arrow[r, "\alpha_{0,1}"] & P^1 \arrow[l, dotted, bend left, "\alpha'_{1,0}"] \arrow[r] & \cdots \arrow[r] & P^{i-1} \arrow[r, "\alpha_{i-1,i}"] & P^i \arrow[l, dotted, bend left, "\alpha'_{i,i-1}"] \arrow[r, "\alpha_{i,i+1}"] & P^{i+1} \arrow[l, dotted, bend left, "\alpha'_{i+1,i}"] \arrow[r] & \cdots \arrow[r] & P^{n-1} \arrow[r] & 0 \end{tikzcd}$$

which is homotopy equivalent to 0, with splittings given by the maps $\alpha'_{i+1,i}=(-1)^{i+n}t_{st}^{-1}\alpha_{i+1,i}$.

Proof. Note that $b_rb_{r+1}=b_{r+1}$ and $b_{r+1}T_1\cdots T_rb_r=T_1\cdots T_rb_r$, hence $\alpha_{i,i+1}=\tau_{n-1}\cdots\tau_{i+2}\tau_{i+1}e_{i+2}$ and $\alpha_{i+1,i}=e_{i+2}\tau_1\tau_2\cdots\tau_{i+1}$.

We have

$$\alpha_{i-1,i}\alpha_{i,i+1}=e_i\tau_{n-1}\cdots\tau_i\tau_{n-1}\cdots\tau_{i+1}e_{i+2}=e_i\tau_{n-2}\cdots\tau_i\tau_{n-1}\cdots\tau_ie_{i+2}=0.$$

It follows that the maps $\alpha_{i-1,i}$ provide a differential.

We have

$$\alpha_{i,i+1}\alpha_{i+1,i}=\tau_{n-1}\cdots\tau_{i+1}\tau_1\cdots\tau_{i+1}e_{i+1}=\tau_1\cdots\tau_{i-1}\tau_{n-1}\cdots\tau_{i+2}\tau_{i+1}\tau_i\tau_{i+1}e_{i+1}$$

[2]Also affine Hecke version?

and

$$\alpha_{i,i-1}\alpha_{i-1,i} = \tau_1\cdots\tau_i\tau_{n-1}\cdots\tau_i e_{i+1} = \tau_1\cdots\tau_{i-1}\tau_{n-1}\cdots\tau_{i+2}\tau_i\tau_{i+1}\tau_i e_{i+1}.$$

It follows that

$$\alpha_{i,i+1}\alpha_{i+1,i} - \alpha_{i,i-1}\alpha_{i-1,i} = \partial_{s_1\cdots s_{i-1}s_{n-1}\cdots s_{i+2}}\Big((x_{i+2}-x_i)^{-1}\big(Q_{st}(x_{i+2},x_{i+1}) - Q_{st}(x_i,x_{i+1})\big)\Big).$$

Write $Q_{st}(u,v) = \sum_{a,b} q_{ab}u^a v^b$ with $q_{a,b} \in\in \mathbf{Z}$. We have

$$(x_{i+2}-x_i)^{-1}\big(Q_{st}(x_{i+2},x_{i+1}) - Q_{st}(x_i,x_{i+1})\big) = \sum_{a\geq 1, b\geq 0} q_{ab}x_{i+1}^b(x_{i+2}^{a-1} + x_{i+2}^{a-2}x_i + \cdots + x_i^{a-1}),$$

hence

$$\alpha_{i,i+1}\alpha_{i+1,i} - \alpha_{i,i-1}\alpha_{i-1,i} = \sum_{a\geq 1,b\geq 0} q_{ab}x_{i+1}^b \sum_{c=i-1}^{a-n+i+1} \partial_{s_1\cdots s_{i-1}}(x_i^c)\partial_{s_{n-1}\cdots s_{i+2}}(x_{i+2}^{a-c-1}) = (-1)^{n+i}q_{n-2,0}$$

and finally $\alpha_{i,i+1}\alpha'_{i+1,i} + \alpha'_{i,i-1}\alpha_{i-1,i} = 1$. ☐

Assume C is a symmetrizable Cartan matrix, *i.e.*, there is a family $(d_i)_{i\in I}$ of positive integers with $\mathrm{lcm}(\{d_i\}) = 1$ and such that (b_{ij}) is symmetric, for $b_{ij} = d_i a_{ij}$.

Let $\mathbf{k}^\bullet$ be the quotient of $\mathbf{k}$ by the ideal generated by those $t_{i,j,r,s}$ such that $d_i r + d_j s \neq -2b_{ij}$. Let $\tilde{H}_n^\bullet(C) = \mathbf{k}^\bullet \otimes_{\mathbf{k}} \tilde{H}_n(C)$. The algebra $\tilde{H}_n^\bullet(C)$ is graded with $\deg 1_\nu = 0$, $\deg x_{i,\nu} = 2d_{\nu_i}$ and $\deg \tau_{i,\nu} = -b_{\nu_i,\nu_{i+1}}$.

Remark 3.15. The description of the basis S for $\tilde{H}_n^\bullet(C)$ (cf Theorem 3.7) shows that the rank of the sum of the homogeneous components of $1_\nu \tilde{H}_n^\bullet(C)1_\nu$ with degree less than a given integer is finite.

3.2.4. *Quivers with automorphism.* Let Γ be an oriented quiver with a compatible automorphism [Lu, §12.1.1]: this is the data of

- a set $\tilde{I}$ (vertices)
- a set H (edges) and a map with finite fibers $h \mapsto [h]$ from H to the set of two-element subsets of $\tilde{I}$
- maps $s : H \to \tilde{I}$ (source) and $t : H \to \tilde{I}$ (target) such that $\{s(h), t(h)\} = [h]$ for any $h \in H$
- automorphisms $a : \tilde{I} \to \tilde{I}$ and $a : H \to H$ such that $s(a(h)) = a(s(h))$ and $t(a(h)) = a(t(h))$ and such that $s(h)$ and $t(h)$ are not in the same a-orbit for $h \in H$.

We put $I = \tilde{I}/a$. We define $i \cdot i = 2\#(i)$ and $i \cdot j = -\#\{h \in H | [h] \in i \cup j\}$ for $i \neq j$ in I (note that this uses only the graph structure, not the orientation). This defines a Cartan datum and $\big(2\frac{i\cdot j}{i\cdot i}\big)_{i,j}$ is a symmetrizable Cartan matrix.

Given $i,j \in I$, let d_{ij} be the number of orbits of a in $\{h \in H | s(h) \in i \text{ and } t(h) \in j\}$. We have $d_{ij} + d_{ji} = -2(i\cdot j)/\mathrm{lcm}(i\cdot i, j\cdot j)$ for $i \neq j$.

Define

$$P_{ij}(u,v) = \big(v^{l/(j\cdot j)} - u^{l/(i\cdot i)}\big)^{d_{ij}} \text{ where } l = \mathrm{lcm}(i\cdot i, j\cdot j), \text{ for } i \neq j \text{ and } P_{ii} = 0.$$

We have

$$Q_{ij} = (-1)^{d_{ij}}\big(u^{l/(i\cdot i)} - v^{l/(j\cdot j)}\big)^{-2(i\cdot j)/l} \text{ for } i \neq j.$$

We put $k = \mathbf{Z}$ and $H_n(\Gamma) = H_n(Q)$. This is a specialization of the algebra $\tilde{H}_n(C)$ introduced in §3.2.3.

The algebra $H_n(\Gamma)$ is graded with $\deg 1_\nu = 0$, $\deg x_{i,\nu} = \nu_i \cdot \nu_i$ and $\deg \tau_{i,\nu} = -\nu_i \cdot \nu_{i+1}$. As a graded algebra, it is a specialization of $\tilde{H}_n^\bullet(C)$ (here, $d_i = (i \cdot i)/2$).

Consider another choice of orientation s', t' of the graph $(\tilde{I}, H, h \mapsto [h])$, compatible with the automorphism a. Given $i \neq j$, define

$$\beta_{ij} = \begin{cases} (-1)^{d_{ij}+d'_{ij}} & \text{if } d_{ij} \geq d'_{ij} \\ 1 & \text{otherwise.} \end{cases}$$

We have an isomorphism

$$H_n(\Gamma) \xrightarrow{\sim} H_n(\Gamma'),\ 1_\nu \mapsto 1_\nu,\ x_{i,\nu} \mapsto x_{i,\nu},\ \tau_{i,\nu} \mapsto \beta_{\nu_i,\nu_{i+1}} \tau_{i,\nu}.$$

It follows that, up to isomorphism, the graded algebra $H_n(\Gamma)$ depends only on the Cartan datum. Note nevertheless that the system of isomorphisms constructed above between the algebras corresponding to different orientations is not a transitive system. Consequently, we do not define "the" algebra associated to a Cartan datum (or a graph with automorphism) Note finally that, up to isomorphism, $H_n(\Gamma)$ depends only on the Cartan matrix and a change of Cartan datum corresponds to a rescaling of the grading.

Note that if Γ is the disjoint union of full subquivers Γ_1 and Γ_2, then $H_n(\Gamma) = H_n(\Gamma_1) \otimes H_n(\Gamma_2)$.

3.2.5. *Type A graphs.* Let k be a field and $q \in k^\times$.

Assume first $q = 1$. Given I a subset of k, we denote by I_1 the quiver with set of vertices I and with an arrow $i + 1 \to i$, whenever $i, i + 1 \in I$.

Assume now $q \neq 1$. Given I a subset of $k^\times$, we denote by I_q the quiver with set of vertices I and with an arrow $qi \to q$, whenever $i, qi \in I$.

Note that I_q has type A and we put $\mathfrak{sl}_{I_q} = \mathfrak{g}_{I_q}$. Let us assume I_q is connected. Let us describe the possible type for the underlying graph.

Assume $q = 1$. Type:

- A_n if $|I| = n$ and k has characteristic 0 or $p > n$.
- $\tilde{A}_{p-1}$ if $|I| = p$ is the characteristic of k.
- A_∞ if I is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if I is unbounded in both directions.

Assume $q \neq 1$. Denote by e the multiplicative order of q. Type:

- A_n if $|I| = n < e$.
- $\tilde{A}_{e-1}$ if $|I| = e$.
- A_∞ if I is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if I is unbounded in both directions.

3.2.6. *Idempotents and representations.* Let k be a field Let Γ be a quiver. Given $a \in k$, we denote by $kH_n(\Gamma)\text{-Mod}_a$ the category of $H_n(\Gamma)$-modules M such that $M = \bigoplus_\nu 1_\nu M$ and for every ν, the elements $x_{i,\nu}$ act locally nilpotently on $1_\nu M$ for $1 \leq i \leq n$.

Note that there is an automorphism of $kH_n(\Gamma)$ defined by $\tau_{i,\nu} \mapsto \tau_{i,\nu}$ and $x_{i,\nu} \mapsto x_{i,\nu} + a$. It induces an equivalence between the categories $kH_n(\Gamma)\text{-Mod}_a$ and $kH_n(\Gamma)\text{-Mod}_0$.

Let I be a subset of k and let $\Gamma = I_1$.

Let

$$\bar{\mathcal{O}}' = \bigoplus_{\nu \in I^n} k[X_1, \ldots, X_n][\{(X_i - X_j)^{-1}\}_{i \neq j, \nu_i \neq \nu_j}, \{(X_i - X_j + 1)^{-1}\}_{i \neq j, \nu_i + 1 \neq \nu_j}],$$

a non-unitary ring. Note that this is a subring of

$$\bigoplus_{\nu \in I^n} k[X_1, \ldots, X_n][(X_i - X_j - a)^{-1}]_{i \neq j, a \neq \nu_i - \nu_j}.$$

We denote by 1_ν the unit of the summand of $\bar{\mathcal{O}}'$ corresponding to ν. We put a structure of non-unitary algebra on $\bar{\mathcal{O}}'\bar{H}_n = \bar{\mathcal{O}}' \otimes_{\mathbf{Z}[X_1, \ldots, X_n]} \bar{H}_n$ by setting

$$T_i 1_\nu - 1_{s_i(\nu)} T_i = \begin{cases} (X_{i+1} - X_i)^{-1}(1_\nu - 1_{s_i(\nu)}) & \text{if } \nu_i \neq \nu_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\tilde{\bar{\mathcal{O}}}' = \bigoplus_{\nu \in I^n} k[x_1, \ldots, x_n][\{(\nu_i - \nu_j + x_i - x_j)^{-1}\}_{i \neq j, \nu_i \neq \nu_j}, \{(\nu_i - \nu_j + 1 + x_i - x_j)^{-1}\}_{i \neq j, \nu_i + 1 \neq \nu_j}],$$

a subring of

$$\bigoplus_{\nu \in I^n} k[x_1, \ldots, x_n][(x_i - x_j - a)^{-1}]_{i \neq j, a \neq 0}.$$

From Proposition 3.12 and §3.1.7, we obtain the following proposition.

Proposition 3.16. *We have an isomorphism of non-unitary algebras*

$$\tilde{\bar{\mathcal{O}}}' H_n(\Gamma) \xrightarrow{\sim} \bar{\mathcal{O}}'\bar{H}_n, \ x_i 1_\nu \mapsto (X_i - \nu_i) 1_\nu,$$

$$\tau_i 1_\nu \mapsto \begin{cases} (X_i - X_{i+1} + 1)^{-1}(T_i - 1) 1_\nu & \textit{if } \nu_i = \nu_{i+1} \\ ((X_i - X_{i+1}) T_i + 1) 1_\nu & \textit{if } \nu_i = \nu_{i+1} + 1 \\ \frac{X_i - X_{i+1}}{X_i - X_{i+1} + 1}(T_i - 1) 1_\nu + 1_\nu & \textit{otherwise.} \end{cases}$$

Let M be a $k\bar{H}_n$-module. Given $a \in k^n$, we denote by M_a the $k[X_1, \ldots, X_n]$-submodule of M of elements with support contained in the closed point of $\mathbf{A}^n_k$ given by a.

We denote by $\bar{\mathcal{C}}_\Gamma$ the category of $k\bar{H}_n$-modules M such that

$$M = \bigoplus_{a \in \Gamma^n} M_a.$$

Theorem 3.17. *We have an equivalence of categories*

$$kH_n(\Gamma)\text{-Mod}_0 \xrightarrow{\sim} \bar{\mathcal{C}}_\Gamma, \ M \mapsto M$$

where X_i acts on $1_\nu M$ by $(x_i + \nu_i)$ and T_i acts on $1_\nu M$ by

- $(x_i - x_{i+1} + 1)\tau_i + 1$ *if* $\nu_i = \nu_{i+1}$
- $(x_i - x_{i+1} - 1)^{-1}(\tau_i - 1)$ *if* $\nu_i = \nu_{i+1} + 1$
- $(x_i - x_{i+1} + \nu_{i+1} - \nu_i + 1)(x_i - x_{i+1} + \nu_{i+1} - \nu_i)^{-1}(\tau_i - 1) + 1$ *otherwise.*

Let k be a field that is a $\mathbf{Z}[q^{\pm 1},(q-1)^{-1}]$-algebra. Let I be a subset of $k^\times$ and let $\Gamma=I_q$. Let

$$\mathcal{O}'=\bigoplus_{\nu\in I^n} k[X_1^{\pm 1},\ldots,X_n^{\pm 1}][\{(X_i-X_j)^{-1}\}_{i\neq j,\nu_i\neq\nu_j},\{(qX_i-X_j)^{-1}\}_{i\neq j,q\nu_i\neq\nu_j}],$$

a non-unitary $k[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$-algebra. Note that this is a subring of

$$\bigoplus_{\nu\in I^n} k[X_1^{\pm 1},\ldots,X_n^{\pm 1}][(X_i-aX_j)^{-1}]_{i\neq j,a\in k-\{0,\nu_i\nu_j^{-1}\}}.$$

We denote by 1_ν the unit of the summand of $\mathcal{O}'$ corresponding to ν. We put a structure of non-unitary algebra on $\mathcal{O}'H_n=\mathcal{O}'\otimes_{\mathbf{Z}[q^{\pm 1},X_1^{\pm 1},\ldots,X_n^{\pm 1}]}H_n$ by setting

$$T_i1_\nu-1_{s_i(\nu)}T_i=\begin{cases}(1-q)X_{i+1}(X_i-X_{i+1})^{-1}(1_\nu-1_{s_i(\nu)}) & \text{if } \nu_i\neq\nu_{i+1}\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$\tilde{\mathcal{O}}'=\bigoplus_{\nu\in I^n} k[x_1^{\pm 1},\ldots,x_n^{\pm 1}][\{(\nu_i\nu_j^{-1}x_i-x_j)^{-1}\}_{i\neq j,\nu_i\neq\nu_j},\{(q\nu_i\nu_j^{-1}x_i-x_j)^{-1}\}_{i\neq j,q\nu_i\neq\nu_j}],$$

a subring of

$$\bigoplus_{\nu\in\mathbf{Z}^n} k[q^{\pm 1},x_1^{\pm 1},\ldots,x_n^{\pm 1}][(x_i-ax_j)^{-1}]_{i\neq j,a\in k-\{0,1\}}.$$

From Proposition 3.12 and §3.1.7, we obtain the following proposition.

Proposition 3.18. *We have an isomorphism of non-unitary algebras*

$$\tilde{\mathcal{O}}'H_n(\Gamma)\xrightarrow{\sim}\mathcal{O}'H_n,\ x_i1_\nu\mapsto\nu_i^{-1}X_i1_\nu,$$

$$\tau_i1_\nu\mapsto\begin{cases}\nu_i(qX_i-X_{i+1})^{-1}(T_i-q)1_\nu & \textit{if } \nu_i=\nu_{i+1}\\ \nu_i^{-1}((X_i-X_{i+1})T_i+(q-1)X_{i+1}))1_\nu & \textit{if } \nu_i=q\nu_{i+1}\\ (\frac{X_i-X_{i+1}}{qX_i-X_{i+1}}(T_i-q)+1)1_\nu & \textit{otherwise.}\end{cases}$$

Let M be a kH_n-module. Given $a\in(k^\times)^n$, we denote by M_a the $k[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$-submodule of M of elements with support contained in the closed point of $\mathbf{A}_k^n$ given by a.

We denote by $\mathcal{C}_\Gamma$ the category of kH_n-modules M such that

$$M=\bigoplus_{a\in\Gamma^n}M_a.$$

Theorem 3.19. *We have an equivalence of categories*

$$kH_n(\Gamma)\text{-Mod}_1\xrightarrow{\sim}\mathcal{C}_\Gamma,\ M\mapsto M$$

where X_i acts on $1_\nu M$ by ν_ix_i and T_i acts on $1_\nu M$ by

- $(qx_i-x_{i+1})\tau_i+q$ *if* $\nu_i=\nu_{i+1}$
- $(q^{-1}x_i-x_{i+1})^{-1}(\tau_i+(1-q)x_{i+1})$ *if* $\nu_i=q\nu_{i+1}$
- $(\nu_ix_i-\nu_{i+1}x_{i+1})^{-1}\left((q\nu_ix_i-\nu_{i+1}x_{i+1})\tau_i+(1-q)\nu_{i+1}x_{i+1}\right)$ *otherwise.*

Remark 3.20. The equivalences in Theorems 3.17 and 3.19 restrict to equivalences between full subcategories for which $k[X_1,\ldots,X_n]$ (resp. $k[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$) act through a specific quotient and $k[x_1,\ldots,x_n]$ acts through the corresponding quotient. This provides a realization of (possibly degenerate) cyclotomic Hecke algebras in terms of $H_n(\Gamma)$. This has been studied independently in detail by Brundan and Kleshchev [BrKl]. The apparition of affine Hecke algebras in relation with $\hat{\mathfrak{sl}}_p$-categorifications goes back to [Gro].

4. 2-categories

4.1. **Construction.**

4.1.1. *Half Kac-Moody algebras.* Let I be a set and $C=(a_{ij})_{i,j\in I}$ a Cartan matrix. We consider the ring $\mathbf{k}$ and the matrix Q of §3.2.3.

Define $\mathcal{B}=\mathcal{B}(C)$ as the free strict monoidal $\mathbf{k}$-linear category with a unit generated by objects E_s for $s\in I$ and by arrows

$$x_s : E_s \to E_s \text{ and } \tau_{st} : E_sE_t \to E_tE_s \text{ for } s,t\in I$$

with relations

(1) $\tau_{st}\circ\tau_{ts}=Q_{st}(E_tx_s,x_tE_s)$

(2) $\tau_{tu}E_s\circ E_t\tau_{su}\circ\tau_{st}E_u-E_u\tau_{st}\circ\tau_{su}E_t\circ E_s\tau_{tu}=\begin{cases}\frac{Q_{st}(x_sE_t,E_sx_t)E_s-E_sQ_{st}(E_tx_s,x_tE_s)}{x_sE_tE_s-E_sE_tx_s}E_s & \text{if } s=u\\ 0 & \text{otherwise.}\end{cases}$

(3) $\tau_{st}\circ x_sE_t-E_sx_t\circ\tau_{st}=\delta_{st}$

(4) $\tau_{st}\circ E_sx_t-x_sE_t\circ\tau_{st}=-\delta_{st}$

These relations state that the maps x_s and τ_{st} give an action of the nil affine Hecke algebra associated with C on powers of E. More precisely, we have an isomorphism of (non-unitary) algebras

$$\begin{aligned}\tilde{H}_n(C) &\xrightarrow{\sim} \bigoplus_{\nu,\nu'\in I^n}\mathrm{Hom}_{\mathcal{B}}(E_{\nu_n}\cdots E_{\nu_1},E_{\nu'_n}\cdots E_{\nu'_1})\\ 1_\nu &\mapsto \mathrm{id}_{E_{\nu_n}\cdots E_{\nu_1}}\\ x_{i,\nu} &\mapsto E_{\nu_n}\cdots E_{\nu_{i+1}}x_{\nu_i}E_{\nu_{i-1}}\cdots E_{\nu_1}\\ \tau_{i,\nu} &\mapsto E_{\nu_n}\cdots E_{\nu_{i+2}}\tau_{\nu_{i+1},\nu_i}E_{\nu_{i-1}}\cdots E_{\nu_1}\end{aligned}$$

Let $s\in I$ and $n\geq 0$. We have an isomorphism of algebras $\mathbf{k}({}^0H_n)\xrightarrow{\sim}\mathrm{End}_{\mathcal{B}_0}(E_s^n)$ and we denote by $E_s^{(n)}=b_nE_s^n\in\mathcal{B}^i$ the image of the idempotent $b_n=T_{w[1,n]}X_1^{n-1}X_2^{n-2}\cdots X_{n-1}$ of ${}^0H_n$ (cf §3.1.6). We denote also by $F_s^{(n)}$ the image of $T_{w[1,n]}X_1^{n-1}X_2^{n-2}\cdots X_{n-1}\in{}^0H_n^{\mathrm{opp}}$. Note that this idempotent corresponds to the idempotent $b'_n=X_1^{n-1}X_2^{n-2}\cdots X_{n-1}T_{w[1,n]}$ of ${}^0H_n$. Thanks to Lemma 3.4, we have the following result (as in [ChRou, Lemma 5.15]).

Lemma 4.1. *The action map is an isomorphism ${}^0H_nb_n\otimes_{P_n^{\mathfrak{S}_n}}E_s^{(n)}\xrightarrow{\sim}E_s^n$. In particular, we have $E_s^n\simeq n!\cdot E_s^{(n)}$. Similarly, we have isomorphisms $b'_n\cdot{}^0H_n\otimes_{P_n^{\mathfrak{S}_n}}F_s^{(n)}\xrightarrow{\sim}F_s^n$. In particular, we have $F_s^n\simeq n!\cdot F_s^{(n)}$.*

The following Proposition is a consequence of Lemma 3.14 (apply $\mathrm{Hom}_{\tilde{H}_n(C)}(P_\bullet,-)$). It gives a categorical version of the Serre relations.

Proposition 4.2. *Consider $s \neq t \in I$ and let $m = m_{st}$. Let $\alpha_{i,i+1} = \tau_{m+1} \cdots \tau_{i+2}\tau_{i+1}$ and $\alpha'_{i+1,i} = (-1)^{i+m} t_{st}^{-1} \tau_1\tau_2 \cdots \tau_{i+1}$. We have a complex*

$$\begin{tikzcd} \cdots \arrow[r] & E_s^{(m-i+2)}E_tE_s^{(i-1)} \arrow[r, "\alpha'_{i-1,i}"] & E_s^{(m-i+1)}E_tE_s^{(i)} \arrow[l, dotted, bend left, "\alpha_{i,i-1}"] \arrow[r, "\alpha'_{i,i+1}"] & E_s^{(m-i)}E_tE_s^{(i+1)} \arrow[l, dotted, bend left, "\alpha_{i+1,i}"] \arrow[r] & \cdots \end{tikzcd}$$

which is homotopy equivalent to 0, with splittings given by the maps $\alpha_{i+1,i}$. In particular,

$$\bigoplus_{i \text{ even}} E_s^{(m-i+1)}E_tE_s^{(i)} \simeq \bigoplus_{i \text{ odd}} E_s^{(m-i+1)}E_tE_s^{(i)}.$$

Remark 4.3. The first part of Proposition 4.2 generalizes [KhoLau]. We will give a different proof of the existence of an isomorphism (second part of the Proposition) in a sequel in the case of integrable 2-representations.

Assume now C is symmetrizable and consider (d_i), (b_{ij}) and $\mathbf{k}^\bullet$ as in §3.2.3. We put $\mathcal{B}_0^\bullet = \mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^\bullet$.

The category $\mathcal{B}_0^\bullet$ can be enriched in graded abelian groups by setting $\deg x_s = 2d_s$ and $\deg \tau_{st} = -b_{st}$. We denote by $\mathcal{B}^\bullet$ the corresponding graded category. It follows from Theorem 3.7 and Remark 3.15 that Hom-spaces in $\mathcal{B}^\bullet$ are free $\mathbf{k}^\bullet$-modules of finite rank.

We put $E_s^{(n)} = b_n E_s^n(\frac{n(n-1)}{2}d_s)$. Note that $P_n(\frac{n(n-1)}{2}d_s)$ is self-dual as a graded $P_n^{\mathfrak{S}_n}$-module and we have

$$E_s^n \simeq v^{n(n-1)d_s/2}[n]_s! E_s^{(n)}$$

where $[n]_s! = [n]!(v^{d_s})$.

The maps α_{ij} and α'_{ij} of Proposition 4.2 are graded and the proposition remains true in $\mathcal{B}^\bullet$.

Consider finally Γ a quiver with a compatible automorphism and consider the specialization $\mathbf{k}^\bullet \to \mathbf{Z}$ of §3.2.4. We put $\mathcal{B}_{\mathbf{Z}}^\bullet(\Gamma) = \mathcal{B}^\bullet(C) \otimes_{\mathbf{k}^\bullet} \mathbf{Z}$.

4.1.2. *Symmetrizable Kac-Moody algebras.* Let $(I, \cdot)$ be a finite set and a symmetric bilinear pairing on $\mathbf{Z}I$ giving a Cartan datum, *i.e.*, satisfying

- $i \cdot i \in 2\mathbf{Z}_{>0}$
- $2\frac{i \cdot j}{i \cdot i} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$.

We put $a_{ij} = 2\frac{i \cdot j}{i \cdot i}$ and $m(i,j) = -a_{ij}$. The matrix (a_{ij}) is a symmetrizable Cartan matrix.

Let $(X, Y, \langle -, - \rangle, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be a root datum of type $(I, \cdot)$ [Lu, §2.2.1], *i.e.*

- X and Y are finitely generated free abelian groups and $\langle -, - \rangle : Y \times X \to \mathbf{Z}$ is a perfect pairing
- $I \to X,\ i \mapsto \alpha_i$ and $I \to Y,\ i \mapsto \alpha_i^\vee$ are embeddings and $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Associated with this data, there is a Kac-Moody algebra $\mathfrak{g}$, a quantum group $U_v(\mathfrak{g})$, as well as completed versions [Lu]. Let us recall those we will need.

Consider the $\mathbf{Q}(v)$-algebra $'U_v^+(\mathfrak{g})$ generated by elements e_i for $i \in I$ with relations

$$\sum_{a+b=1-a_{ij}} (-1)^a e_i^{(a)} e_j e_i^{(b)} = 0 \tag{5}$$

for any $i \neq j \in I$, where $e_i^{(a)} = \frac{e_i^a}{a!}$. We denote by $U_v^+(\mathfrak{g})$ the $\mathbf{Z}[v^{\pm 1}]$-subalgebra generated by the $E_i^{(a)}$ for $i \in I$ and $a \geq 0$. We define an algebra $U_v^-(\mathfrak{g})$ isomorphic to $U_v^+(\mathfrak{g})$ with E_i replaced by F_i.

Let $'U_v(\mathfrak{g})$ be the category enriched in $\mathbf{Q}(v)$-vector spaces with set of objects X and morphisms generated by $e_i : \lambda \to \lambda + \alpha_i$ and $f_i : \lambda \to \lambda - \alpha_i$ subject to the following relations:

- the relation (5) and its version with e_r replaced by f_r
- $[e_i, f_j] = 0$ if $i \neq j$
- $[e_i, f_i]1_\lambda = \langle \alpha_i^\vee, \lambda\rangle 1_\lambda$.

Let $U_v(\mathfrak{g})$ be the subcategory enriched in $\mathbf{Z}[v^{\pm 1}]$-modules of $'U_v(\mathfrak{g})$ with same objects as $'U_v(\mathfrak{g})$ and with morphisms generated by $e_i^{(r)}$ and $f_i^{(r)}$ for $i \in I$ and $r \geq 0$.

We put $U_1(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathbf{Z}[v^{\pm 1}]} \mathbf{Z}[v^{\pm 1}]/(v-1)$, etc.

Note that $\bigoplus_{\lambda,\mu\in X} \mathrm{Hom}_{U_v(\mathfrak{g})}(\lambda,\mu)$ is the non-unitary ring ${}_{\mathcal{A}}\dot{\mathbf{U}}$ of [Lu, §23.2]. The category of functors (compatible with the $\mathbf{Z}[v^{\pm 1}]$-structure) $U_v(\mathfrak{g}) \to \mathbf{Z}[v^{\pm 1}]$-Mod is equivalent to the category of unital ${}_{\mathcal{A}}\dot{\mathbf{U}}$-modules via $V \mapsto \bigoplus_\lambda V(\lambda)$.

4.1.3. 2-*Kac Moody algebras.* Let $\mathcal{B}_1$ be the strict monoidal $\mathbf{k}$-linear category obtained from $\mathcal{B}$ by adding F_s right dual to E_s for every $s \in I$. Define

$$\varepsilon_s = \varepsilon_{E_s} : E_sF_s \to \mathbf{1} \text{ and } \eta_s = \eta_{E_s} : \mathbf{1} \to F_sE_s.$$

The dual pairs (E_s, F_s) provides dual pairs (E_s^n, F_s^n) and the action of ${}^0H_n$ on E_s^n induces an action of $({}^0H_n)^{\mathrm{opp}}$ on F_s^n. We denote by x_s the endomorphism of F_s induced by $x_s \in \mathrm{End}(E_s)$ and denote also by $\tau_{st} : F_sF_t \to F_tF_s$ the morphism induced by $\tau_{st} \in \mathrm{Hom}(E_sE_t, E_tE_s)$.

We define a morphism of monoids

$$h : \mathrm{Ob}(\mathcal{B}_1) \to X,\ E_s \mapsto \alpha_s,\ F_s \mapsto -\alpha_s.$$

Consider the strict 2-category $\mathfrak{A}_1$ with set of objects X and $\mathcal{H}om(\lambda,\lambda') = h^{-1}(\lambda'-\lambda)$, a full subcategory of $\mathcal{B}_1$. We write $E_{s,\lambda}$ for $E_s\mathbf{1}_\lambda$, $\varepsilon_{s,\lambda}$ for $\varepsilon_{s,\lambda}\mathbf{1}_\lambda$, etc.

Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{g})$ be the $\mathbf{k}$-linear strict 2-category deduced from $\mathfrak{A}_1$ by inverting the following 2-arrows:

- when $\langle \alpha_s^\vee, \lambda\rangle \geq 0$,

$$\rho_{s,\lambda} = \sigma_{ss} + \sum_{i=0}^{\langle \alpha_s^\vee,\lambda\rangle - 1} \varepsilon_s \circ (x_s^i F_s) : E_sF_s\mathbf{1}_\lambda \to F_sE_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{\langle \alpha_s^\vee,\lambda\rangle}$$

- when $\langle \alpha_s^\vee, \lambda\rangle \leq 0$,

$$\rho_{s,\lambda} = \sigma_{ss} + \sum_{i=0}^{-1-\langle \alpha_s^\vee,\lambda\rangle} (F_s x_s^i) \circ \eta_s : E_sF_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{-\langle \alpha_s^\vee,\lambda\rangle} \to F_sE_s\mathbf{1}_\lambda$$

- $\sigma_{st} : E_sF_t\mathbf{1}_\lambda \to F_tE_s\mathbf{1}_\lambda$ for all $s \neq t$ and all λ

where we define

$$\sigma_{st} = (F_tE_s\varepsilon_t) \circ (F_t\tau_{ts}F_s) \circ (\eta_tE_sF_t) : E_sF_t \to F_tE_s.$$

Remark 4.4. The inversion of maps in the definition of $\mathfrak{A}$ accounts for the Lie algebra relations $[e_s, f_s] = h_s$ and $[e_s, f_t] = 0$ for $s \neq t$. The elements h_ζ for $\zeta \in Y$ appear only through their action as multiplication by $\langle \zeta, \lambda\rangle$ on the λ-weight space.

We proceed now as in §4.1.1 to define graded versions. Let $\mathfrak{A}_0^\bullet = \mathfrak{A} \otimes_{\mathbf{k}} \mathbf{k}^\bullet$. The category $\mathfrak{A}_0^\bullet$ can be enriched in graded abelian groups by setting

$$\deg \varepsilon_{s,\lambda} = d_s(1 - \langle \alpha_s^\vee, \lambda\rangle) \text{ and } \deg \eta_{s,\lambda} = d_s(1 + \langle \alpha_s^\vee, \lambda\rangle).$$

We denote by $\mathfrak{A}^\bullet$ the corresponding graded 2-category.

Note that σ_{st} is a graded map (for all $s, t \in I$), while $\rho_{s,\lambda}$ carries shifts:

$$\rho_{s,\lambda} : E_sF_s\mathbf{1}_\lambda \xrightarrow{\sim} F_sE_s\mathbf{1}_\lambda \oplus \bigoplus_{i=0}^{\langle\alpha_s^\vee,\lambda\rangle-1} \mathbf{1}_\lambda\big(d_s(2i+1-\langle\alpha_s^\vee,\lambda\rangle)\big) \text{ when } \langle\alpha_s^\vee,\lambda\rangle \geq 0,$$

$$\rho_{s,\lambda} : E_sF_s\mathbf{1}_\lambda \oplus \bigoplus_{i=0}^{-1-\langle\alpha_s^\vee,\lambda\rangle} \mathbf{1}_\lambda\big(-d_s(2i+1+\langle\alpha_s^\vee,\lambda\rangle)\big) \xrightarrow{\sim} F_sE_s\mathbf{1}_\lambda \text{ when } \langle\alpha_s^\vee,\lambda\rangle \leq 0.$$

We have a dual pair in $\mathfrak{A}^\bullet$

$$\Big(E_s\mathbf{1}_\lambda, \mathbf{1}_\lambda F_s\Big(d_s(1+\langle\alpha_s^\vee,\lambda\rangle)\Big)\Big).$$

Finally, given a quiver Γ with a compatible automorphism and associated Cartan matrix C, we put $\mathfrak{A}_{\mathbf{Z}}^\bullet(\Gamma) = \mathfrak{A}^\bullet \otimes_{\mathbf{k}^\bullet} \mathbf{Z}$ (cf §3.2.4). We put also $\mathfrak{A}_{\mathbf{Z}} = \mathfrak{A} \otimes_{\mathbf{k}} \mathbf{Z}$.

Let us summarize: we have constructed several 2-categories with set of objects X and with $\mathrm{Hom}(\lambda, \lambda') = h^{-1}(\lambda'-\lambda)$. Given a root datum, we have a $\mathbf{k}$-linear 2-category $\mathcal{A}$ and a specialization $\mathcal{A}^\bullet$ that is $\mathbf{k}^\bullet$-linear and graded. Given in addition a quiver with compatible automorphism affording the Cartan matrix, we have a further specialization $\mathcal{A}_{\mathbf{Z}}^\bullet$ that is graded and $\mathbf{Z}$-linear.

Remark 4.5. The action of ${}^0H_n$ on E_s^n is given by

$$X_i \mapsto E_s^{n-i}x_sE_s^{i-1} \text{ and } T_i \mapsto E_s^{n-i-1}\tau_{ss}E_s^{i-1}$$

while the action of ${}^0H_n^{\mathrm{opp}}$ on F_s^n is given by

$$X_i \mapsto F_s^{i-1}x_sF_s^{n-i} \text{ and } T_i \mapsto F_s^{i-1}\tau_{ss}F_s^{n-i-1}.$$

4.1.4. *Other versions.* We define here categories related to the ones defined in the previous section by removing adding generators or imposing extra symmetry conditions and relations.

We define $\mathcal{B}_1^l$ as the strict monoidal $\mathbf{k}$-linear category obtained from $\mathcal{B}$ by adding F_s left and right adjoint to E_s for every $s \in I$. Define

$$\varepsilon_s^l = \varepsilon_{F_s} : F_s \cdot E_s \to \mathbf{1}, \text{ and } \eta_s^l = \eta_{F_s} : \mathbf{1} \to E_s \cdot F_s.$$

Define also $\mathfrak{A}_1^l$ and $\mathfrak{A}^l$ as $\mathfrak{A}_1$ and $\mathfrak{A}$ were defined from $\mathcal{B}_1$. Now, we define $\mathfrak{A}'$ as the $\mathbf{k}$-linear strict 2-category obtained from $\mathfrak{A}_2$ by adding the relations

(1) when $\langle\alpha_s^\vee,\lambda\rangle \geq 0$, the composition

$$F_sE_s\mathbf{1}_\lambda \xrightarrow{(\mathrm{id},0,\ldots,0)} F_sE_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{\oplus\langle\alpha_s^\vee,\lambda\rangle} \xrightarrow{\rho_{s,\lambda}^{-1}} E_sF_s\mathbf{1}_\lambda \xrightarrow{(-1)^{\langle\alpha_s^\vee,\lambda\rangle+1}(X^{\langle\alpha_s^\vee,\lambda\rangle}F_s)} E_sF_s\mathbf{1}_\lambda \xrightarrow{\varepsilon_s} \mathbf{1}_\lambda$$

is equal to ε_s^l [3]

[3]We should replace x_s by $-x_s$ in nil Hecke of graphs to get rid of the sign here

(2) when $\langle\alpha_s^\vee,\lambda\rangle \le 0$, the composition

$$E_sF_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{-\langle\alpha_s^\vee,\lambda\rangle} \xrightarrow[\sim]{\rho_{s,\lambda}} F_sE_s\mathbf{1}_\lambda \xrightarrow{\varepsilon_s^l} \mathbf{1}_\lambda$$

is equal to $(0,\dots,0,(-1)^{1+\langle\alpha_s^\vee,\lambda\rangle})$.

Remark 4.6. The last two relations show that ε_s^l can be expressed in terms of the maps x_s, $\tau_{s,s}$, η_s and ε_s. As a consequence, the adjunction (F_s,E_s) is determined by the adjunction (E_s,F_s) and the maps x_s and $\tau_{s,s}$.

We define specializations of $\mathfrak{A}'$ in the same way as those defined for $\mathfrak{A}$. Note that

$$\deg\eta_{s,\lambda}^l = d_s(1-\langle\alpha_s^\vee,\lambda\rangle) \text{ and } \deg\varepsilon_{s,\lambda}^l = d_s(1+\langle\alpha_s^\vee,\lambda\rangle)$$

and we have a dual pair in $\mathfrak{A}'^\bullet$

$$\Big(\mathbf{1}_\lambda F_s\Big(-d_s(1+\langle\alpha_s^\vee,\lambda\rangle)\Big), E_s\mathbf{1}_\lambda\Big).$$

We define $\bar{\mathfrak{A}}'$ to be the largest monoidal additive category quotient[4] of $\mathfrak{A}'$ that is strictly sovereign and such that

- relation (2) holds when $\langle\alpha_s^\vee,\lambda\rangle = -1$
- when $\langle\alpha_s^\vee,\lambda\rangle \ge 0$, the composition

$$F_sE_s\mathbf{1}_\lambda \xrightarrow{(\mathrm{id},0,\dots,0)} F_sE_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{\oplus\langle\alpha_s^\vee,\lambda\rangle} \xrightarrow{\rho_{s,\lambda}^{-1}} E_sF_s\mathbf{1}_\lambda \xrightarrow{(-1)^{\langle\alpha_s^\vee,\lambda\rangle+1}(X^{\langle\alpha_s^\vee,\lambda\rangle}F_s)} E_sF_s\mathbf{1}_\lambda \xrightarrow{\varepsilon_s^r} \mathbf{1}_\lambda$$

  is equal to ε_s^l
- when $\langle\alpha_s^\vee,\lambda\rangle \le 0$, the composition

$$\mathbf{1}_\lambda \xrightarrow{\eta_s^r} F_sE_s\mathbf{1}_\lambda \xrightarrow{(-1)^{\langle\alpha_s^\vee,\lambda\rangle}(FX^{-\langle\alpha_s^\vee,\lambda\rangle})} F_sE_s\mathbf{1}_\lambda \xrightarrow{\rho_{s,\lambda}^{-1}} E_sF_s\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{\oplus-\langle\alpha_s^\vee,\lambda\rangle} \xrightarrow{(\mathrm{id},0,\dots,0)} E_sF_s\mathbf{1}_\lambda$$

  is equal to η_s^l

Remark 4.7. One can also also consider the category $\mathfrak{A}^l$ as well as the largest additive monoidal category quotient of $\mathfrak{A}^l$ that is strictly sovereign.

There are canonical strict 2-functors

$$\mathfrak{A} \to \mathfrak{A}' \to \bar{\mathfrak{A}}'.$$

4.1.5. *Completion.* Consider $\mathcal{A}_{\mathcal{A}}^\vee = 2\lim_I \mathcal{A}/(\mathcal{A}\mathbf{1}_\lambda\mathcal{A})_{\lambda\in I}$, where λ runs over ideals in the poset X.

The various quotients are integrable representations of $\mathcal{A}$. So, on them, we know the Θ_s are invertible and "the relations $[E_i,F_j]=0$ are automatic for $i\neq j$".

4.2. **Properties.**

[4]Needs to be changed to 2-category!

4.2.1. *Symmetries.* The map σ_{st} can be defined using the Hecke action on F^2 instead of E^2:

Lemma 4.8. *Given $s,t \in I$, we have $\sigma_{st} = (E_sF_t \xrightarrow{E_sF_t\eta_s^r} E_sF_tF_sE_s \xrightarrow{E_s\tau_{ts}E_s} E_sF_sF_tE_s \xrightarrow{\varepsilon_s^rF_tE_s} F_tE_s)$.*

Proof. The lemma follows from the commutativity of the following diagram

$$\begin{tikzcd}
E_sF_t \arrow[r, "\bullet\eta"] \arrow[d, "\eta\bullet"'] \arrow[dd, bend right, "\eta\bullet"'] & E_sF_tF_sE_s \arrow[r, "E_s\tau_{ts}E_s"] & E_sF_sF_tE_s \arrow[r, "\varepsilon\bullet"] & F_tE_s \\
F_tE_tE_sF_t \arrow[r, "\bullet\eta"] & F_tE_tE_sF_tF_sE_s \arrow[r, "\bullet\tau_{ts}E_s"] & F_tE_tE_sF_sF_tE_s \arrow[r, "F_tE_t\varepsilon F_tE_s"] & F_tE_tF_tE_s \arrow[u, "F_t\varepsilon E_s"'] \\
F_tE_tE_sF_t \arrow[r, "\bullet\eta"] \arrow[drr, "F_t\tau_{ts}F_t"'] & F_tE_tE_sF_tF_sE_s \arrow[r, "F_t\tau_{ts}\bullet"] & F_tE_sE_tF_tF_sE_s \arrow[r, "F_tE_s\varepsilon F_sE_s"] & F_tE_sF_sE_s \arrow[uu, bend right, "F_t\varepsilon E_s"'] \\
 & & F_tE_sE_tF_t \arrow[uuur, bend right=60, "\bullet\varepsilon"'] &
\end{tikzcd}$$

□

We define the *Chevalley involution*, a strict equivalence of 2-categories $I : \mathfrak{A}^{\mathrm{opp}} \xrightarrow{\sim} \mathfrak{A}$ satisfying $I^2 = \mathrm{Id}$ by

$$I(\mathbf{1}_\lambda) = \mathbf{1}_{-\lambda},\ I(E_s) = F_s,\ I(\varepsilon_s^r) = \eta_s^r,\ I(\tau_{st}) = \tau_{st} \text{ and } I(x_s) = x_s.$$

Note that $I(\sigma_{st}) = \sigma_{st}$ (Lemma 4.8) and $I(\rho_{s,\lambda}) = \rho_{s,-\lambda}$. [5]

We define the *Chevalley duality*, a strict equivalence of 2-categories $D : \mathfrak{A}^{\mathrm{rev}} \xrightarrow{\sim} \mathfrak{A}$ satisfying $D^2 = \mathrm{Id}$ by

$$\mathbf{1}_\lambda \mapsto \mathbf{1}_\lambda,\ E_s \mapsto F_s,\ x_s \mapsto x_s, \tau_{st} \mapsto \tau_{st}, \varepsilon_s^r \mapsto \varepsilon_s^r,\ \eta_s^r \mapsto \eta_s^r.$$

Note that D fixes σ_{st} (Lemma 4.8) and $\rho_{s,\lambda}$. [6].

There is also a strict equivalence of monoidal categories

$$\tilde{\mathcal{B}}'^{\mathrm{rev}} \xrightarrow{\sim} \tilde{\mathcal{B}}',\ E_s \mapsto E_s,\ x_s \mapsto x_s,\ \tau_{st} \mapsto -\tau_{ts}.$$

4.2.2. *Relations in $\mathfrak{sl}_2$.* We provide isomorphisms between sums of objects of type $E_s^nF_s^n$ and sum of objects of type $F_s^nE_s^n$.

In this section, we work in the category $\mathcal{A}$ associated with $\mathfrak{g} = \mathfrak{sl}_2$: $I = \{s\}$ with $s \cdot s = 2$, $X = Y = \mathbf{Z}$, $\alpha_s^\vee = 1$ and $\alpha_s = 2$.

We put $E = E_s$ and $F = F_s$. We put $\varepsilon = \varepsilon_s^r$ and $\eta = \eta_s^r$. Let $i \in \mathbf{Z}_{\geq 0}$. We define by induction $\varepsilon_m : E^mF^m \to \mathbf{1}$ and $\eta_m : \mathbf{1} \to F^mE^m$ in $\mathcal{B}_1$. We put $\varepsilon_0 = \eta_0 = \mathrm{id}$ and $\varepsilon_m = \varepsilon_{m-1} \circ (E^{m-1}\varepsilon F^{m-1})$ and $\eta_m = (F^{m-1}\eta E^{m-1}) \circ \eta_{m-1}$.

Given $a, b \in \mathbf{Z}_{\geq 0}$, we denote by $\mathcal{P}(a,b)$ the set of partitions with at most a non-zero parts, all of which are at most b. Given $\mu = (\mu_1 \geq \cdots \mu_a \geq 0) \in \mathcal{P}(a,b)$, we denote by $m_\mu(X_1,\ldots,X_a) = \sum_\sigma X_1^{\mu_{\sigma(1)}} \cdots X_a^{\mu_{\sigma(a)}}$ the corresponding monomial symmetric function (here, σ runs over $\mathfrak{S}_a$ modulo the stabilizer of μ).

[5]What about η^l?

[6]What about η^l?

Let $m,n,i\in\mathbf{Z}_{\geq 0}$ with $i\leq m$ and $i\leq n$ and let $\lambda\in X$. Let $r=m-n+\lambda$. Assume $r<0$. We put

$$L(m,n,i,\lambda)=\bigoplus_{\substack{w\in\mathfrak{S}_m^{m-i}\\ w'\in{}^{n-i}\mathfrak{S}_n\\ \mu\in\mathcal{P}(i,-r-i)}}(T_w\otimes m_\mu(X_{n-i+1},\ldots,X_n)X_{n-i+1}^{i-1}X_{n-i+2}^{i-2}\cdots X_{n-1}T_{w'})\mathbf{Z}\subset{}^0H_m\otimes_{{}^0H_i}{}^0H_n,$$

where the right action (resp. the left action) of ${}^0H_i$ on ${}^0H_m$ (resp. on ${}^0H_n$) is via $X_r\mapsto X_{r+m-i}$ and $T_r\mapsto T_{r+m-i}$ (resp. $X_r\mapsto X_{r+n-i}$ and $T_r\mapsto T_{r+n-i}$). The sum is direct since $T_{w[1,i]}X_1^{i-1}\cdots X_{i-1}T_{w[1,i]}\neq 0$ (cf §3.1.6).

Note that $\bigoplus_{\mu\in\mathcal{P}(i,-r-i)}m_\mu(X_1,\ldots,X_i)\mathbf{Z}$ is the subspace of $\mathbf{Z}[X_1,\ldots,X_i]^{\mathfrak{S}_i}$ of symmetric polynomials whose degree in any of the variables is at most $-r-i$. It has dimension $\binom{-r}{i}$. Note that $L(m,n,0,\lambda)=\mathbf{Z}$ and $L(m,n,i,\lambda)=0$ if $i>0$ and $r=0$.

Let $\bar{L}(m,n,i,\lambda)=L(m,n,i,\lambda)({}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}})$, a $(({}^0H^f_m\otimes({}^0H^f_n)^{\mathrm{opp}}),({}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}}))$-subbimodule of ${}^0H_m\otimes_{{}^0H_i}{}^0H_n$.

When needed, we will also consider the modules $L([a,b],[a',b'],i,\lambda)$ and $\bar{L}([a,b],[a',b'],i,\lambda)$ where $1\leq a\leq b\leq m$ and $1\leq a'\leq b'\leq n$, which are defined similarly.

Lemma 4.9. *The multiplication map induces an isomorphism*

$$L(m,n,i,\lambda)\otimes({}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}})\xrightarrow{\sim}\bar{L}(m,n,i,\lambda).$$

The $(({}^0H^f_m\otimes({}^0H^f_n)^{\mathrm{opp}}),({}^0H_{m-i}\otimes({}^0H_{n-i})^{\mathrm{opp}}))$*-subbimodule* $L(m,n,i,\lambda)({}^0H_{m-i}\otimes({}^0H_{n-i})^{\mathrm{opp}})$ *of* ${}^0H_m\otimes_{{}^0H_i}{}^0H_n$ *is projective.*

Proof. The first statements is clear. The $(({}^0H^f_m\otimes({}^0H^f_n)^{\mathrm{opp}}),({}^0H_{m-i}\otimes({}^0H_{n-i})^{\mathrm{opp}}))$-bimodule $L(m,n,i,\lambda)({}^0H_{m-i}\otimes({}^0H_{n-i})^{\mathrm{opp}})$ is isomorphic to $|\mathcal{P}(i,-r-i)|$ copies of

$${}^0H^f_n\mathbf{Z}[X_1,\ldots,X_{m-i}]^{\mathfrak{S}_{m-i}}\otimes\mathbf{Z}[X_1,\ldots,X_{n-i}]^{\mathfrak{S}_{n-i}}X_{n-i+1}^{i-1}X_{n-i+2}^{i-2}\cdots X_{n-1}{}^0H^f_n.$$

On the other hand, ${}^0H_d$ is projective as a $({}^0H^f_d,{}^0H_d)$-bimodule (cf §3.1.6) and the last statement of the lemma follows. ∎

Let $L'(m,n,i,\lambda)=\mathrm{Hom}_{\mathbf{Z}}(L(n,m,i,-\lambda),\mathbf{Z})$ and

$$\bar{L}'(m,n,i,\lambda)=\mathrm{Hom}_{{}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}}}(\bar{L}(n,m,i,-\lambda),{}^0H^f_{m-i}\otimes{}^0H^f_{n-i}).$$

The canonical isomorphism

$$L(n,m,i,-\lambda)\otimes_{\mathbf{Z}}({}^0H^f_{m-i}\otimes{}^0H^f_{n-i})\xrightarrow{\sim}\bar{L}(n,m,i,-\lambda)$$

induces an isomorphism

$$\mathrm{Hom}_{\mathbf{Z}}(L(n,m,i,-\lambda),{}^0H^f_{m-i}\otimes{}^0H^f_{n-i})\xrightarrow{\sim}\bar{L}'(m,n,i,\lambda)$$

and composing with the canonical isomorphism

$$L'(m,n,i,\lambda)\otimes_{\mathbf{Z}}({}^0H^f_{m-i}\otimes{}^0H^f_{n-i})\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{Z}}(L(n,m,i,-\lambda),{}^0H^f_{m-i}\otimes{}^0H^f_{n-i}),$$

we obtain an isomorphism of right $({}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}})$-modules

$$L'(m,n,i,\lambda)\otimes_{\mathbf{Z}}({}^0H^f_{m-i}\otimes{}^0H^f_{n-i})\xrightarrow{\sim}\bar{L}'(m,n,i,\lambda).$$

Given $m,n\in\mathbf{Z}_{\geq 0}$, we define by induction a map $\sigma_{m,n}:E^mF^n\to F^nE^m$. The maps $\sigma_{m,0}$ and $\sigma_{0,n}$ are identities. We put $\sigma_{m,1}=(\sigma E^{m-1})\circ(E\sigma_{m-1,1})$ and $\sigma_{m,n}=(F^{n-1}\sigma_{m,1})\circ(\sigma_{m,n-1}F)$.

Lemma 4.10. *The map $\sigma_{m,n}$ is a morphism of $(H_m^f \otimes (H_n^f)^{\mathrm{opp}})$-modules. We have*

$$\sigma_{m,1} = (E^mF \xrightarrow{\eta E^mF} FE^{m+1}F \xrightarrow{F(T_1\cdots T_m)F} FE^{m+1}F \xrightarrow{FE^m\varepsilon} FE^m)$$

and

$$\sigma_{1,n} = (EF^n \xrightarrow{EF^n\eta} EF^{m+1}E \xrightarrow{E(T_1\cdots T_n)E} EF^{n+1}E \xrightarrow{\varepsilon F^nE} F^nE).$$

Given $a, b \in \mathbf{Z}_{\geq 0}$, we have a commutative diagram

$$\begin{tikzcd}
E^aF^b \arrow[r, "\eta\bullet"] \arrow[d, "\sigma_{a,b}"'] & FE^{a+1}F^b \arrow[r, "F\sigma_{a+1,b}"] & F^{b+1}E^{a+1} \\
F^bE^a \arrow[r, "F^b\eta\bullet"'] & F^{b+1}E^{a+1} \arrow[ur, "(T_b\cdots T_1)E^{a+1}"'] &
\end{tikzcd}$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
E^mF \arrow[r, "E\eta\bullet"] \arrow[d, "\eta\bullet"'] & EFE^mF \arrow[r, "EF(T_1\cdots T_{m-1})F"] & EFE^mF \arrow[r, "\bullet\varepsilon"] & EFE^{m-1} \arrow[r, "\eta\bullet"] & FE^2FE^{m-1} \arrow[d, "FT\bullet"] \\
FE^{m+1}F \arrow[d, "FT\bullet"'] & & & & FE^2FE^{m-1} \arrow[d, "FE\varepsilon\bullet"] \\
FE^{m+1}F \arrow[r, "FE^2\eta\bullet"] \arrow[rr, bend right, "\mathrm{id}"'] & FE^2FE^mF \arrow[r, "FE\varepsilon\bullet"] & FE^{m+1}F \arrow[r, "\bullet(T_1\cdots T_{m-1})F"] & FE^{m+1}F \arrow[r, "\bullet\varepsilon"] & FE^m
\end{tikzcd}$$

and the second statement follows by induction. The third statement follows from the second one by applying the Chevalley duality (cf §4.2.1).

Let $i \in [1, m-1]$. Since $T_{i+1}T_1 \cdots T_m = T_1 \cdots T_m T_i$, we have a commutative diagram

$$\begin{tikzcd}
E^mF \arrow[r, "\eta\bullet"] \arrow[dd, "T_iF"'] & FE^{m+1}F \arrow[r, "F(T_1\cdots T_m)F"] \arrow[dr, "F(T_{i+1}T_1\cdots T_m)F"'] & FE^{m+1}F \arrow[r, "\bullet\varepsilon"] & FE^m \arrow[d, "FT_i"] \\
& & FE^{m+1}F \arrow[r, "\bullet\varepsilon"] & FE^m \\
E^mF \arrow[r, "\eta\bullet"] & FE^{m+1}F \arrow[ur, "F(T_1\cdots T_m)F"'] & &
\end{tikzcd}$$

It follows that $\sigma_{m,1}$ commutes with the action of ${}^0H_m^f$ and by induction we deduce that $\sigma_{m,n}$ commutes with ${}^0H_m^f$. The commutation with $({}^0H_n^f)^{\mathrm{opp}}$ follows by applying the Chevalley duality.

We have a commutative diagram

$$\begin{tikzcd}
E^aF \arrow[r, "\eta\bullet"] \arrow[d, "\eta\bullet"] \arrow[ddd, bend right, "\sigma_{a,1}"'] & FE^{a+1}F \arrow[rr, "F\sigma_{a+1,1}"] \arrow[d, "F\eta\bullet"] & & F^2E^{a+1} \\
FE^{a+1}F \arrow[dr, "F\eta\bullet"'] & F^2E^{a+2}F \arrow[r, "F^2T\bullet"] & F^2E^{a+2}F \arrow[r, "F^2E(T_1\cdots T_a)F"] & F^2E^{a+2}F \arrow[u, "\bullet\varepsilon"] \\
& F^2E^{a+1}F \arrow[r, "T\bullet"] & F^2E^{a+1}F \arrow[ur, "F^2E(T_1\cdots T_a)F"'] & \\
FE^a \arrow[rrr, "F\eta\bullet"'] & & & F^2E^{a+1} \arrow[uuu, bend right, "T\bullet"']
\end{tikzcd}$$

hence, we obtain a commutative diagram

$$\begin{tikzcd}
E^aF^b \arrow[r, "\eta\bullet"] \arrow[d, "\sigma_{a,1}\bullet"'] & FE^{a+1}F^b \arrow[rr, "F\sigma_{a+1,1}\bullet"] & & F^2E^{a+1}F^{b-1} \arrow[rr, "\bullet\sigma_{a+1,b-1}"] & & F^{b+1}E^{a+1} \\
FE^aF^{b-1} \arrow[rr, "F\eta\bullet"'] & & F^2E^{a+1}F^{b-1} \arrow[ur, "T\bullet"'] \arrow[rr, "F^2\sigma_{a+1,b-1}"'] & & F^{b+1}E^{a+1} \arrow[ur, "T\bullet"'] &
\end{tikzcd}$$

The last part of the Lemma follows now by induction on b. $\square$

Given $P \in \mathbf{Z}[X_1, \ldots, X_n]$, we denote by $\deg_*(P)$ the maximum of the degrees in any of the variables of P. Given μ a partition and l a non-negative integer, we denote by $\mu \cup \{l\}$ the partition obtained by adding l to μ.

Lemma 4.11. *Let $a, b \in \mathbf{Z}_{\geq 0}$, $P \in \mathbf{Z}[X_1, \ldots, X_a]^{\mathfrak{S}_a}$ and $Q \in \mathbf{Z}[X_{a+1}, \ldots, X_{a+b}]^{\mathfrak{S}_{[a+1,a+b]}}$. Then,* $\deg_* \partial_{w[1,a+b]w[1,a]w[a+1,b]}(PQ) \leq \max(\deg_*(P) + b, \deg_*(Q) + a)$.

Let $\mu \in \mathcal{P}(a, d)$ for some $d \in \mathbf{Z}_{\geq 0}$ and let $l \geq d + a$. We have

$$\partial_{s_1\cdots s_a}(X_{a+1}^l m_\mu(X_1, \ldots, X_a)) = m_{\mu\cup\{l-a\}}(X_1, \ldots, X_{a+1}) + R,$$

where R is a symmetric polynomial with $\deg_* R < l - a$.

Proof. Let us first show by induction on $n \geq 1$ that given $a_1, \ldots, a_n \in \mathbf{Z}_{\geq 0}$, we have

$$\deg_*(\partial_{w[1,n]}(X_1^{a_1} \cdots X_n^{a_n})) \leq \max(\{a_i\}) - n + 1. \tag{6}$$

This clear for $n = 1$. Applying a permutation of $[1, n]$ if necessary, we can assume that $a_n = \min(\{a_i\})$. Then,

$$\begin{aligned}
\partial_{w[1,n]}(X_1^{a_1} \cdots X_n^{a_n}) &= (X_1 \cdots X_n)^{a_n} \partial_{s_1\cdots s_{n-1}} \partial_{w[1,n-1]}(X_1^{a_1-a_n} \cdots X_{n-1}^{a_{n-1}-a_n}) \\
&= (X_1 \cdots X_n)^{a_n} \partial_{s_1\cdots s_{n-1}}(R)
\end{aligned}$$

where R is a polynomial in $X_1, \ldots, X_{n-1}$ whose degree in X_{n-1} is at most $\max(\{a_i\}) - a_n - n + 2$ by induction. It follows that the degree in X_n of $\partial_{s_1\cdots s_{n-1}}(R)$ is at most $\max(\{a_i\}) - a_n - n + 1$ and (6) follows from the fact that $\partial_{w[1,n]}(X_1^{a_1} \cdots X_n^{a_n})$ is a symmetric polynomial.

We have $\partial_{w[1,a+b]w[1,a]w[a+1,b]}(PQ) = \partial_{w[1,a+b]}(PX_1^{a-1} \cdots X_{a-1}QX_{a+1}^{b-1} \cdots X_{a+b-1})$ and the first part of the lemma follows from (6).

We prove the lemma by induction on a. We write $k \subset \mu$ if there is i such that $\mu_i = k$ and we denote by $\mu \setminus k$ the partition obtained by removing k to μ. We have

$$\partial_{s_1\cdots s_a}(X_{a+1}^l m_\mu(X_1,\ldots,X_a)) = \sum_{k\subset\mu} \partial_{s_1}(X_1^k \partial_{s_2\cdots s_a}(X_{a+1}^l m_{\mu\setminus k}(X_2,\ldots,X_a))).$$

By induction, we have

$$\partial_{s_2\cdots s_a}(X_{a+1}^l m_{\mu\setminus k}(X_2,\ldots,X_a)) = X_2^{l-a+1} m_{\mu\setminus k}(X_3,\ldots,X_{a+1}) + R,$$

where the degree in X_2 of R is strictly less than $l-a+1$. It follows that

$$\partial_{s_1\cdots s_a}(X_{a+1}^l m_\mu(X_1,\ldots,X_a)) = \sum_{k\subset\mu} X_1^{l-a} X_2^k m_{\mu\setminus k}(X_3,\ldots,X_{a+1}) + R' = X_1^{l-a} m_\mu(X_2,\ldots,X_{a+1}) + R',$$

where the degree in X_1 of R' is strictly less than $l-a$. The lemma follows. $\square$

The following Lemma is clear.

Lemma 4.12. *Let $\mathcal{C}$ be a k-linear category, X, Y two objects of $\mathcal{C}$, L and L' two right $\operatorname{End}(X)$-modules and $f : L \to \operatorname{Hom}(X,Y)$ and $f' : L' \to \operatorname{Hom}(X,Y)$ two morphisms of right $\operatorname{End}(X)$-modules. Let $\phi : L \otimes_{\operatorname{End}(X)} X \to Y$ and $\phi' : L \otimes_{\operatorname{End}(X)} X \to Y$ be the associated morphisms.*

Consider finite filtrations on L and on L' such that $f(L^{<i}) = f'(L'^{<i})$ for all i. Assume there are isomorphisms $L^{\le i}/L^{<i} \xrightarrow{\sim} L'^{\le i}/L'^{<i}$ for all i such that the following diagram commutes

$$\begin{tikzcd}
L^{\le i}/L^{<i} \arrow[dd, "\sim"'] \arrow[dr] & \\
& \operatorname{Hom}(X,Y)/L^{<i}\operatorname{End}(X) \\
L'^{\le i}/L'^{<i} \arrow[ur] &
\end{tikzcd}$$

Then, ϕ is an isomorphism if and only if ϕ' is an isomorphism.

Lemma 4.13. *Assume $m-n+\lambda \le 0$. We have an isomorphism $\sum_i \operatorname{act}\circ\Big(\operatorname{id}\otimes\big((F^{n-i}\eta_i E^{m-i})\circ \sigma_{m-i,n-i}\big)\Big)$:*

$$\bigoplus_{i=0}^{\min(m,n)} L(m,n,i,\lambda)\otimes_{\mathbf{Z}} E^{m-i}F^{n-i}\mathbf{1}_\lambda(i(i-2m-\lambda)) \xrightarrow{\sim} F^n E^m \mathbf{1}_\lambda.$$

It induces an isomorphism of $({}^0H_m^f \otimes ({}^0H_n^f)^{\mathrm{opp}})$-modules:

$$\bigoplus_{i=0}^{\min(m,n)} \bar{L}(m,n,i,\lambda)\otimes_{{}^0H_{m-i}^f\otimes({}^0H_{n-i}^f)^{\mathrm{opp}}} E^{m-i}F^{n-i}\mathbf{1}_\lambda(i(i-2m-\lambda)) \xrightarrow{\sim} F^n E^m.$$

Assume $m-n+\lambda \ge 0$. We have an isomorphism $\sum_i(\operatorname{id}\otimes(\sigma_{m-i,n-i}\circ(E^{m-i}\varepsilon_i F^{n-i})))\circ\operatorname{act}^*$:

$$E^m F^n \mathbf{1}_\lambda \xrightarrow{\sim} \bigoplus_{i=0}^{\min(m,n)} L'(m,n,i,\lambda)\otimes_{\mathbf{Z}} F^{n-i}E^{m-i}\mathbf{1}_\lambda(i(2n-\lambda-i)).$$

It induces an isomorphism of $({}^0H^f_m \otimes ({}^0H^f_n)^{\mathrm{opp}})$*-modules:*

$$E^mF^n\mathbf{1}_\lambda \xrightarrow{\sim} \bigoplus_{i=0}^{\min(m,n)} \bar{L}'(m,n,i,\lambda) \otimes_{{}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}}} F^{n-i}E^{m-i}\mathbf{1}_\lambda(i(2n-\lambda-i)).$$

Proof. Note first that the statements for (m,n,λ) where $m-n+\lambda\le 0$ are transformed into the statements for $(n,m,-\lambda)$ by the Chevalley involution. It is immediate to check that the maps are graded and it is enough to prove the Lemma in the non-graded setting.

Assume $m-n+\lambda\le 0$. Note that the first statement is equivalent to the second one (Lemma 4.9), whose map makes sense thanks to Lemma 4.10. We will drop the idempotents $\mathbf{1}_\lambda$ to simplify notations. Note that the result holds for $m=n=1$ as $\rho_{s,\lambda}$ is invertible by definition.

Since $\bar{L}(m,n,i,\lambda)\otimes_{{}^0H^f_{m-i}\otimes({}^0H^f_{n-i})^{\mathrm{opp}}}({}^0H_{m-i}\otimes{}^0H_{n-i})$ is projective as a $({}^0H^f_m\otimes({}^0H^f_n)^{\mathrm{opp}}, {}^0H_{m-i}\otimes {}^0H^{\mathrm{opp}}_{n-i})$-bimodule (Lemma 4.9), it is enough to show that the second map is an isomorphism after multiplication by $T_{w[1,m]}\otimes T_{w[1,n]}$ (Lemma 3.3).

We prove the Lemma by induction on $n+m$. Note that the Lemma holds trivially when $n=0$ or $m=0$ as well as when $(m,n)=(1,1)$. So, we can assume $m+n\ge 3$.

• Let us first consider the case $m-n+\lambda=0$. Applying the Chevalley duality if necessary, we can assume that $n>1$. By induction, we have isomorphisms

$$\begin{aligned}(7)\quad E^mF^n \oplus \bar{L}(m,n-1,1,\lambda-2)\otimes_{{}^0H^f_{m-1}\otimes({}^0H^f_{n-2})^{\mathrm{opp}}} E^{m-1}F^{n-1} &\xrightarrow{\sim} F^{n-1}E^mF \\ &\xrightarrow{\sim} F^nE^m \oplus \bar{L}'(m,1,1,\lambda)\otimes_{{}^0H^f_{m-1}} F^{n-1}E^{m-1}.\end{aligned}$$

Applying $I\circ D$ to the diagram of Lemma 4.10, we obtain a commutative diagram

$$\begin{tikzcd}
E^{m-1}F \arrow[r, "\eta\bullet"] \arrow[d, "\sigma_{m-1,1}"'] & FE^mF \arrow[r, "F\sigma_{m,1}"] & F^2E^m \\
FE^{m-1} \arrow[r, "F\eta\bullet"'] & F^2E^m \arrow[ur, "TE^m"'] &
\end{tikzcd}$$

It follows that the composition of maps in (7) has one of its components equal to

$$\begin{aligned}(8)\quad \mathrm{act}\circ(T_{n-1}E^m)\circ(F^{n-1}\eta E^{m-1})\circ(F^{n-2}\sigma_{m-1,n-1}):& \\ \bar{L}(m,n-1,1,\lambda-2)\otimes_{{}^0H^f_{m-1}\otimes({}^0H^f_{n-2})^{\mathrm{opp}}} E^{m-1}F^{n-1} &\to F^nE^m.\end{aligned}$$

We have $(T_{w[1,m]}\otimes T^{\mathrm{opp}}_{w[1,n]})\bar{L}(m,n-1,1,\lambda-2)=(T_{w[1,m]}\otimes T_{w[1,n]})\mathbf{Z}$ and it follows that the map in (8) vanishes after multiplication by $(T_{w[1,m]}\otimes T^{\mathrm{opp}}_{w[1,n]})$. We deduce that the component $\sigma_{m,n}: E^mF^n\to F^nE^m$ of the composition of maps in (7) is an isomorphism.

• We consider now the case $n=1$ and $m+\lambda\le 0$. By induction, we have an isomorphism

$$E^{m-1}FE\oplus L([2,m],1,1,\lambda+2)\otimes E^{m-1}\xrightarrow{\sim} FE^m.$$

So, we have an isomorphism

$$(9)\qquad E^mF\oplus L(1,1,1,\lambda)\otimes E^{m-1}\oplus L([2,m],1,1,\lambda+2)\otimes E^{m-1}\xrightarrow{\sim} FE^m$$

Taking the image under the Chevalley duality of the commutative diagram of Lemma 4.10, we obtain a commutative diagram

$$\begin{tikzcd} E^{m-1} \arrow[r, "\bullet\eta"] \arrow[dr, "\eta\bullet"'] & E^{m-1}FE \arrow[r, "\sigma_{m-1,1}E"] & FE^m \\ & FE^m \arrow[ur, "F(T_1\cdots T_{m-1})"'] & \end{tikzcd}$$

It follows that the isomorphism (9) induces an isomorphism $(\sigma_{m,1}, \mathrm{act}\circ(\mathrm{id}\otimes(\eta E^{m-1})), (\mathrm{act}\circ(\mathrm{id}\otimes(\eta E^{m-2})))E)$:

$$E^mF \oplus \left(\bigoplus_{0\le i<-\lambda} X_1^iT_1\cdots T_{m-1}\mathbf{Z}\right)\otimes E^{m-1} \oplus L([2,m],1,1,\lambda+2)\otimes E^{m-1} \xrightarrow{\sim} FE^m.$$

Let

$$M = \left(\bigoplus_{0\le i<-\lambda} X_1^iT_1\cdots T_{m-1}{}^0H_{m-1}\right) \oplus \bigoplus_{\substack{w\in\mathfrak{S}_{[2,m]}^{[2,m-1]}\\ l<-m-\lambda}} T_w(X_m^l)^0H_{m-1}.$$

This is a ${}^0H_m^f$-submodule of ${}^0H_m$. We have

$$\begin{aligned} T_{w[1,m]}M &= T_{w[1,m]}\left(\sum_{i<-\lambda}\partial_{s_{m-1}\cdots s_1}(X_1^i)^0H_{m-1} + \sum_{i<-m-\lambda}(X_m^i)^0H_{m-1}\right)\\ &= T_{w[1,m]}\sum_{i\le-\lambda-m}(X_m^i)^0H_{m-1}\\ &= T_{w[1,m]}\bar{L}(m,1,1,\lambda)^0H_{m-1}. \end{aligned}$$

Note that M is generated by $\dim_{\mathbf{Z}} L(m,1,1,\lambda)$ elements as a right ${}^0H_{m-1}$-module. Since $\bar{L}(m,1,1,\lambda)^0H_{m-1}$ is a free right ${}^0H_{m-1}$-module of rank $\dim_{\mathbf{Z}} L(m,1,1,\lambda)$, it follows that M is a free right ${}^0H_{m-1}$-module of that rank. We have an isomorphism

$$(\sigma_{m,1}, \mathrm{act}\circ(\mathrm{id}\otimes(\eta E^{m-1}))) : E^mF \oplus M\otimes_{{}^0H_{m-1}} E^{m-1} \xrightarrow{\sim} FE^m. \tag{10}$$

The morphism

$$(\sigma_{m,1}, \mathrm{act}\circ(\mathrm{id}\otimes(\eta E^{m-1}))) : E^mF \oplus \bar{L}(m,1,1,\lambda)\otimes_{{}^0H_{m-1}^f} E^{m-1} \xrightarrow{\sim} FE^m \tag{11}$$

becomes an isomorphism after multiplication by $T_{w[1,m]}$, since it coincides with the multiplication by $T_{w[1,m]}$ of the isomorphism (10). It follows from Lemmas 4.9 and 3.3 that the morphism (11) is an isomorphism and the lemma is proven when $n=1$.

- We consider finally the case $n>1$ and $m-n+\lambda<0$. We have an isomorphism

$$F\bigoplus_{i=0}^{n-1} L(m,n-1,i,\lambda)\otimes E^{m-i}F^{n-i-1} \xrightarrow{\sim} F^nE^m.$$

The case $n=1$ of the lemma gives isomorphisms

$$\left(L(m-i,1,1,\lambda-2(n-i-1))\otimes E^{m-i-1}\right)F^{n-i-1} \oplus E^{m-i}F^{n-i} \xrightarrow{\sim} FE^{m-i}F^{n-i-1}.$$

Combining the previous two isomorphisms, we obtain a isomorphism

$$\bigoplus_{i=0}^{n}\Big(L(m,[2,n],i,\lambda)\oplus L(m,[2,n],i-1,\lambda)\otimes L(m-i+1,1,1,\lambda-2(n-i))\Big)\otimes E^{m-i}F^{n-i}\xrightarrow{\sim}F^nE^m.$$

In that isomorphism, the map $L(m,[2,n],i,\lambda)\otimes E^{m-i}F^{n-i}\to F^nE^m$ is $\mathrm{act}\circ\Big(\mathrm{id}\otimes\big((F^{n-i}\eta_iE^{m-i})\circ\sigma_{m-i,n-i}\big)\Big)$. It follows from Lemma 4.10 that the map

$$L(m,[2,n],i-1,\lambda)\otimes L(m-i+1,1,1,\lambda-2(n-i))\otimes E^{m-i}F^{n-i}\to F^nE^m$$

is

$$\mathrm{act}\circ\Bigg(\mathrm{id}\otimes\Big(\mathrm{act}\circ\big((T_{n-i}\cdots T_1)E^{m-i+1}\big)\Big)\Bigg)\circ\Big(\mathrm{id}\otimes\mathrm{id}\otimes\big((F^{n-i}\eta_iE^{m-i})\circ\sigma_{m-i,n-i}\big)\Big).$$

Let $i>0$ and

$$M_i=L(m,[2,n],i,\lambda)\oplus\left(\bigoplus_{l<-r+n-i}T_{n-i}\cdots T_1X_1^l\mathbf{Z}\right)\cdot L(m,[2,n],i-1,\lambda)\cdot\left(\bigoplus_{1\le j\le m-i}T_j\cdots T_{m-i}\mathbf{Z}\right),$$

a subgroup of ${}^0H_m\otimes_{{}^0H_i}{}^0H_n$. We have shown that there is an isomorphism

$$\mathrm{act}\circ\Big(\mathrm{id}\otimes\big((F^{n-i}\eta_iE^{m-i})\circ\sigma_{m-i,n-i}\big)\Big):\bigoplus_{i=0}^{n}M_i\otimes E^{m-i}F^{n-i}\xrightarrow{\sim}F^nE^m.$$

Let

$$N_i=\bigoplus_{\substack{w'\in{}^{[2,n-i+1]}\mathfrak{S}_{[2,n]}\\ \mu\in\mathcal{P}(i-1,-r-i)\\ l<-r+n-i}}T_{n-i}\cdots T_1X_1^lm_\mu(X_{n-i+2},\ldots,X_n)X_{n-i+2}^{i-2}\cdots X_{n-1}T_{w'}\mathbf{Z}$$

and

$$N_i'=\bigoplus_{\substack{w'\in{}^{[2,n-i]}\mathfrak{S}_{[2,n]}\\ \mu\in\mathcal{P}(i,-r-1-i)}}m_\mu(X_{n-i+1},\ldots,X_n)X_{n-i+1}^{i-1}\cdots X_{n-1}T_{w'}\mathbf{Z}.$$

We have

$$M_i=\left(\bigoplus_{w\in\mathfrak{S}_m^{m-i}}T_w\mathbf{Z}\right)\otimes\left(N_i'\oplus N_i\right).$$

We have

$$T_{w[n-i+1,n]}N_i'T_{w[1,n]}=\sum_{\mu\in\mathcal{P}(i,-r-1-i)}m_\mu(X_{n-i+1},\ldots,X_n)T_{w[1,n]}\mathbf{Z}$$

and

$$\begin{aligned}T_{w[n-i+1,n]}N_iT_{w[1,n]}&=\\ &=\sum_{\substack{\mu\in\mathcal{P}(i-1,-r-i)\\ l<-r+n-i}}\partial_{s_{n-1}\cdots s_{n-i+1}}(\partial_{s_{n-i}\cdots s_1}(X_1^l)m_\mu(X_{n-i+2},\ldots,X_n))T_{w[1,n]}\mathbf{Z}\\ &=\sum_{\substack{\mu\in\mathcal{P}(i-1,-r-i)\\ l<-r+n-i}}(\sum_{k\le l-n+i}P_{k,l}(X_1,\ldots,X_{n-i})R_{k,\mu}(X_{n-i+1},\ldots,X_n))T_{w[1,n]}\mathbf{Z}\end{aligned}$$

where $P_{k,l}$ is a symmetric polynomial and $R_{k,\mu}=\partial_{s_{n-1}\cdots s_{n-i+1}}(X_{n-i+1}^k m_\mu(X_{n-i+2},\ldots,X_n))$ satisfies $\deg_* R_{k,\mu}\le \max(k-i+1,-r-i-1)$ by Lemma 4.11.

Let us fix k and l. By induction, the composite morphism

$$E^{m-i}F^{n-i}\xrightarrow{\sigma_{m-i,n-i}}F^{n-i}E^{m-i}\xrightarrow{P_{k,l}E^{m-i}}F^{n-i}E^{m-i}$$

is equal to

$$\sum_{\substack{j\ge i\\ \mu'\in\mathcal{P}(j-i,-r-j+i)}}((m_{\mu'}(X_{n-j+1},\ldots,X_{n-i})X_{n-j+1}^{j-i-1}\cdots X_{n-i-1})E^{m-i})\circ(\mathrm{id}\otimes((F^{n-j}\eta_{j-i}E^{m-j})\circ\sigma_{m-j,n-j}))\circ f_{j,\mu'}$$

for some $f_{j,\mu'}:E^{m-i}F^{n-i}\to E^{m-j}F^{m-j}$. We have

$$\begin{aligned}T_{w[n-j+1,n]w[n-i+1,n]}m_{\mu'}(X_{n-j+1},\ldots,X_{n-i})X_{n-j+1}^{j-i-1}\cdots X_{n-i-1}R_{k,\mu}(X_{n-i+1},\ldots,X_n)T_{w[1,n]}=\\ =\partial_{w[n-j+1,n]w[n-i+1,n]w[n-j+1,n-i]}(m_{\mu'}(X_{n-j+1},\ldots,X_{n-i})R_{k,\mu}(X_{n-i+1},\ldots,X_n))T_{w[1,n]}=\\ =S_{k,\mu,\mu'}(X_{n-j+1},\ldots,X_n)T_{w[1,n]},\end{aligned}$$

where $S_{k,\mu,\mu'}$ is a symmetric polynomial and $\deg_* S_{k,\mu,\mu'}\le -r-j$ by Lemma 4.11. Note that if $j=i$ and $k\ne -r-1$, then $\deg_* S_{k,\mu,\mu'}\le -r-i-1$.

Assume $l=-r+n-i-1$ and $k=l-n+i=-r-1$. We have

$$R_{k,\mu}=\partial_{s_{n-1}\cdots s_{n-i+1}}(X_{n-i+1}^{-r-1}m_\mu(X_{n-i+2},\ldots,X_n))=m_{\mu\cup\{-r-i\}}(X_{n-i+1},\ldots,X_n)+T(X_{n-i+1},\ldots,X_n),$$

where T is a symmetric polynomial with $\deg_* T\le -r-i-1$ (Lemma 4.11).

We have shown that the images of $L(m,n,i,\lambda)$ and of M_i in $\mathrm{Hom}(E^{m-i}F^{n-i},F^{(n)}E^{(m)})$ coincide modulo maps that factor through

$$\bigoplus_{j>i}(T_{w[1,m]}T_{w[1,n]}^{\mathrm{opp}})L(m,n,i,\lambda)\otimes E^{m-j}F^{n-j}\to F^{(n)}E^{(m)}.$$

Using Lemma 4.12, we deduce by descending induction on i that the lemma holds, using that $\dim_{\mathbf{Z}} M_i=\dim_{\mathbf{Z}} L(m,n,i,\lambda)$ as in the case $n=1$ considered earlier. $\square$

Remark 4.14. Let $\hat{\mathcal{B}}_1$ the k-linear category $\mathcal{B}\times\mathcal{B}^{\mathrm{opp}}$. Denote by F_s the object E_s of $\mathcal{B}^{\mathrm{opp}}$ and define $\hat{h}:\mathrm{Ob}(\hat{\mathcal{B}}_1)\to X,\ (M,N)\mapsto h(M)+h(N)$. Consider the 2-category $\hat{\mathfrak{A}}_1$ with set of objects X and $\mathcal{H}om(\lambda,\lambda')=\hat{h}^{-1}(\lambda'-\lambda)$. The isomorphisms of Lemma 4.13, together with σ_{st} for $s\ne t$, are the first steps to provide a direct construction of a tensor structure on the homotopy category of $\hat{\mathfrak{A}}_1$ (after adding maps $(M\otimes E_s,F_s\otimes N)\to(M,N)$).

4.2.3. *Decomposition of* $[E_s^{(m)},F_t^{(n)}]$.

Lemma 4.15. *Let $s\in I$ and $m,n\in\mathbf{Z}_{\ge0}$. Let $r=m-n+\langle\alpha_s^\vee,\lambda\rangle$. We have the following isomorphisms in $\mathfrak{A}^i$ and in $\mathfrak{A}^{\bullet i}$:*

$$E_s^{(m)}F_s^{(n)}\simeq\bigoplus_{i\in\mathbf{Z}_{\ge0}}\begin{bmatrix}r\\ i\end{bmatrix}_s F_s^{(n-i)}E_s^{(m-i)}\ \text{if } r\ge0$$

$$E_s^{(m)}F_s^{(n)}\oplus\bigoplus_{i\in1+2\mathbf{Z}_{\ge0}}\begin{bmatrix}i-1-r\\ i\end{bmatrix}_s F_s^{(n-i)}E_s^{(m-i)}\simeq\bigoplus_{i\in2\mathbf{Z}_{\ge0}}\begin{bmatrix}i-1-r\\ i\end{bmatrix}_s F_s^{(n-i)}E_s^{(m-i)}\ \text{if } r<0$$

$$F_s^{(n)}E_s^{(m)}\oplus\bigoplus_{i\in1+2\mathbf{Z}_{\ge0}}\begin{bmatrix}i-1+r\\ i\end{bmatrix}_s E_s^{(m-i)}F_s^{(n-i)}\simeq\bigoplus_{i\in2\mathbf{Z}_{\ge0}}\begin{bmatrix}i-1+r\\ i\end{bmatrix}_s E_s^{(m-i)}F_s^{(n-i)}\ \text{if } r>0$$

$$F_s^{(n)}E_s^{(m)} \simeq \bigoplus_{i\in\mathbf{Z}_{\geq 0}} \begin{bmatrix} -r \\ i \end{bmatrix}_s E_s^{(m-i)}F_s^{(n-i)} \text{ if } r \leq 0$$

Let $t \in I - \{s\}$ and $m, n \in \mathbf{Z}_{\geq 0}$. We have the following isomorphisms in $\mathfrak{A}^i$ and in $\mathfrak{A}^{\bullet i}$:

$$E_s^{(m)}F_t^{(n)} \simeq F_t^{(n)}E_s^{(m)}.$$

Proof. The first isomorphism follows from the isomorphism of $({}^0H_m \otimes {}^0H_n^{\mathrm{opp}})$-modules in Lemma 4.13. Assume $r < 0$. Given $l \in \mathbf{Z}_{>0}$, we have (cf e.g. [Lu, 1.3.1(e) p.9])

$$\sum_i (-1)^i \begin{bmatrix} i-1-r \\ i \end{bmatrix} \cdot \begin{bmatrix} -r \\ l-i \end{bmatrix} = 0.$$

It follows that

$$\bigoplus_{\substack{i\in 1+2\mathbf{Z}_{\geq 0} \\ j\geq 0}} \begin{bmatrix} i-1-r \\ i \end{bmatrix} \cdot \begin{bmatrix} -r \\ j \end{bmatrix} E^{(m-i-j)}F^{(n-i-j)} \simeq \bigoplus_{\substack{i\in 2\mathbf{Z}_{>0} \\ j\geq 0}} \begin{bmatrix} i-1-r \\ i \end{bmatrix} \cdot \begin{bmatrix} -r \\ j \end{bmatrix} E^{(m-i-j)}F^{(n-i-j)} \oplus$$
$$\oplus \bigoplus_{i\geq 1} \begin{bmatrix} -r \\ i \end{bmatrix} E^{(m-i)}F^{(n-i)},$$

hence

$$\bigoplus_{i\in 1+2\mathbf{Z}_{\geq 0}} \begin{bmatrix} i-1-r \\ i \end{bmatrix} F^{(n-i)}E^{(m-i)} \simeq \bigoplus_{i\in 2\mathbf{Z}_{>0}} \begin{bmatrix} i-1-r \\ i \end{bmatrix} F^{(n-i)}E^{(m-i)} \oplus \bigoplus_{i\geq 1} \begin{bmatrix} -r \\ i \end{bmatrix} E^{(m-i)}F^{(n-i)}$$

using the first isomorphism of the lemma for $(m-i, n-i)$. The second isomorphism of the lemma follows by applying again the first isomorphism.

The third and fourth isomorphism follow from the second and first by applying the Chevalley involution.

The isomorphisms σ_{st} induce an isomorphism $E_s^mF_t^n \xrightarrow{\sim} F_t^nE_s^m$ compatible with the action of ${}^0H_m \otimes {}^0H_n$ (the proof in Lemma 4.10 works when $s \neq t$). It follows that $E_s^{(m)}F_t^{(n)} \simeq F_t^{(n)}E_s^{(m)}$. ∎

4.2.4. *Decategorification.* Proposition 4.2 shows that we have a morphism of algebras

$$U_1^+(\mathfrak{g}) \to \mathcal{B}^i_{\leq 1},\ e_s^{(r)} \mapsto [E_s^{(r)}]$$

and a morphism of $\mathbf{Z}[v^{\pm 1}]$-algebras

$$U_v^+(\mathfrak{g}) \to \mathcal{B}^{\bullet i}_{\leq 1},\ e_s^{(r)} \mapsto [E_s^{(r)}].$$

The defining relations for $\mathfrak{A}$ show that we have a functor compatible with the $\mathbf{Z}[v^{\pm 1}]$-structure:

$$U_1(\mathfrak{g}) \to \mathfrak{A}^i_{\leq 1},\quad \lambda \mapsto \lambda,\ e_s^{(r)} \mapsto [E_s^{(r)}],\ f_s^{(r)} \mapsto [F_s^{(r)}]$$

and a functor compatible with the $\mathbf{Z}[v^{\pm 1}]$-structure:

$$U_v(\mathfrak{g}) \to \mathfrak{A}^{\bullet i}_{\leq 1},\quad \lambda \mapsto \lambda,\ e_s^{(r)} \mapsto [E_s^{(r)}],\ f_s^{(r)} \mapsto [F_s^{(r)}].$$

5. 2-Representations

We assume in this section that the set I is always taken to be finite. All results are stated over $\mathbf{k}$ and are related to representations of $\mathfrak{g}$. They generalize immediately to the graded case over $\mathbf{k}^\bullet$ and relate then to representations of $U_v(\mathfrak{g})$.

5.1. **Integrable representations.**

5.1.1. *Definition.* Let $\mathfrak{B}$ be a $\mathbf{k}$-linear 2-category.

Given $R : \mathfrak{A} \to \mathfrak{B}$ a 2-functor, we have a collection of $\{R(\lambda)\}$ of objects of $\mathfrak{B}$. We say that R gives a 2-representation of $\mathfrak{A}$ on $\{R(\lambda)\}$. If this makes sense, we put $\mathcal{V} = \bigoplus_{\lambda \in X} R(\lambda)$ and say that we have a 2-representation of $\mathfrak{A}$ on $\mathcal{V}$.

The data of a strict 2-functor $R : \mathfrak{A} \to \mathfrak{B}$ is the same as the data of

- a family $(\mathcal{V}_\lambda)_{\lambda \in X}$ of objects of $\mathfrak{B}$
- 1-arrows $E_{s,\lambda} : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha_s}$ and $F_{s,\lambda} : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_s}$ for $s \in I$
- $x_{s,\lambda} \in \mathrm{End}(E_{s,\lambda})$ and $\tau_{s,t,\lambda} \in \mathrm{Hom}(E_{s,\lambda+\alpha_t} E_{t,\lambda}, E_{t,\lambda+\alpha_s} E_{s,\lambda})$ for $s, t \in I$
- an adjunction $(E_{s,\lambda}, F_{s,\lambda+\alpha_s})$

such that

- relations (1)-(4) in §4.1.1 hold
- the maps $\rho_{s,\lambda}$ and σ_{st} for $s \neq t$ are isomorphisms
- the map $\eta^l_{s,\lambda} : \mathbf{1}_\lambda \to E_{s,\lambda-\alpha_s} F_{s,\lambda}$ defined by
$$\eta^l_{s,\lambda} = \rho^{-1}_{s,\lambda} \circ (0, \ldots, 0, (-1)^{\langle \alpha_s^\vee, \lambda \rangle})$$
for $\langle \alpha_s^\vee, \lambda \rangle > 0$ gives an adjunction $(F_{s,\lambda}, E_{s,\lambda+\alpha_s}))$
- the map $\varepsilon^l_{s,\lambda} : F_{s,\lambda+\alpha_s}, E_{s,\lambda} \to \mathbf{1}_\lambda$ defined by
$$\varepsilon^l_{s,\lambda} = (0, \ldots, 0, (-1)^{\alpha_s^\vee, \lambda}) \circ \rho^{-1}_{s,\lambda}$$
for $\langle \alpha_s^\vee, \lambda \rangle < -1$ gives an adjunction $(F_{s,\lambda+\alpha_s}, E_{s,\lambda})$.

From now on, we assume $\mathfrak{B}$ is a locally full 2-subcategory of $\mathfrak{Lin}_{\mathbf{k}}$.

Definition 5.1. *A 2-representation $\mathfrak{A} \to \mathfrak{B}$ is* integrable *if E_s and F_s are locally nilpotent for all s,* i.e., *for any λ and any object M of the category $\mathcal{V}_\lambda$, there is an integer n such that $E_{s,\lambda+n\alpha_s} \cdots E_{s,\lambda+\alpha_s} E_{s,\lambda}(M) = 0$ and $F_{s,\lambda-n\alpha_s} \cdots F_{s,\lambda-\alpha_s} F_{s,\lambda}(M) = 0$.*

Our main object of study is the 2-category of integrable 2-representations of $\mathfrak{A}$ in $\mathbf{k}$-linear, abelian, triangulated and dg-categories.

Lemma 5.2. *Assume $\mathfrak{g}$ is finite-dimensional. Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}(\mathfrak{g})$ in $\mathfrak{Lin}_{\mathbf{k}}$. Let $\lambda \in X$ and $M \in \mathcal{V}_\lambda$. Then, there exists a full sub-2-representation $\mathcal{W}$ of $\mathcal{V}$ containing M and such that there are finitely many $\mu \in X$ with $\mathcal{W}_\mu \neq 0$.*

Proof. We define $\mathcal{W}$ as the full subcategory of $\mathcal{V}$ with objects of the form XM with $X \in \mathrm{Hom}_{\tilde{\mathcal{A}}'}(\lambda, \mu)$ for some μ. It follows from ???[7] that... ☐

Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}(\mathfrak{g})$ in $\mathfrak{Lin}_{\mathbf{k}}$. There is an induced action of $K^b(\mathfrak{A})$ on $K^b(\mathcal{V})$.

Lemma 5.3. *Let $s \in I$.*

[7] Check ref integrable and finitely generated implies finite-dimensional

- *Let $C \in K^b(\mathcal{V})$. If* $\mathrm{Hom}(E_s^i M, C) = 0$ *in $K^b(\mathcal{V})$ for all $M \in \mathcal{V}$ such that $F_s M = 0$, then $C = 0$.*
- *Let X be a 1-arrow of $K^b(\mathfrak{A}')$. If $X E_s^i(M) = 0$ for all $M \in K^b(\mathcal{V})$ such that $F_s M = 0$, then $X(N) = 0$ for all $N \in K^b(\mathcal{V})$.*
- *Let f a 2-arrow of $K^b(\mathfrak{A}')$. If $f(E_s^i M)$ is an isomorphism for all $M \in K^b(\mathcal{V})$ such that $F_s M = 0$, then $f(N)$ is an isomorphism for all $N \in K^b(\mathcal{V})$.*

Proof. Let i be a maximal integer such that $F_s^i C \neq 0$. We have

$$\mathrm{End}(F_s^i C) \simeq \mathrm{Hom}(E_s^i F_s^i C, C) = 0,$$

hence a contradiction and consequently $C = 0$.

Let $X^\vee$ be a right dual of X. Let $M, N \in K^b(\mathcal{V})$ such that $F_s M = 0$ and let $i \geq 0$. We have

$$\mathrm{Hom}(E_s^i(M), X^\vee X(N)) \simeq \mathrm{Hom}(X E_s^i(M), X(N)) = 0$$

and we deduce from the first statement of the Lemma that $C(N) = 0$.

The last assertion follows from the second one by taking for X the cone of f. ☐

The discussion above extends to the case of 2-representations of $\mathfrak{A}^\bullet$ in a graded $\mathbf{k}^\bullet$-linear 2-category. When $\mathcal{V}$ has trivial grading (*i.e.*, the self-equivalence is the identity), an action of $\mathfrak{A}^\bullet$ is an action of $\mathfrak{A}_0^\bullet$.

5.1.2. *Simple 2-representations.* We assume that the root datum is Y-regular, *i.e.*, the image of the embedding $I \to Y$ is linearly independent in Y (cf [Lu, §2.2.2]). Let $X^+ = \{\lambda \in X | \langle \alpha_i^\vee, \lambda\rangle \in \mathbf{Z}_{\geq 0}$ for all $i \in I\}$. The set X is endowed with a poset structure defined by $\lambda \geq \mu$ if $\lambda - \mu \in \bigoplus_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i^\vee$.

Let $\lambda \in -X^+$. Consider the 2-functor $\mathfrak{Hom}(\lambda, -) : \mathfrak{A} \to \mathfrak{Lin}_{\mathbf{k}}$ and let $R : \mathfrak{A} \to \mathfrak{Lin}_{\mathbf{k}}$ be the 2-subfunctor generated by the $F_{s,\lambda}$ for $s \in I$, *i.e.*, $R(\mu)$ is the $\mathbf{k}$-linear full subcategory of $\mathfrak{Hom}(\lambda, \mu)$ with objects in $h^{-1}(\mu - \lambda + \alpha_s)F_s$. We denote by $\mathcal{L}(\lambda)$ the quotient 2-functor, viewed as a $\mathbf{k}$-linear category endowed with a decomposition $\mathcal{L}(\lambda) = \bigoplus_{\mu \in X} \mathcal{L}(\lambda)_\mu$ and endowed with an action of $\mathfrak{A}$.

Denote by $\bar{\mathbf{1}}_\lambda$ the identity functor of $\mathcal{L}(\lambda)$. It follows from Lemma 4.13 that $F_s E_s^{\langle\alpha_s^\vee, -\lambda\rangle+1}\mathbf{1}_\lambda$ is isomorphic to a direct summand of $E_s^{\langle\alpha_s^\vee, -\lambda\rangle+1}F_s\mathbf{1}_\lambda$. In particular, $F_s E_s^{\langle\alpha_s^\vee, -\lambda\rangle+1}\bar{\mathbf{1}}_\lambda = 0$. The isomorphism

$$\mathrm{End}(E_s^{\langle\alpha_s^\vee, -\lambda\rangle+1}\bar{\mathbf{1}}_\lambda) \simeq \mathrm{Hom}(E_s^{\langle\alpha_s^\vee, -\lambda\rangle}\bar{\mathbf{1}}_\lambda, F_s E_s^{\langle\alpha_s^\vee, \lambda\rangle+1}\bar{\mathbf{1}}_\lambda)$$

shows that

$$E_s^{\langle\alpha_s^\vee, -\lambda\rangle+1}\bar{\mathbf{1}}_\lambda = 0.$$

Since $F_s E_t \mathbf{1}_\mu$ is a direct summand of $E_t F_s \mathbf{1}_\mu$ plus a multiple of $\mathbf{1}_\mu$, it follows that every object of $\mathcal{L}(\lambda)$ is isomorphic to a sum of objects of the form $E_{s_1} \cdots E_{s_n}\bar{\mathbf{1}}_\lambda$ for some $s_1, \ldots, s_n \in I$. In particular, every object of $\mathcal{L}(\lambda)_\lambda$ is isomorphic to a multiple of $\bar{\mathbf{1}}_\lambda$. Since $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$ is a quotient of $\mathrm{End}(\mathbf{1}_\lambda)$, it is commutative and $\mathcal{L}(\lambda)_\lambda$ is equivalent to the category of free $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$-mod-modules of finite rank.

Note that $\mathbf{C} \otimes K_0(\mathcal{L}(\lambda))$ is isomorphic to the simple integrable representation of $\mathfrak{g}$ with highest weight λ [Kac, Corollary 10.4], or it is 0. We will show in a sequel to this paper that it is indeed non zero and determine $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$.

5.1.3. *Lowest weights.* Let A be an $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$-algebra. Let $\mathcal{V} = \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} A$, given by $\mathcal{V}_\mu = \mathcal{L}(\lambda)_\mu \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} A$, where the map $\mathrm{End}(\bar{\mathbf{1}}_\lambda) \to Z(\mathcal{L}(\lambda)_\mu)$ is given by right multiplication. The action of $\mathfrak{A}$ on $\mathcal{L}(\lambda)$ extends to an action on $\mathcal{V}$. Similarly, if $\mathcal{A}$ is a $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$-linear category, we have an action of $\mathfrak{A}$ on $\mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{A}$.

Let $\mathcal{V}$ be a 2-representation of $\mathfrak{A}$ in $\mathfrak{Lin}_{\mathbf{k}}$. Given $\lambda \in X$, we denote by $\mathcal{V}_\lambda^{\mathrm{lw}}$ the full subcategory of $\mathcal{V}_\lambda$ of objects M such that $F_s M = 0$ for all $s \in I$.

Lemma 5.4. *If $\mathcal{V}_\lambda^{\mathrm{lw}} \neq 0$, then, $\lambda \in -X^+$.*

Proof. Assume there is $s \in I$ such that $\langle \alpha_s^\vee, \lambda \rangle > 0$ and let $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$. Then, M is a direct summand of $E_s F_s M = 0$. $\square$

Assume $\lambda \in -X^+$. The canonical morphism of 2-representations $\mathcal{H}om(\lambda, -) \to \mathcal{V}$ factors through a morphism $R_M : \mathcal{L}(\lambda) \to \mathcal{V}$. Note that $R_M(\bar{\mathbf{1}}_\lambda) = M$. So, we have a morphism of algebras $\mathrm{End}(\bar{\mathbf{1}}_\lambda) \to \mathrm{End}(M)$ and this shows that the morphism above extends to a morphism of 2-representations $R_M : \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathrm{End}(M) \to \mathcal{V}$. We have also a canonical morphism of 2-representations $R_\lambda : \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}} \to \mathcal{V}$ that extends R_M.

Proposition 5.5. *The morphism of 2-representations*

$$\sum_{\lambda \in -X^+} R_\lambda : \bigoplus_{\lambda \in -X^+} \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}} \to \mathcal{V}$$

is fully faithful.

Proof. Let $\lambda \in -X^+$ and $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$. Let $\mathcal{L}_M(\lambda) = \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathrm{End}(M)$. Let X be an object of $\mathcal{H}om(\lambda, \mu)$. There is an object Y of $\mathcal{H}om(\mu, \lambda)$ left dual to X. We have a commutative diagram of canonical maps

$$\begin{tikzcd} \mathrm{End}_{\mathcal{L}_M(\lambda)}(X\bar{\mathbf{1}}_\lambda) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathcal{L}_M(\lambda)}(YX\bar{\mathbf{1}}_\lambda, \bar{\mathbf{1}}_\lambda) \arrow[d] \\ \mathrm{End}_{\mathcal{V}}(XM) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathcal{V}}(YXM, M) \end{tikzcd}$$

Since $YX\bar{\mathbf{1}}_\lambda$ is isomorphic to a multiple of $\bar{\mathbf{1}}_\lambda$, the right vertical map is an isomorphism, hence the left vertical map is an isomorphism as well. It follows that R_M is fully faithful, hence R_λ is fully faithful as well.

Consider now $\mu \in -X^+$ with $\mu \neq \lambda$. Let $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$ and $N \in \mathcal{V}_\mu^{\mathrm{lw}}$. Let $s_1, \ldots, s_m, t_1, \ldots, t_n \in I$ such that $\alpha_{s_1} + \cdots + \alpha_{s_m} + \lambda = \alpha_{t_1} + \cdots + \alpha_{t_n} + \mu$. If $m = 0$, then we have

$$\mathrm{Hom}(M, E_{t_1} \cdots E_{t_n} N) \simeq \mathrm{Hom}(F_{t_1} M, E_{t_2} \cdots E_{t_n} N) = 0.$$

Assume $m > 0$. Since $F_{s_1} E_{t_1} \cdots E_{t_n} N$ is isomorphic to a direct sum of objects of the form $E_{t_1} \cdots E_{t_{i-1}} E_{t_{i+1}} \cdots E_{t_n} N$ for $t_i = s_1$, it follows by induction on m that

$$\mathrm{Hom}(E_{s_1} \cdots E_{s_m} M, E_{t_1} \cdots E_{t_n} N) = 0.$$

So, there are no non-zero maps between an object in the image of R_λ and an object in the image of R_μ. $\square$

An immediate consequence of Proposition 5.5 is a decomposition result for additive 2-representations generated by lowest weight objects.

Corollary 5.6. *Assume $\mathcal{V}$ is idempotent complete and every object of $\mathcal{V}$ is a direct summand of a multiple of XM for some object X of $\tilde{\mathcal{A}}'$ and $M \in \mathcal{V}$ with $F_i M = 0$ for all i.*

Then, there is an equivalence of 2-representations

$$\sum_{\lambda \in -X^+} R_\lambda : \bigoplus_{\lambda \in -X^+} \left(\mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}}\right)^i \xrightarrow{\sim} \mathcal{V}.$$

5.1.4. *Jordan-Hölder series.* We denote by $\mathfrak{O}^{\mathrm{int}}(\mathfrak{B})$ the 1, 2-full subcategory of 2-representations $\mathcal{V}$ in $\mathfrak{B}$ which are integrable and such that $\{\lambda \in -X^+ | \mathcal{V}_\lambda \neq 0\}$ is bounded below (*i.e.*, a sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ of elements of $-X^+$ with $\mathcal{V}_{\lambda_i} \neq 0$ for all i must be stationary).

Theorem 5.7. *Let $\mathcal{V}$ be an idempotent complete 2-representation in $\mathfrak{O}^{\mathrm{int}}(\mathfrak{Lin}_{\mathbf{k}})$. There is a filtration by thick 2-subrepresentations*

$$0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \cdots \subset \cdots \subset \mathcal{V}\{n\} = \mathcal{V},$$

there are $\mathrm{End}(\bar{\mathbf{1}}_\lambda)$-linear categories $\mathcal{M}_{\lambda,l}$ for $\lambda \in -X^+$ and isomorphisms of 2-representations

$$\mathcal{V}\{l\}/\mathcal{V}\{l-1\} \xrightarrow{\sim} \bigoplus_{\lambda \in -X^+} \left(\mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{M}_{\lambda,l}\right)^i.$$

Proof. We proceed by induction on the maximal length of a sequence $\lambda_1 < \cdots < \lambda_n$ of elements of $-X^+$ such that $\mathcal{V}_{\lambda_i} \neq 0$. Let L be the set of minimal elements $\lambda \in -X^+$ such that $\mathcal{V}_\lambda \neq 0$. Proposition 5.5 gives a fully faithful morphism of 2-representations

$$\bigoplus_{\lambda \in L} \mathcal{L}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}} \to \mathcal{V}$$

that is an equivalence on λ-weight spaces for $\lambda \in L$. By induction, its cokernel satisfies the conclusion of the Theorem and we are done. ☐

This theorem extends to abelian and (dg) triangulated settings, cf [Rou3].

5.1.5. *Bilinear forms.* Assume $\mathcal{V}$ is a 2-representation of $\tilde{\mathcal{A}}'$ in $\mathfrak{Tri}_k$, where k is a field endowed with a $\mathbf{k}$-algebra structure.

The action of $\tilde{\mathcal{A}}'$ on $\mathcal{V}$ induces an action of $U_1(\mathfrak{g})$ on $K_0(\mathcal{V})$. The same holds for 2-representations in abelian or exact categories.

Assume $\mathcal{V}$ is Ext-finite, *i.e.*, $\dim_k \bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{V}}(M, N[i]) < \infty$ for all $M, N \in \mathcal{V}$.

We have a pairing on $K_0(\mathcal{V})$:

$$K_0(\mathcal{V}) \times K_0(\mathcal{V}) \to \mathbf{Z},\ \langle [M], [N] \rangle = \sum_i (-1)^i \dim_k \mathrm{Hom}(M, N[i]).$$

We have

$$\langle e_s(v), v' \rangle = \langle v, f_s(v') \rangle \text{ and } \langle f_s(v), v' \rangle = \langle v, e_s(v') \rangle.$$

Note in particular that if L is a field such that the pairing is perfect on $L \otimes K_0(\mathcal{V})$, then $L \otimes K_0(\mathcal{V})$ is a semi-simple representation of $L \otimes_{\mathbf{Z}} U_1(\mathfrak{g})$.

5.2. **Simple 2-representations of $\mathfrak{sl}_2$.**

5.2.1. *Symmetrizing forms.* We put $P_i = k[X_1, \ldots, X_i]$. Fix a positive integer n. Let i be an integer with $0 \le i \le n$. We denote by $H_{i,n}$ the subalgebra of ${}^0H_n$ generated by $T_1, \ldots, T_{i-1}$ and $P_n^{\mathfrak{S}[i+1,n]}$. This is the same as the subalgebra generated by ${}^0H_i$ and $P_n^{\mathfrak{S}_n}$. We have a decomposition as abelian groups

$$H_{i,n} = {}^0H_i^f \otimes_{\mathbf{Z}} P_n^{\mathfrak{S}[i+1,n]}.$$

and a decomposition as algebras

$$H_{i,n} = {}^0H_i \otimes_{\mathbf{Z}} \mathbf{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}[i+1,n]}. \tag{12}$$

Lemma 5.8. *The algebra $H_{i,n}$ has a symmetrizing form over $P_n^{\mathfrak{S}_n}$*

$$t_i : H_{i,n} \to P_n^{\mathfrak{S}_n}(2i(i-n))$$
$$P \cdot T_w \cdot w[1,i] \mapsto \partial_{w[1,n]\cdot w[i+1,n]}(P)\delta_{w,w[1,i]}$$

for $w \in \mathfrak{S}_i$ and $P \in P_n^{\mathfrak{S}[i+1,n]}$.

Proof. The decomposition (12) shows that $H_{i,n}$ has a symmetrizing form over $P_n^{\mathfrak{S}[1,i]\times\mathfrak{S}[i+1,n]}$ given by $PT_w w[1,i] \mapsto \partial_{w[1,i]}(P)\delta_{w,w[1,i]}$ for $w \in \mathfrak{S}_i$ and $P \in P_n^{\mathfrak{S}[i+1,n]}$.

The algebra P_n has a symmetrizing form over $P_n^{\mathfrak{S}_n}$ given by $\partial_{w[1,n]}$ and a symmetrizing form over $P_n^{\mathfrak{S}[1,i]\times\mathfrak{S}[i+1,n]}$ given by $\partial_{w[1,i]}\partial_{w[i+1,n]}$. It follows from Lemma 2.12 that the algebra $P_n^{\mathfrak{S}[1,i]\times\mathfrak{S}[i+1,n]}$ has a symmetrizing form over $P_n^{\mathfrak{S}_n}$ given by $\partial_{w[1,n]\cdot w[1,i]\cdot w[i+1,n]}$. The lemma follows from Lemma 2.10. ◻

Let $e_i(\cdots)$ (resp. $h_i(\cdots)$) denote the elementary (resp. complete) symmetric functions and put $e_i = h_i = 0$ for $i < 0$.

Lemma 5.9. *The morphism $\partial_{s_{n-1}\cdots s_{i+1}}$ is a symmetrizing form for the $P_n^{\mathfrak{S}[i+1,n]}$-algebra $P_n^{\mathfrak{S}[i+2,n]}$. The set $\{X_{i+1}^j\}_{0\le j\le n-i-1}$ is a basis, with dual basis $\{(-1)^j e_{n-i-1-j}(X_{i+2}, \ldots, X_n)\}$.*

Proof. The first statement follows as in the proof of Lemma 5.8 from Lemma 2.12. We have $\partial_{s_m}(h_j(X_1, \ldots, X_m)) = -h_{j-1}(X_1, \ldots, X_{m+1})$ and $\partial_{s_m}(e_j(X_{m+1}, \ldots, X_n)) = e_{j-1}(X_m, \ldots, X_n)$.

Let $k, j \in [0, n-i-1]$. We have $e_k(X_{i+2}, \ldots, X_n)) = e_k(X_{i+1}, \ldots, X_n) - X_{i+1}e_{k-1}(X_{i+2}, \ldots, X_n)$, hence

$$\begin{aligned}\partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^j e_k(X_{i+2}, \ldots, X_n)) = &(-1)^{n+i+1} h_{j-n+i+1}(X_{i+1}, \ldots, X_n) e_k(X_{i+1}, \ldots, X_n) - \\ &- \partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^{j+1} e_{k-1}(X_{i+2}, \ldots, X_n)\end{aligned}$$

By induction, we obtain

$$\begin{aligned}\partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^j e_k(X_{i+2}, \ldots, X_n)) = &(-1)^{n+i+1}(h_{j-n+i+1}e_k - h_{j-n+i+2}e_{k-1} + \cdots + \\ &+ (-1)^k h_{j-n+i+1+k} e_0),\end{aligned}$$

where we wrote e_j and h_j for the functions in the variables $X_{i+1}, \ldots, X_n$. It follows from the fundamental relation between elementary and complete symmetric functions that

$$\partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^j e_k(X_{i+2}, \ldots, X_n)) = 0 \text{ if } j + k \neq n - i - 1$$

while

$$\partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^j e_{n-i-1-j}(X_{i+2}, \ldots, X_n)) = (-1)^j.$$

◻

5.2.2. *Induction and restriction.* We have the usual canonical adjoint pair $(\mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}}, \mathrm{Res}_{H_{i,n}}^{H_{i+1,n}})$. The symmetric forms on the algebras $H_{i,n}$ and $H_{i+1,n}$ described in Lemma 5.8 provide an adjoint pair $(\mathrm{Res}_{H_{i,n}}^{H_{i+1,n}}, \mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}})$ and we will now describe the units and counits of that pair, in terms of morphisms of bimodules.

The following proposition gives a Mackey decomposition for nil affine Hecke algebras.

Proposition 5.10. *Assume $i \le n/2$. We have an isomorphism of graded $(H_{i,n}, H_{i,n})$-bimodules*

$$\rho_i : H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \oplus \bigoplus_{j=0}^{n-2i-1} H_{i,n}(-2j) \xrightarrow{\sim} H_{i+1,n}$$

$$(a \otimes a', a_0, \ldots, a_{n-2i-1}) \mapsto aT_i a' + \sum_{j=0}^{n-2i-1} a_j X_{i+1}^j.$$

Assume $i \ge n/2$. We have an isomorphism of graded $(H_{i,n}, H_{i,n})$-bimodules

$$\rho_i : H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \xrightarrow{\sim} H_{i+1,n} \oplus \bigoplus_{j=0}^{2i-n-1} H_{i,n}(2j+2)$$

$$a \otimes a' \mapsto (aT_i a', aa', aX_i a', \ldots, aX_i^{2i-n-1} a').$$

Proof. By [ChRou, Proposition 5.32], we know that the maps above are isomorphisms after applying $-\otimes_{P_n^{\mathfrak{S}_n}} k$, where k is any field. So, the maps are isomorphisms after applying $-\otimes_{P_n^{\mathfrak{S}_n}} \mathbf{Z}$. The proposition follows now from Nakayama's Lemma. □

Let $\mathcal{B}_i$ be a basis for $H_{i,n}$ over $P_n^{\mathfrak{S}_n}$ and $\{b^\vee\}_{b \in \mathcal{B}_i}$ be the dual basis. The symmetrizing forms on $H_{i,n}$ and $H_{i+1,n}$ induce a canonical morphism of $(H_{i,n}, H_{i,n})$-bimodules, which is the Frobenius form of $H_{i+1,n}$ as an $H_{i,n}$-algebra:

$$\varepsilon_i : H_{i+1,n} \to H_{i,n}(-2(n-2i-1))$$

and a canonical morphism of $(H_{i+1,n}, H_{i+1,n})$-bimodules

$$\eta_i : H_{i+1,n} \to H_{i+1,n} \otimes_{H_{i,n}} H_{i+1,n}(2(n-2i-1)).$$

They give rise to the counit and unit of the adjoint pair $(\mathrm{Res}_{H_{i,n}}^{H_{i+1,n}}, \mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}})$. Note that $t_i \circ \varepsilon_i = t_{i+1}$.

Lemma 5.11. *Let $P \in P_n^{\mathfrak{S}[i+2,n]}$ and $w \in \mathfrak{S}_{i+1}$. We have*

$$\varepsilon_i(PT_w s_1 \cdots s_i) = \begin{cases} \partial_{s_{n-1}\cdots s_{i+1}}(P) T_{ws_1\cdots s_i} & \text{if } w \in \mathfrak{S}_i s_i \cdots s_1 \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\varepsilon_i(P) = \partial_{s_{n-1}\cdots s_{i+1}} \left(P(X_1 - X_{i+1}) \cdots (X_i - X_{i+1}) \right)$$
$$\text{and } \varepsilon_i(PT_i) = -\partial_{s_{n-1}\cdots s_{i+1}} \left(P(X_1 - X_{i+1}) \cdots (X_{i-1} - X_{i+1}) \right).$$

When $i < n/2$, we have

$$\varepsilon_i(T_i) = \varepsilon_i(X_{i+1}^j) = 0 \text{ for } j < n-2i-1 \text{ and } \varepsilon_i(X_{i+1}^{n-2i-1}) = (-1)^{n+1}.$$

When $i \geq n/2$, we have

$$\varepsilon_i(T_i) = (-1)^{n+1} X_i^{2i-n} \quad (\mathrm{mod} \bigoplus_{j=0}^{2i-n-1} P_n^{\mathfrak{S}[1,i]\times\mathfrak{S}[i+1,n]} X_i^j).$$

Proof. Let us consider the first equality. Let $f : H_{i+1,n} \to H_{i,n}$ be the linear map sending $PT_w s_1 \cdots s_i$ to the second term of the equality. Note that $f(Pa) = Pf(a)$ for all $P \in P_n^{\mathfrak{S}[i+1,n]}$ and $a \in H_{i+1,n}$.

Let $j < i$, let $P \in P_n^{\mathfrak{S}[i+2,n]}$ and let $w \in \mathfrak{S}_{i+1}$. If $w \notin \mathfrak{S}_i s_i \cdots s_1$, then

$$f(T_j PT_w s_1 \cdots s_i) = 0 = T_j f(PT_w s_1 \cdots s_i).$$

Assume now $w \in \mathfrak{S}_i s_i \cdots s_1$. Then,

$$\begin{aligned} f(T_j PT_w s_1 \cdots s_i) &= \partial_{s_{n-1}\cdots s_{i+1}}(s_j(P)) T_j T_{ws_1\cdots s_i} + \partial_{s_{n-1}\cdots s_{i+1}s_j}(P) T_{ws_1\cdots s_i} \\ &= T_j \partial_{s_{n-1}\cdots s_{i+1}}(P) T_{ws_1\cdots s_i} \\ &= T_j f(PT_w s_1 \cdots s_i). \end{aligned}$$

It follows that f is left $H_{i,n}$-linear. Since $t_i \circ f = t_{i+1}$, we obtain the first equality from Lemma 2.11.

We have

$$s_i \cdots s_1 = (X_1 - X_{i+1}) \cdots (X_i - X_{i+1}) T_i \cdots T_1 \mod F^{<(*,2i)}$$

hence $\varepsilon_i(P) = \partial_{s_{n-1}\cdots s_{i+1}}(P(X_1 - X_{i+1}) \cdots (X_i - X_{i+1}))$.

We have

$$T_i s_i \cdots s_1 = -T_i s_{i-1} \cdots s_1 = -(X_1 - X_{i+1}) \cdots (X_{i-1} - X_{i+1}) T_i \cdots T_1 \mod F^{<(*,2i)}$$

hence $\varepsilon_i(PT_i) = -\partial_{s_{n-1}\cdots s_{i+1}}(P(X_1 - X_{i+1}) \cdots (X_{i-1} - X_{i+1}))$.

The vanishing statements follow immediately from degree considerations.

Let $P = X_{i+1}^{n-2i-1}(X_1 - X_{i+1})(X_2 - X_{i+1}) \cdots (X_i - X_{i+1})$. We have

$$P = \sum_{j=0}^{i} (-1)^j X_{i+1}^{n-2i-1+j} e_{i-j}(X_1, \ldots, X_i).$$

We have $\partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^r) = 0$ for $r < n - i - 1$. It follows that

$$\varepsilon_i(X_{i+1}^{n-2i-1}) = \partial_{s_{n-1}\cdots s_{i+1}}(P) = (-1)^i \partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^{n-i-1}) = (-1)^{n+1}$$

by Lemma 5.9.

Assume $i \geq n/2$. We have

$$\begin{aligned} \varepsilon_i(T_i) &= -\partial_{s_{n-1}\cdots s_{i+1}}((X_1 - X_{i+1}) \cdots (X_{i-1} - X_{i+1})) \\ &= \sum_{j=n-i-1}^{i-1} (-1)^{j+1} \partial_{s_{n-1}\cdots s_{i+1}}(X_{i+1}^j) e_{i-1-j}(X_1, \ldots, X_{i-1}). \end{aligned}$$

By induction, we see that $e_k(X_1, \ldots, X_{i-1}) \in (-1)^k X_i^k + \sum_{j<k} P_i^{\mathfrak{S}_i} X_i^j$. It follows that

$$\varepsilon_i(T_i) = (-1)^{n+1} X_i^{2i-n} \quad (\mathrm{mod} \bigoplus_{j=0}^{2i-n-1} P_n^{\mathfrak{S}[1,i]\times\mathfrak{S}[i+1,n]} X_i^j).$$

$\square$

Lemma 5.12. *We have*

$$\eta_i(1) = T_i \cdots T_1 s_1 \cdots s_i \pi + \cdots + T_1 s_1 \cdots s_i \pi T_i \cdots T_2 + s_1 \cdots s_i \pi T_i \cdots T_1,$$

where $\pi = \sum_{j=0}^{n-i-1} (-1)^j e_{n-i-1-j}(X_{i+2}, \ldots, X_n) \otimes X_{i+1}^j$.

Let $P \in P_n^{\mathfrak{S}_i}$. *We have*

$$m((1 \otimes P \otimes 1 \otimes 1)\eta_i(1)) = (-1)^{n+1} \partial_{s_1 \cdots s_i}(P(X_{i+1} - X_{i+2}) \cdots (X_{i+1} - X_n))$$

and

$$m((1 \otimes T_{i+1} P \otimes 1 \otimes 1)\eta_i(1)) = (-1)^n \partial_{s_1 \cdots s_i}(P(X_{i+1} - X_{i+3}) \cdots (X_{i+1} - X_n)).$$

When $i > n/2 - 1$, *we have*

$$m((1 \otimes X_{i+1}^j \otimes 1 \otimes 1)\eta_i(1)) = m((1 \otimes T_{i+1} \otimes 1 \otimes 1)\eta_i(1)) = 0 \textit{ for } j < 2i - n + 1$$
$$\textit{and } m((1 \otimes X_{i+1}^{2i-n+1} \otimes 1 \otimes 1)\eta_i(1)) = (-1)^{n+1}.$$

When $i \leq n/2 - 1$, *we have*

$$m((1 \otimes T_{i+1} \otimes 1 \otimes 1)\eta_i(1)) = (-1)^n X_{i+2}^{n-2i-2} \pmod{\bigoplus_{j=0}^{n-2i-3} P_n^{\mathfrak{S}[1,i+1] \times \mathfrak{S}[i+2,n]} X_{i+2}^j}.$$

Proof. Let $\mathcal{B}$ be a basis for $\mathbf{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}[i+2,n]}$ over $\mathbf{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}[i+1,n]}$ and $\mathcal{B}^\vee$ the dual basis for the symmetrizing form $\partial_{s_{n-1} \cdots s_{i+1}}$. Let $\pi = \sum_{a \in \mathcal{B}} a^\vee \otimes a$ be the Casimir element. Let $R = \{1, T_i, \ldots, T_i \cdots T_1\}$, a basis of ${}^0H_{i+1}$ over ${}^0H_i$. Its dual basis for the Frobenius form

$$T_w \mapsto \begin{cases} T_{ws_1 \cdots s_i} & \text{if } w \in \mathfrak{S}_i s_i \cdots s_1 \\ 0 & \text{otherwise} \end{cases}$$

is given by $\{1^\vee = T_i \cdots T_1, \ldots, (T_i \cdots T_2)^\vee = T_1, (T_i \cdots T_1)^\vee = 1\}$. It follows from Lemmas 5.11 and 2.11 that

$$T_w s_1 \cdots s_i \mapsto \begin{cases} T_{ws_1 \cdots s_i} & \text{if } w \in \mathfrak{S}_i s_i \cdots s_1 \\ 0 & \text{otherwise.} \end{cases}$$

extends to a Frobenius form for the $\left({}^0H_i \otimes \mathbf{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}[i+2,n]}\right)$-algebra $H_{i+1,n}$ for which the basis dual to R is $\{h^\vee s_1 \cdots s_i\}_{h \in R}$. Then, $\{ah\}_{a \in \mathcal{B}, h \in R}$ is a basis of $H_{i+1,n}$ as an $H_{i,n}$-module. Furthermore, the dual basis for the Frobenius form ε_i is $\{h^\vee s_1 \cdots s_i a^\vee\}_{a \in \mathcal{B}, h \in R}$ (cf Lemma 5.11). We have

$$\eta_i(1) = T_i \cdots T_1 s_1 \cdots s_i \pi + \cdots + T_1 s_1 \cdots s_i \pi T_i \cdots T_2 + s_1 \cdots s_i \pi T_i \cdots T_1$$

and the first statement of the lemma follows from Lemma 5.9. We deduce that

$$T_{w[1,i+2]} \eta_i(1) = T_{w[1,i+2]} s_1 \cdots s_i \pi T_i \cdots T_1 = (-1)^i T_{w[1,i+2]} \pi T_i \cdots T_1.$$

Let $b \in H_{i+2,n}^{H_{i,n}}$. Define

$$f(b) = m((1 \otimes b \otimes 1 \otimes 1)\eta_i(1)) = \sum_{a \in \mathcal{B}, h \in R} h^\vee s_1 \cdots s_i a^\vee b a h \in H_{i+2,n}.$$

We have

$$T_{w[1,i+2]} f(b) = (-1)^i T_{w[1,i+2]} \sum_{a \in \mathcal{B}} a^\vee b a T_i \cdots T_1.$$

Since $\deg(\pi) = 2(n-i-1)$, it follows that

$$T_{w[1,i+2]} f(b) = 0 \text{ for } b \in F^{<(2i-n+1,*)}.$$

We have

$$\left(\mathbf{Q}(X_1,\ldots,X_n)^{\mathfrak{S}[i+3,n]} \rtimes \mathfrak{S}_{i+2}\right)^{\mathbf{Q}(X_1,\ldots,X_n)^{\mathfrak{S}[i+2,n]} \rtimes \mathfrak{S}_{i+1}} = \mathbf{Q}(X_1,\ldots,X_n)^{\mathfrak{S}[1,i+1]\times\mathfrak{S}[i+3,n]}$$

and

$$H_{i+2,n}^{H_{i+1,n}} = P_n^{\mathfrak{S}[1,i+1]\times\mathfrak{S}[i+3,n]}.$$

We deduce that given $b \in H_{i+2,n}^{H_{i,n}}$, then $f(b) \in P_n$. Note that left multiplication by $T_{w[1,i+2]}$ is injective on P_n.

We have $m(\pi) = (X_{i+2} - X_{i+1}) \cdots (X_n - X_{i+1})$ by Lemma 3.1. Let $P \in P_n^{\mathfrak{S}_i}$. We have $T_{w[1,i+2]} f(P) = (-1)^i T_{w[1,i+2]} \partial_{s_1\cdots s_i}(P(X_{i+2}-X_{i+1})\cdots(X_n-X_{i+1}))$, hence

$$f(P) = (-1)^{n+1} \partial_{s_1\cdots s_i}(P(X_{i+1}-X_{i+2})\cdots(X_{i+1}-X_n)).$$

We take now $\mathcal{B} = \{X_{i+1}^j\}_{0\le j\le n-i-1}$, cf Lemma 5.9. We have

$$T_{w[1,i+2]} f(T_{i+1}P) = (-1)^i T_{w[1,i+2]} \sum_{j=0}^{n-i-2} (-1)^j e_{n-i-2-j}(X_{i+3},\ldots,X_n) P X_{i+1}^j T_i \cdots T_1$$

hence

$$f(T_{i+1}P) = (-1)^n \partial_{s_{n-1}\cdots s_i}(P(X_{i+1}-X_{i+3})\cdots(X_{i+1}-X_n)).$$

Assume $i > n/2 - 1$. The vanishing statements are immediate consequences of the previous two equalities of the Lemma.

We have

$$\begin{aligned} f(X_{i+1}^{2i-n+1}) &= (-1)^{n+1} \partial_{s_1\cdots s_i}(X_{i+1}^{2i-n+1}(X_{i+1}-X_{i+2})\cdots(X_{i+1}-X_n)) \\ &= (-1)^{n+1} \partial_{s_1\cdots s_i}(X_{i+1}^i) = (-1)^{n+1}. \end{aligned}$$

Assume $i \le n/2 - 1$. We have

$$\begin{aligned} f(T_{i+1}) &= (-1)^n \partial_{s_1\cdots s_i}((X_{i+1}-X_{i+3})\cdots(X_{i+1}-X_n)) \\ &= \sum_{j=i}^{n-i-2} (-1)^{i-j} \partial_{s_1\cdots s_i}(X_{i+1}^j) e_{n-i-2-j}(X_{i+3},\ldots,X_n). \end{aligned}$$

By induction, we see that $e_k(X_{i+3},\ldots,X_n) \in (-1)^k X_{i+2}^k + \sum_{j<k} \mathbf{Z}[X_{i+2},\ldots,X_n]^{\mathfrak{S}[i+2,n]} X_{i+2}^j$. Consequently,

$$f(T_{i+1}) = (-1)^n X_{i+2}^{n-2i-2} \pmod{\bigoplus_{j=0}^{n-2i-3} P_n^{\mathfrak{S}[1,i+1]\times\mathfrak{S}[i+2,n]} X_{i+2}^j}.$$

□

As a consequence of Lemmas 5.11 and 5.12, we obtain a description of the units and counits η_i and ε_i through the isomorphisms of Proposition 5.10.

Proposition 5.13. *If $i < n/2$ then we have a commutative diagram*

$$\begin{tikzcd} H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \oplus H_{i,n} \oplus H_{i,n}(-2) \oplus \cdots \oplus H_{i,n}(-2(n-2i-1)) \arrow[r, "\rho_i", "\sim"'] \arrow[dr, "{(0,0,\ldots,0,(-1)^{n+1})}"'] & H_{i+1,n} \arrow[d, "\varepsilon_i"] \\ & H_{i,n}(-2(n-2i-1)) \end{tikzcd}$$

If $i \geq n/2$ then the image of $\varepsilon_i \circ \rho_i$ in

$$\mathrm{Hom}_{H_{i,n},H_{i,n}}(H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2), H_{i,n}(2(2i-n+1)))/ \bigoplus_{j=0}^{2i-n-1} (a \otimes a' \mapsto aX_i^j a') \cdot Z(H_{i,n})_{2(2i-n-j)}$$

is equal to the image of the map $a \otimes a' \mapsto (-1)^{n+1} a X_i^{2i-n} a'$.

If $i \leq n/2 - 1$ then the image of $\rho_{i+1} \circ \eta_i$ in

$$\mathrm{Hom}_{H_{i+1,n},H_{i+1,n}}(H_{i+1,n}, H_{i+2,n}(2(n-2i-2)))/ \bigoplus_{j=0}^{n-2i-3} X_{i+2}^j Z(H_{i+1,n})_{2(n-2i-2-j)}$$

is equal to $(-1)^n X_{i+2}^{n-2i-2}$.

If $i > n/2 - 1$ then we have a commutative diagram

$$\begin{tikzcd} H_{i+1,n} \arrow[r, "\eta_i"] \arrow[dr, "{(0,0,\ldots,0,(-1)^{n+1})}"'] & H_{i+1,n} \otimes_{H_{i,n}} H_{i+1,n}(2(n-2i-1)) \arrow[d, "\rho_{i+1}", "\sim"'] \\ & H_{i+2,n}(2(n-2i-2)) \oplus H_{i+1,n}(2(n-2i-1)) \oplus H_{i+1,n}(2(n-2i+1)) \oplus \cdots \oplus H_{i+1,n} \end{tikzcd}$$

5.2.3. $\mathfrak{sl}_2$-*action.* Let $\tilde{\mathcal{V}}(-n)_\lambda = H_{(n+\lambda)/2,n}$-free for $\lambda \in \{-n, -n+2, \ldots, n-2, n\}$. We define $E = \bigoplus_{i=0}^{n-1} \mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}}$ and $F = \bigoplus_{i=0}^{n-1} \mathrm{Res}_{H_{i,n}}^{H_{i+1,n}}$. We have a canonical adjunction (E, F). Multiplication by X_{i+1} induces an endomorphism of $\mathrm{Ind}_{H_{i,n}}^{H_{i+1,n}}$ and taking the sum over all i, we obtain an endomorphism x of E. Similarly, T_{i+1} induces an endomorphism of $\mathrm{Ind}_{H_{i,n}}^{H_{i+2,n}}$ and we obtain an endomorphism τ of E^2. Propositions 5.10 and 5.13 show that this endows $\tilde{\mathcal{V}}(-n) = \bigoplus_\lambda \tilde{\mathcal{V}}(-n)_\lambda$ with an action of $\bar{\mathfrak{A}}'$.

Let $R : \mathcal{V}(-n) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_{-n})} P_n^{\mathfrak{S}_n} \to \tilde{\mathcal{V}}(-n)$ be the morphism of 2-representations associated with $M = P_n^{\mathfrak{S}_n} \in \tilde{\mathcal{V}}(-n)_{-n}$.

Proposition 5.14. *The canonical map* $\mathrm{End}(\bar{\mathbf{1}}_{-n}) \to P_n^{\mathfrak{S}_n}$ *is an isomorphism and R induces an isomorphism of* 2*-representations of* $\mathfrak{A}$

$$\mathcal{V}(-n) \xrightarrow{\sim} \tilde{\mathcal{V}}(-n).$$

In particular, the action of $\mathfrak{A}$ on $\mathcal{V}(-n)$ extends uniquely to an action of $\bar{\mathfrak{A}}'$.

Proof. The canonical map $\bar{\mathbf{1}}_{-n} \to F^{(n)} E^{(n)} \bar{\mathbf{1}}_{-n}$ is an isomorphism by Lemma 4.13. It follows that the canonical map is an isomorphism $\mathrm{End}(\bar{\mathbf{1}}_{-n}) \to \mathrm{End}(E^{(n)} \bar{\mathbf{1}}_{-n})$. We have a commutative diagram of canonical morphisms of $\mathrm{End}(\bar{\mathbf{1}}_{-n})$-algebras

$$\begin{tikzcd}
& \mathrm{End}(\bar{\mathbf{1}}_{-n}) \arrow[dl, "\sim"'] \arrow[d] \arrow[dr] & \\
\mathrm{End}(E^{(n)}\bar{\mathbf{1}}_{-n}) \arrow[r] & \mathrm{End}(E^{(n)}M) & P_n^{\mathfrak{S}_n} \arrow[l, "\sim"'] \arrow[ll, bend left, "\sim"]
\end{tikzcd}$$

and it follows that all maps in the diagram are isomorphisms. The proposition follows. □

5.3. **Construction of representations.** In this section, we show that, for integrable representations, certain axioms are consequences of others.

5.3.1. *Biadjointness.*

Theorem 5.15. *The canonical strict 2-functor $\mathfrak{A} \to \mathfrak{A}'$ induces an equivalence from the 2-category of integrable 2-representations of $\mathfrak{A}'$ to the 2-category of integrable 2-representations of $\mathfrak{A}$.*

Proof. It is enough to consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Assume $\lambda \geq 0$. Let $\tilde{\varepsilon}_{\mathrm{Id}_{\mathcal{V}_\lambda} F} : FE\,\mathrm{Id}_{\mathcal{V}_\lambda} \to \mathrm{Id}_{\mathcal{V}_\lambda}$ be the map whose image under

$$\mathrm{Hom}(\sigma_\lambda, \mathrm{Id}_{\mathcal{V}_\lambda}) : \mathrm{Hom}(FE\,\mathrm{Id}_{\mathcal{V}_\lambda}, \mathrm{Id}_{\mathcal{V}_\lambda}) \xrightarrow{\sim} \mathrm{Hom}(EF\,\mathrm{Id}_{\mathcal{V}_\lambda}, \mathrm{Id}_{\mathcal{V}_\lambda})/\left(\bigoplus_{i=0}^{\lambda-1} Z(\mathcal{V}_\lambda)\cdot\varepsilon\circ(x^iF)\right)$$

coincides with $(-1)^{\lambda+1}\varepsilon\circ(x^\lambda F)$.

Assume $\lambda \geq -1$. Let $\hat{\eta}_{\mathrm{Id}_{\mathcal{V}_\lambda} F} : \mathrm{Id}_{\mathcal{V}_{\lambda+2}} \to EF\,\mathrm{Id}_{\mathcal{V}_{\lambda+2}}$ be the unique morphism such that

$$\rho_{\lambda+2}\circ\eta_{\mathrm{Id}_{\mathcal{V}_\lambda} F} = (0, 0, \ldots, 0, (-1)^{\lambda+1}).$$

Assume $\lambda \leq -2$. Let $\hat{\eta}_{\mathrm{Id}_{\mathcal{V}_\lambda} F} : \mathrm{Id}_{\mathcal{V}_{\lambda+2}} \to EF\,\mathrm{Id}_{\mathcal{V}_{\lambda+2}}$ be the map whose image under

$$\mathrm{Hom}(\mathrm{Id}_{\mathcal{V}_{\lambda+2}}, \rho_{\lambda+2}) : \mathrm{Hom}(\mathrm{Id}_{\mathcal{V}_{\lambda+2}}, EF\,\mathrm{Id}_{\mathcal{V}_{\lambda+2}}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Id}_{\mathcal{V}_{\lambda+2}}, FE\,\mathrm{Id}_{\mathcal{V}_{\lambda+2}})/\left(\bigoplus_{i=0}^{-3-\lambda}(Fx^i)\circ\eta\cdot Z(\mathcal{V}_{\lambda+2})\right)$$

coincides with $(-1)^\lambda(Fx^{-\lambda-2})\circ\eta$.

Assume $\lambda < 0$. Let $\tilde{\varepsilon}_{\mathrm{Id}_{\mathcal{V}_\lambda} F} : FE\,\mathrm{Id}_{\mathcal{V}_\lambda} \to \mathrm{Id}_{\mathcal{V}_\lambda}$ be the unique morphism such that

$$\varepsilon_{\mathrm{Id}_{\mathcal{V}_\lambda} F}\circ\rho_\lambda = (0, 0, \ldots, 0, (-1)^{\lambda+1}).$$

The theorem will follow from the fact that the maps $\tilde{\varepsilon}_{\mathrm{Id}_{\mathcal{V}_\lambda} F}$ are the units of an adjoint pair $(\mathrm{Id}_{\mathcal{V}_\lambda} F, E\,\mathrm{Id}_{\mathcal{V}_\lambda})$. Note that the same will hold for the maps $\hat{\eta}_{\mathrm{Id}_{\mathcal{V}_\lambda} F}$.

It is enough to show that

$$(\tilde{\varepsilon}F)\circ(F\hat{\eta}_F) \text{ and } (E\tilde{\varepsilon}_F)\circ(\hat{\eta}_F E) \text{ are invertible.} \tag{13}$$

Note that this holds for $\mathcal{V} = \mathcal{V}(\lambda)$ by Proposition 5.13.

Let $M \in \mathcal{V}_\lambda$ such that $FM = 0$. Proposition 5.5 provides a fully faithful morphism of 2-representations

$$R : \mathcal{V}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathrm{End}(M) \to \mathcal{V}$$

with $R(\bar{\mathbf{1}}_\lambda) \simeq M$, hence (13) holds on $E^i M$, for all i. Applying this to $K^b(\mathcal{V})$ shows that (13) holds (Lemma 5.3). $\square$

5.3.2. *$\mathfrak{sl}_2$-categorifications.* Let k be a field.

Definition 5.16. *Let $\mathcal{V} \in \mathfrak{Ab}_k^f$. An $\mathfrak{sl}_2$-categorification on $\mathcal{V}$ is the data of*
- *an adjoint pair (E, F) of exact functors $\mathcal{V} \to \mathcal{V}$*
- *$X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$*

such that
- *the action of $[E]$ and $[F]$ on $K_0(\mathcal{V})$ give a locally finite representation of $\mathfrak{sl}_2$*
- *classes of simple objects are weight vectors*
- *F is isomorphic to a left adjoint of E*
- *X has a single eigenvalue*
- *the action on E^n of $X_i = E^{n-i}XE^{i-1}$ for $1 \leq i \leq n$ and of $T_i = E^{n-i-1}TE^{i-1}$ for $1 \leq i \leq n-1$ induce an action of an affine Hecke algebra with $q \neq 1$, a degenerate affine Hecke algebra or a nil affine Hecke algebra of GL_n.*

Note that the three types of actions (affine Hecke with $q \neq 1$, degenerate affine Hecke and nil affine Hecke) are equivalent by Theorems 3.17 and 3.19. Only T needs to be changed, as follows:

$$\text{affine} \longleftrightarrow \text{nil} \qquad\qquad \text{degenerate} \longleftrightarrow \text{nil}$$

$$T \longmapsto (qEX - XE)T + q \qquad\qquad T \longmapsto (EX - XE + 1)T + 1$$

Note also that, in the nil case, if a is the eigenvalue of X, then by replacing X by $X - a$ one reaches the case where 0 is the eigenvalue of X. As a consequence, given an $\mathfrak{sl}_2$-categorification, one can construct a new categorification by modifying X and T as above so that the action of X and T induce an action of the nil affine Hecke algebra ${}^0H_n$ on $\mathrm{End}(E^n)$ and X is locally nilpotent.

In [ChRou], the case of nil affine Hecke algebras wasn't considered. The equivalence of the definitions explained above shows that the results of [ChRou] generalize to this setting. It can also be seen directly that all constructions, results and proofs in [ChRou] involving degenerate affine Hecke algebras carry over to nil affine Hecke algebras. A key point is the commutation relation between T_i and a polynomial: that relation is the same for the degenerate affine Hecke algebra and the nil affine Hecke algebra. The definition of c_n^τ [ChRou, §3.1.4] needs to be modified: we define $c_n = T_{w[1,n]}$. Note that $T_{w[1,n]}^2 = 0$ for $n \geq 2$. Given M a projective $k({}^0H_n^f)$-module, we have $c_n M = \{m \in M \mid T_w m = 0 \text{ for all } w \in \mathfrak{S}_n - \{1\}\}$.

Remark 5.17. We haven't included the parameters a and q in the definition, as they are not needed here.

Theorem 5.18. *Let k be a field and $\mathcal{V} \in \mathfrak{Ab}_k^f$ Assume given an $\mathfrak{sl}_2$-categorification on $\mathcal{V}$. Let $x = X$ and*

$$\tau = \begin{cases} (qEX - XE)^{-1}(T - q) & \textit{affine case} \\ (EX - XE + 1)^{-1}(T - 1) & \textit{degenerate affine case} \\ T & \textit{nil affine case.} \end{cases}$$

This defines a 2*-representation of* $\mathfrak{A}(\mathfrak{sl}_2)$ *on* $\mathcal{V}$.

Conversely, a integrable action of $\mathfrak{A}(\mathfrak{sl}_2)$ *on* $\mathcal{V}$ *gives rise to an* $\mathfrak{sl}_2$*-categorification on* $\mathcal{V}$.

This provides an equivalence between the 2*-category of* $\mathfrak{sl}_2$*-categorifications and the* 2*-category of integrable* 2*-representations of* $\mathfrak{A}(\mathfrak{sl}_2)$ *in* $\mathfrak{Ab}_k^f$.

Proof. By [ChRou, Theorem 5.27], the maps $\rho_{s,\lambda}$ are invertible. So, the result follows from Theorem 5.15. □

In the isotypic case, we have a stronger result:

Theorem 5.19. *Let* k *be a field and* $\mathcal{V} \in \mathfrak{Ab}_k^f$. *Assume given an* $\mathfrak{sl}_2$*-categorification on* $\mathcal{V}$ *such that* $\mathbf{C} \otimes K_0(\mathcal{V})$ *is a multiple of an irreducible representation of* $\mathfrak{sl}_2(\mathbf{C})$. *Then, the construction of Theorem 5.18 gives rise to an action of* $\bar{\mathfrak{A}}'(\mathfrak{sl}_2)$ *on* $\mathcal{V}$.

Proof. Theorems 5.18 and 5.15 provide an action of $\mathfrak{A}'$. Let $\lambda \in X$ minimum such that $\mathcal{V}_\lambda \neq 0$. Note that the Theorem holds for $\mathcal{V}(\lambda)$ by Proposition 5.13.

Let $N \in \mathcal{V}_{\lambda+2i}$ for some $i \geq 0$. Let N' be the cokernel of $\varepsilon_i(N) : E^i F^i N \to N$. We have $F^i N' = 0$, hence $[N'] = 0$ in $K_0(\mathcal{V})$ since the only non-zero elements of $\mathbf{C} \otimes K_0(\mathcal{V})$ killed by $[F]$ are in the λ-weight space. So, $N' = 0$ and we deduce that N is a quotient of $E^i(F^i(N))$.

Let $M \in \mathcal{V}_\lambda$ such that $FM = 0$. Proposition 5.5 provides a fully faithful morphism of 2-representations

$$R : \mathcal{V}(\lambda) \otimes_{\mathrm{End}(\bar{\mathbf{1}}_\lambda)} \mathrm{End}(M) \to \mathcal{V}$$

with $R(\bar{\mathbf{1}}_\lambda) \simeq M$. Since the Theorem holds for $\mathcal{V}(\lambda)$, it follows that the relations defining $\bar{\mathfrak{A}}'$ hold when applied to $E^i M$, for every i. It follows that they hold for every quotient of $E^i M$. We deduce that the relations hold on $\mathcal{V}$. □

5.3.3. *Involution* ι. Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}'$ in $\mathfrak{Lin}_{\mathbf{k}}$.

Let $(\mathcal{V}^\iota)_\lambda = \mathcal{V}_{-\lambda}$, let $E_s^\iota = F_s$ and $F_s^\iota = E_s$. Let $x_s^\iota \in \mathrm{End}(E_s^\iota)$ corresponding to $x_s \in \mathrm{End}(E_s) \xrightarrow{\sim} \mathrm{End}(F_s)$ and let $\tau_{st}^\iota \in \mathrm{Hom}(E_s^\iota E_t^\iota, E_t^\iota E_s^\iota)$ corresponding to $-\tau_{st} \in \mathrm{Hom}(E_s E_t, E_t E_s) \xrightarrow{\sim} \mathrm{Hom}(F_s F_t, F_t F_s)$.

The adjunction (F_s, E_s) gives an adjoint pair (E_s^ι, F_s^ι).

Proposition 5.20. *The construction above defines a* 2*-representation of* $\mathfrak{A}'$ *on* $\mathcal{V}^\iota$.

Proof. The relations (1)-(4) in §4.1.1 are clear. Let us show that the maps $\rho_{s,\lambda}$ on $\mathcal{V}^\iota$ are isomorphisms. As in the proof of Theorem 5.15, it is enough to do so for $\mathcal{V} = \mathcal{V}(-n)$ for some $n > 0$.

Given a field k, consider the canonical 2-representation of $\mathfrak{A}'$ on $\mathcal{W} = \bigoplus_i \left(H_{i,n} \otimes_{P_n^{\mathfrak{S}_n}} k\right)$-mod. The category $\mathcal{W}^\iota$ is endowed with a structure of $\mathfrak{sl}_2$-categorification. It follows from Theorem 5.18 that the maps $\rho_{s,\lambda}$ are isomorphisms for $\mathcal{W}^\iota$.

We conclude now as in the proof of Proposition 5.10 that the maps $\rho_{s,\lambda}$ are isomorphisms for $\mathcal{V}(-n)^\iota$.

We are left with proving the invertibility of σ_{st} for $s \neq t$. This is a consequence of Theorem 5.21 below. □

5.3.4. *Relation* $[E_s, F_t] = 0$ *for* $s \neq t$. Let $\{\mathcal{V}\}_{\lambda \in X}$ be a family of $\mathbf{k}$-linear categories endowed with the data of

- functors $E_s : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha_s}$ and $F_s : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_s}$ for $s \in I$
- $x_s \in \mathrm{End}(E_s)$ and $\tau_{st} \in \mathrm{Hom}(E_s E_t, E_t E_s)$ for $s, t \in I$

- an adjunction (E_s, F_s)

such that

- relations (1)-(4) in §4.1.1 hold
- the maps $\rho_{s,\lambda}$ are isomorphisms

Theorem 5.21. *The data above defines a 2-representation of $\mathfrak{A}'$ on $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$.*

Theorem 5.15 provides maps ε^l and η^l and we only have to show the invertibility of the maps $\sigma_{s,t}$ for any $s \neq t \in I$. Note that the construction of §5.3.3 provide a category $\mathcal{V}^\iota$ satisfying the same properties as the category $\mathcal{V}$.

Let $s \neq t \in I$. We write $Q_{ts}(u,v) = \sum_{a,b} q_{ab} u^a v^b$ with $q_{a,b} \in k$. Let $\lambda \in X$ and $r \geq 0$. Consider the morphism

$$\psi : {}^0H_{r+1} \to \mathrm{End}(E_s E_t^r \mathbf{1}_\lambda)$$

$$h \mapsto (E_s E_t^r \xrightarrow{\eta\bullet} F_t E_t E_s E_t^r \xrightarrow{F_t \tau_{ts}\bullet} F_t E_s E_t^{r+1} \xrightarrow{F_t h} F_t E_s E_t^{r+1} \xrightarrow{F_t \tau_{st}\bullet} F_t E_t E_s E_t^r \xrightarrow{\varepsilon^l\bullet} E_s E_t^r).$$

Lemma 5.22. *Let $a \leq -\langle \alpha_t^\vee, \lambda\rangle - r - 1$. We have*

$$\psi(X_{r+1}^a T_{w[1,r+1]}) = \begin{cases} T_{w[1,r]} q_{m_{ts},0} (-1)^{\langle \alpha_t^\vee, \lambda\rangle + m_{ts}+1} & \text{if } a = -\langle \alpha_t^\vee, \lambda\rangle - r - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$(\tau_{st} E_t^r) \circ (E_s \tau_{tt} E_t^{r-1}) \circ (\tau_{ts} E_t^r) =$$
$$= ((E_t \tau_{ts}) \circ (\tau_{tt} E_t) \circ (E_t \tau_{st})) + \sum_{\substack{\beta \geq 0 \\ m_{ts} > \alpha_1 + \alpha_2 \geq 0}} q_{\alpha_1+\alpha_2+1,\beta} (X^{\alpha_1} E_s E_t)(E_t X^\beta E_t)(E_t E_s X^{\alpha_2}).$$

Let

$$f_a = (E_s E_t^r \xrightarrow{\tau_{st}\bullet} E_t E_s E_t^{r-1} \xrightarrow{\eta\bullet} F_t E_t^2 E_s E_t^{r-1} \xrightarrow{F_t (X^a E_t \circ T)\bullet} F_t E_t^2 E_s E_t^{r-1} \xrightarrow{\varepsilon^l\bullet} E_t E_s E_t^{r-1} \xrightarrow{\tau_{ts}\bullet} E_s E_t^r)$$

and

$$g_b = E_s E_t^r \xrightarrow{\eta\bullet} F_t E_t E_s E_t^r \xrightarrow{F_t X_{r+1}^b\bullet} F_t E_t E_s E_t^r \xrightarrow{\varepsilon^l\bullet} E_s E_t^r.$$

We have $X_{r+1}^a T_{w[1,r+1]} = T_{w[1,r]} X_{r+1}^a T_r \cdots T_1$, hence

$$\psi(X_{r+1}^a T_{w[1,r+1]}) = T_{w[1,r]} f_a T_{r-1} \cdots T_1 + \sum_{\substack{\beta \geq 0 \\ \alpha_1+\alpha_2 \geq 1}} q_{\alpha_1+\alpha_2+1,\beta} T_{w[1,r]} g_{a+\alpha_1} \partial_{s_1 \cdots s_{r-1}}(X_r^{\alpha_2})(X^\beta E_t^r).$$

We have a commutative diagram (Lemma 4.10 and Chevalley duality)

$$\begin{tikzcd} E_t \arrow[r, "\eta E_t"] \arrow[d, "E_t \eta"'] & F_t E_t^2 \arrow[d, "F_t T"] & \\ E_t F_t E_t \arrow[r, "\sigma E_t"'] & F_t E_t^2 \arrow[r, "\varepsilon^l E_t"] & E_t \end{tikzcd} \tag{14}$$

- Assume first $\langle \alpha_t^\vee, \lambda\rangle + 2r - m_{ts} < 0$. The diagram (14) shows that the composition

$$E_t E_s E_t^{r-1} \xrightarrow{\eta\bullet} F_t E_t^2 E_s E_t^{r-1} \xrightarrow{F_t T\bullet} F_t E_t^2 E_s E_t^{r-1} \xrightarrow{\varepsilon^l\bullet} E_t E_s E_t^{r-1}$$

vanishes. Since $X_2^a T = TX_1^a + \sum_{l=0}^{a-1} X_2^l X_1^{a-1-l}$, it follows that

$$E_tE_sE_t^{r-1} \xrightarrow{\eta\bullet} F_tE_t^2E_sE_t^{r-1} \xrightarrow{F_t(X^aE_t\circ T)\bullet} F_tE_t^2E_sE_t^{r-1} \xrightarrow{\varepsilon^l\bullet} E_tE_sE_t^{r-1}$$

equals

$$\sum_{l=0}^{a-1}(\varepsilon^l\circ(F_tX^l)\circ\eta)X^{a-1-l}E_sE_t^{r-1},$$

hence

$$f_a = \sum_{\substack{0\le l\le a-1\\ 0\le\alpha\le m_{ts}\\ 0\le\beta\le m_{st}}} T_{w[1,r]}q_{\alpha,\beta}X_{E_s}^\beta(\varepsilon^l\circ(F_tX^l)\circ\eta)\partial_{s_1\cdots s_{r-1}}(X_r^{a-1-l+\alpha})$$

If $\varepsilon^l\circ(F_tX^l)\circ\eta\neq 0$, then $l\ge -\langle\alpha_t^\vee,\lambda\rangle-2r+m_{ts}-1$. If $\partial_{s_1\cdots s_{r-1}}(X_r^{a-1-l+\alpha})\neq 0$, then $a-1-l+\alpha\ge r-1$. If both of those terms are non zero, then $a\ge -\langle\alpha_t^\vee,\lambda\rangle-r+(m_{ts}-\alpha)-1$, hence $a=-\langle\alpha_t^\vee,\lambda\rangle-r-1$, $\alpha=m_{ts}$ and $a-1-l=r-1-m_{ts}$. In particular, we have $r>m_{ts}$. So, we have

$$f_a = \begin{cases} T_{w[1,r]}q_{m_{ts},0}(-1)^{\langle\alpha_t^\vee,\lambda\rangle+m_{ts}+1} & \text{if } a=-\langle\alpha_t^\vee,\lambda\rangle-r-1 \text{ and } r>m_{ts}\\ 0 & \text{otherwise.}\end{cases}$$

If $g_{a+\alpha_1}\partial_{s_1\cdots s_{r-1}}(X_r^{\alpha_2})\neq 0$, then $a+\alpha_1\ge -\langle\alpha_t^\vee,\lambda\rangle-2r+m_{ts}-1$ and $\alpha_2\ge r-1$, hence $a\ge -\langle\alpha_t^\vee,\lambda\rangle-r-2+(m_{ts}-\alpha_1-\alpha_2)\ge -\langle\alpha_t^\vee,\lambda\rangle-r-1$. We obtain $a=-\langle\alpha_t^\vee,\lambda\rangle-r+(m_{ts}-\alpha)-1$, $\alpha_2=r-1$ and $\alpha_1+\alpha_2=m_{ts}-1$. In particular, $r\le m_{ts}$. So, we have

$$\sum_{\substack{\beta\ge 0\\ \alpha_1+\alpha_2\ge 1}} q_{\alpha_1+\alpha_2+1,\beta}T_{w[1,r]}g_{a+\alpha_1}\partial_{s_1\cdots s_{r-1}}(X_r^{\alpha_2})(X^\beta E_t^r) =$$

$$= \begin{cases} T_{w[1,r]}q_{m_{ts},0}(-1)^{\langle\alpha_t^\vee,\lambda\rangle+m_{ts}+1} & \text{if } a=-\langle\alpha_t^\vee,\lambda\rangle-r-1 \text{ and } r\le m_{ts}\\ 0 & \text{otherwise.}\end{cases}$$

So, we have shown that

$$\psi(X_{r+1}^aT_{w[1,r+1]}) \begin{cases} T_{w[1,r]}q_{m_{ts},0}(-1)^{\langle\alpha_t^\vee,\lambda\rangle+m_{ts}+1} & \text{if } a=-\langle\alpha_t^\vee,\lambda\rangle-r-1\\ 0 & \text{otherwise.}\end{cases}$$

• Assume now $\langle\alpha_t^\vee,\lambda\rangle+2r-m_{ts}\ge 0$. We can assume that $\langle\alpha_t^\vee,\lambda\rangle+r<0$, for otherwise the lemma is empty. So, we have $r>m_{ts}$.

If $\partial_{s_1\cdots s_{r-1}}(X_r^{\alpha_2})\neq 0$, then $\alpha_2\ge r-1$, hence $m_{ts}\ge r$, which is impossible. So,

$$\sum_{\substack{\beta\ge 0\\ \alpha_1+\alpha_2\ge 1}} q_{\alpha_1+\alpha_2+1,\beta}T_{w[1,r]}g_{a+\alpha_1}\partial_{s_1\cdots s_{r-1}}(X_r^{\alpha_2})(X^\beta E_t^r) = 0.$$

Let $\mu=\lambda+r\alpha_t+\alpha_s$. The diagram (14) shows that there are elements $z_i\in Z(\mathcal{V}_\mu)$ with $z_{\langle\alpha_t^\vee,\mu\rangle}=(-1)^{\langle\alpha_t^\vee,\mu\rangle+1}$ such that

$$(E_t\mathbf{1}_\mu \xrightarrow{\eta\bullet} F_tE_t^2\mathbf{1}_\mu \xrightarrow{F_tT\bullet} F_tE_t^2\mathbf{1}_\mu \xrightarrow{\varepsilon^l\bullet} E_t\mathbf{1}_\mu) = \sum_{i=0}^{\langle\alpha_t^\vee,\mu\rangle} z_iX^i.$$

So,

$$f_a = \sum_{\substack{0\le l\le a-1\\ 0\le\alpha\le m_{ts}\\ 0\le\beta\le m_{st}}} T_{w[1,r]} q_{\alpha,\beta} X_{E_s}^{\beta} (\varepsilon^l \circ (F_t X^l) \circ \eta) \partial_{s_1\cdots s_{r-1}} (X_r^{a-1-l+\alpha}) +$$

$$+ \sum_{\substack{0\le\alpha\le m_{ts}\\ 0\le\beta\le m_{st}}} T_{w[1,r]} q_{\alpha,\beta} X_{E_s}^{\beta} \sum_{i=0}^{\langle \alpha_t^\vee,\mu\rangle} z_i \partial_{s_1\cdots s_{r-1}} (X_r^{a+i+\alpha}).$$

We have

$$a-1+m_{ts} \le -\langle \alpha_t^\vee,\lambda\rangle - r - 2 + m_{ts} \le r-2$$

hence $\partial_{s_1\cdots s_{r-1}}(X_r^{a-1-l+\alpha}) = 0$ for all a, $l \ge 0$ and $\alpha \le m_{ts}$.

We have $a + \langle \alpha_t^\vee,\mu\rangle + m_{ts} \le r-1$. If $\partial_{s_1\cdots s_{r-1}}(X_r^{a+i+\alpha}) \ne 0$, then $a = -\langle \alpha_t^\vee,\lambda\rangle - r - 1$, $i = \langle \alpha_t^\vee,\mu\rangle$ and $\alpha = m_{ts}$.

We have shown that

$$f_a = \begin{cases} T_{w[1,r]} q_{m_{ts},0} (-1)^{\langle \alpha_t^\vee,\lambda\rangle + m_{ts}+1} & \text{if } a = -\langle \alpha_t^\vee,\lambda\rangle - r - 1 \text{ and } r > m_{ts} \\ 0 & \text{otherwise.} \end{cases}$$

The lemma follows. □

Proof of Theorem 5.21. Let $N \in \mathcal{V}_\lambda$ such that $F_t N = 0$. Define

$$L = \bigoplus_{\substack{w\in\mathfrak{S}_{r+1}^r\\ i<-\langle\alpha_t^\vee,\lambda\rangle - r}} T_w X_{r+1}^i \mathbf{Z} \text{ and } L' = \bigoplus_{\substack{w\in{}^r\mathfrak{S}_{r+1}\\ i<-\langle\alpha_t^\vee,\lambda\rangle - r}} X_{r+1}^i T_w \mathbf{Z}.$$

We have an isomorphism $L \simeq L(r+1,1,1,\lambda)$ (cf §4.2.2).

We have an isomorphism (Lemma 4.13)

$$\mathrm{act} \circ (\mathrm{id} \otimes \eta E_t^r) : L \otimes_{\mathbf{Z}} E_t^r N \xrightarrow{\sim} F_t E_t^{r+1} N.$$

Similarly, applying Lemma 4.13 to $\mathcal{V}^\iota$, we obtain an isomorphism

$$(\mathrm{id} \otimes \varepsilon^l E_t^r) \circ \mathrm{act}^* : F_t E_t^{r+1} N \xrightarrow{\sim} L'^* \otimes_{\mathbf{Z}} E_t^r N.$$

We have a commutative diagram

$$\begin{tikzcd}
L \otimes E_s E_t^r \arrow[r, "\mathrm{act}\circ(\mathrm{id}\otimes E_s\eta\bullet)"'] \arrow[d, "\mathrm{id}\otimes\eta\bullet"'] & E_s F_t E_t^{r+1} \arrow[r, "\eta\bullet"'] \arrow[rrr, bend left, "\sigma_{st}\bullet"] & F_t E_t E_s F_t E_t^{r+1} \arrow[r, "F_t\tau_{ts}\bullet"'] & F_t E_s E_t F_t E_t^{r+1} \arrow[r, "F_t E_s\varepsilon\bullet"'] & F_t E_s E_t^{r+1} \\
L \otimes F_t E_t E_s E_t^r \arrow[r, "F_t\tau_{ts}\bullet"] & L \otimes F_t E_s E_t^{r+1} \arrow[rr, "F_t E_s E_t \eta\bullet"] \arrow[rrr, bend right, "\mathrm{id}"'] & & L \otimes F_t E_s E_t F_t E_t^{r+1} \arrow[r, "F_t E_s \varepsilon\bullet"] & L \otimes F_t E_s E_t^{r+1} \arrow[u, "\mathrm{act}"']
\end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd}
F_tE_sE_t^{r+1} \arrow[r, "F_tE_s\eta^l\bullet"'] \arrow[rrr, bend left, "\sigma^\iota_{ts}\bullet"] \arrow[d, "\mathrm{act}^*"'] & F_tE_sE_tF_tE_t^{r+1} \arrow[r, "F_t\tau_{st}\bullet"'] & F_tE_tE_sF_tE_t^{r+1} \arrow[r, "\varepsilon^l\bullet"'] & E_sF_tE_t^{r+1} \arrow[d, "(\mathrm{id}\otimes E_s\varepsilon^l\bullet)\circ\mathrm{act}^*"] \\
L'^*\otimes F_tE_sE_t^{r+1} \arrow[r, "\mathrm{id}"] \arrow[d, "F_tE_s\eta^l\bullet"'] & L'^*\otimes F_tE_sE_t^{r+1} \arrow[r, "F_t\tau_{st}\bullet"] & L'^*\otimes E_sF_tE_t^{r+1} \arrow[r, "\varepsilon^l"] & L'^*\otimes E_sE_t^r \\
L'^*\otimes F_tE_sE_tF_tE_t^{r+1} \arrow[ur, "F_tE_sE_t\varepsilon^l\bullet"'] & & &
\end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd}
L\otimes E_sE_t^r \arrow[r, "\mathrm{act}\circ(\mathrm{id}\otimes E_s\eta\bullet)"] \arrow[d, "\eta\bullet"'] & E_sF_tE_t^{r+1} \arrow[r, "\sigma_{st}\bullet"] & F_tE_sE_t^{r+1} \arrow[r, "\sigma^\iota_{ts}\bullet"] & E_sF_tE_t^{r+1} \arrow[r, "(\mathrm{id}\otimes E_s\varepsilon^l\bullet)\circ\mathrm{act}^*"] & L'^*\otimes E_sE_t^r \\
L\otimes F_tE_tE_sE_t^r \arrow[r, "F_t\tau_{ts}\bullet"'] & L\otimes F_tE_sE_t^{r+1} \arrow[r, "\mathrm{act}"'] & F_tE_sE_t^{r+1} \arrow[r, "\mathrm{act}^*"'] & L'^*\otimes F_tE_sE_t^{r+1} \arrow[r, "F_t\tau_{st}\bullet"'] & L'^*\otimes E_sE_t^r \arrow[u, "\varepsilon^l\bullet"']
\end{tikzcd}$$

We will show that the top horizontal composition in the diagram above is an isomorphism when applied to N:

$$f : L\otimes E_sE_t^rN \xrightarrow{\sim} L'^*\otimes E_sE_t^rN.$$

It is enough to show that the map γ obtained by left multiplication by $T_{w[1,r+1]}$ is invertible, as in the proof of Lemma 4.13. Lemma 5.22 shows that the map

$$E_sE_t^{(r)} \xrightarrow{X^a\otimes\mathrm{id}} T_{w[1,r+1]}(L\otimes E_sE_t^r) \xrightarrow{\gamma} T_{w[1,r+1]}(L'^*\otimes E_sE_t^r) \xrightarrow{\langle X^{a'},-\rangle} E_sE_t^{(r)}$$

is 0 for $a+a' < -\langle\alpha_t^\vee,\lambda\rangle - r - 1$ and it is an isomorphism for $a+a' = -\langle\alpha_t^\vee,\lambda\rangle - r - 1$. So, γ is invertible. It follows that f is an isomorphism. Consequently, the composition $\sigma^\iota_{ts}\circ\sigma_{st}$ is an isomorphism when applied to E_t^rN. We conclude as in the proof of Theorem 5.15 that it is an isomorphism on all objects of $\mathcal{V}$.

We apply now the result above to $\mathcal{V}^\iota$: it shows that σ^ι_{ts} has a left inverse. So, σ^ι_{ts} is invertible, hence σ_{st} is invertible as well. ☐

5.3.5. *Control from K_0.*

Theorem 5.23. *Consider a root datum with associated Kac-Moody algebra $\mathfrak{g}$ and associated ring* $\mathbf{k}$.

Let k be a field that is a $\mathbf{k}$-algebra and $\mathcal{V}\in\mathfrak{Ab}_k^f$.

Assume given

- *an adjoint pair (E_s,F_s) of exact functors $\mathcal{V}\to\mathcal{V}$ for every $s\in\Gamma_0$*
- *$x_s\in\mathrm{End}(E_s)$ and $\tau_{st}\in\mathrm{Hom}(E_sE_t,E_tE_s)$ for every $s,t\in\Gamma_0$.*

We assume that

- *F_s is isomorphic to a left adjoint of E_s*
- *for every $s\in I$, then $\{[E_s],[F_s]\}$ induce a locally finite action of $\mathfrak{sl}_2$ on $V=K_0(\mathcal{V})$*
- *relations (1)-(4) in §4.1.1 hold*
- *given S a simple objects of $\mathcal{V}$, then $[S]$ is a weight vector*

Given $\lambda \in X$, let $\mathcal{V}_\lambda = \{M \in \mathcal{V} | [M] \in V_\lambda\}$. Then, $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$ and the data above defines an integrable action of $\mathfrak{A}_\mathfrak{g}$ on $\mathcal{V}$.

Proof. This is a consequence of Theorems 5.18 and 5.21. □

5.3.6. *Type A.* Let k be a field. Let $q \in k^\times$ and let I be a subset of k. Assume $0 \notin I$ if $q \neq 1$ and consider the corresponding Lie algebra $\mathfrak{sl}_{I_q}$ as in §3.2.5.

Let $\mathcal{V}$ be a k-linear category. Consider

- an adjoint pair (E, F) of endofunctors of $\mathcal{V}$
- $X \in \operatorname{End}(E)$ and $T \in \operatorname{End}(E^2)$.

Assume there is a decomposition $E = \bigoplus_{i \in I} E_i$, where $X - i$ is locally nilpotent on E_i.

When $q = 1$, we put $x_i = X - i$ (acting on E_i) and

$$\tau_{ij} = \begin{cases} (E_i X - X E_j + 1)^{-1}(T - 1) & \text{if } i = j \\ (E_i X - X E_j)T + 1 & \text{if } j = i + 1 \\ \frac{E_i X - X E_j}{E_i X - X E_j + 1}(T - 1) + 1 & \text{otherwise} \end{cases}$$

(restricted to $E_i E_j$).

When $q \neq 1$, we put $x_i = i^{-1} X$ (acting on E_i) and

$$\tau_{ij} = \begin{cases} i(qE_i X - X E_j)^{-1}(T - q) & \text{if } i = j \\ i^{-1}(E_i X - X E_j)T + i^{-1}(q - 1) X E_j & \text{if } j = qi \\ \frac{E_i X - X E_j}{qE_i X - X E_j}(T - q) + 1 & \text{otherwise} \end{cases}$$

(restricted to $E_i E_j$).

Let $\lambda \in X$. Let $\mathcal{V}_\lambda$ be the full subcategory of objects M of $\mathcal{V}$ such that for every $i \in I$, the following map is invertible:

- when $\langle \alpha_s^\vee, \lambda \rangle \geq 0$, $\sigma_{ss} + \sum_{i=0}^{\langle \alpha_s^\vee, \lambda \rangle - 1} \varepsilon_s \circ (x_s^i F_s) : E_s F_s(M) \to F_s E_s(M) \oplus M^{\langle \alpha_s^\vee, \lambda \rangle}$
- when $\langle \alpha_s^\vee, \lambda \rangle \leq 0$, $\sigma_{ss} + \sum_{i=0}^{-1-\langle \alpha_s^\vee, \lambda \rangle} (F_s x_s^i) \circ \eta_s : E_s F_s(M) \oplus M^{-\langle \alpha_s^\vee, \lambda \rangle} \to F_s E_s(M)$

Assume that

- E_i and F_i are locally nilpotent
- $\mathcal{V} = \bigoplus_{\lambda \in X} \mathcal{V}_\lambda$

Theorem 5.24. *The data above defines an action of $\mathfrak{A}_\mathbf{Z}(\mathfrak{sl}_{I_q}) \otimes k$ on $\mathcal{V}$.*

Proof. The x_i's and τ_{ij}'s satisfy the relations (1)-(4) in §4.1.1 thanks to Propositions 3.16 and 3.18. The invertibility of σ_{st} for $s \neq t$ follows from Theorem 5.21. □

5.3.7. *$\mathfrak{sl}$-categorifications.* Let k be a field. Let $q \in k^\times$ and let I be a subset of k. Assume $0 \notin I$ if $q \neq 1$ and consider the corresponding Lie algebra $\mathfrak{sl}_{I_q}$ as in §3.2.5.

Let $\mathcal{V} \in \mathfrak{A}b_k^f$.

Definition 5.25 (Chuang-Rouquier)**.** *An $\mathfrak{sl}_{I_q}$-categorification on $\mathcal{V}$ is the data of*

- *an adjoint pair (E, F) of exact functors $\mathcal{V} \to \mathcal{V}$*
- *$X \in \operatorname{End}(E)$ and $T \in \operatorname{End}(E^2)$.*

Given $i \in k$, let E_i (resp. F_i) be the generalized i-eigenspace of X acting on E (resp. F). We assume that

- *$E = \bigoplus_{i \in I} E_i$*

- *the action of the* $[E_i]$ *and* $[F_i]$ *on* $K_0(\mathcal{V})$ *gives an integrable highest weight representation of* $\mathfrak{sl}_{I_q}$
- *classes of simple objects are weight vectors*
- F *is isomorphic to a left adjoint of* E
- *the action on* E^n *of* $X_i = E^{n-i}XE^{i-1}$ *for* $1 \le i \le n$ *and of* $T_i = E^{n-i-1}TE^{i-1}$ *for* $1 \le i \le n-1$ *induce an action of*
  - *an affine Hecke algebra if* $q \neq 1$
  - *a degenerate affine Hecke if* $q = 1$.

Consider an $\mathfrak{sl}_{I_q}$-categorification on $\mathcal{V}$.

Theorem 5.26. *Assume given an* $\mathfrak{sl}_{I_q}$*-categorification on* $\mathcal{V}$*. The construction above gives rise to an action of* $\tilde{\mathfrak{A}}_{\mathfrak{sl}_{I_q}}$ *on* $\mathcal{V}$*.*

Conversely, an integrable action of $\tilde{\mathfrak{A}}_{\mathfrak{sl}_{I_q}}$ *on* $\mathcal{V}$ *gives rise to an* $\mathfrak{sl}_{I_q}$*-categorification on* $\mathcal{V}$*.*

Proof. The theorem follows now from Theorem 5.23. □

5.4. **Examples.** We give examples of actions of $\mathfrak{A}_{\mathfrak{sl}}$, via Theorem 5.26. These examples have all been studied in [ChRou, §7] in the context of $\mathfrak{sl}_2$-categorifications.

5.4.1. *Symmetric groups.* Let p be a prime number, $k = \mathbf{F}_p$ and $I = k$, viewed as a type $\tilde{A}_{p-1}$ quiver (here, $q = 1$). Let $\mathcal{V} = \bigoplus_{n\ge 0} k\mathfrak{S}_n$-mod. Let $E = \bigoplus_{n\ge 0}\operatorname{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$, let X be its endomorphism corresponding to right multiplication by $(1, n+1) + \cdots + (n, n+1)$ on the $(k\mathfrak{S}_{n+1}, k\mathfrak{S}_n)$-bimodule $k\mathfrak{S}_{n+1}$ and let T corresponding to right multiplication by $(n+1, n+2))$ on the $(k\mathfrak{S}_{n+2}, k\mathfrak{S}_n)$-bimodule $k\mathfrak{S}_{n+2}$. This defines an action of $\mathfrak{A}_{\mathfrak{sl}_{\tilde{A}_{p-1}}}$ on $\mathcal{V}$ (cf [ChRou, §7.1]).

5.4.2. *Cyclotomic Hecke algebras.* Consider $q \neq 1$ and k a field and $v_1, \ldots, v_d \in k^\times$. Let $I = \{q^m v_r\}_{m\in\mathbf{Z}, 1\le r\le i}$, a disjoint union of quivers of type $A_{\infty,\infty}$ (q not a root of unity) or of type $\tilde{A}_{e-1}$ (q a primitive e-th root of 1).

Let $H_n(v, q)$ be the quotient of $kH_n(q)$ by the two-sided ideal generated by $(X_1 - v_1)\cdots(X_d - v_d)$ and let $\mathcal{V} = \bigoplus_{n\ge 0} H_n(v, q)$-mod. Let $E = \bigoplus_{n\ge 0}\operatorname{Ind}_{H_n(v,q)}^{H_{n+1}(v,q)}$, let X be its endomorphism corresponding to right multiplication by X_{n+1} on the $(H_{n+1}(v, q), H_n(v, q))$-bimodule $H_{n+1}(v, q)$ and let T corresponding to right multiplication by T_{n+1} on the $(H_{n+2}(v, q), H_n(v, q))$-bimodule $H_{n+2}(v, q)$. This defines an action of $\mathfrak{A}_{\mathfrak{sl}_{I_q}}$ on $\mathcal{V}$ (cf [ChRou, §7.1]).

New proof of Ariki's Theorem if we compare with geometric realization.

5.4.3. *General linear groups over finite fields.* Let q be a prime power, k a field of characteristic $\ell > 0$ that does not divide $q(q-1)$. Let A_n be the sum of the unipotent blocks of $k\,\mathrm{GL}_n(q)$ and $\mathcal{V} = \bigoplus_{n\ge 0} A_n$-mod. Let

As in the proof of [ChRou, Lemma 7.16], one checks that the E_i's and their adjoints induce an action of $\mathfrak{sl}_{I_q}$ on $K_0(\mathcal{V})$. So, we have constructed an action of $\mathfrak{A}_{\mathfrak{sl}_{I_q}}$ on $\mathcal{V}$.

5.4.4. *Rational representations.*

5.4.5. *Soergel bimodules.*

5.4.6. *Rational Cherednik algebras.*

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