QUIVER HECKE ALGEBRAS AND 2-LIE ALGEBRAS

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Contents

Abstract. We provide an introduction to the 2-representation theory of Kac-Moody algebras, starting with basic properties of nil Hecke algebras and quiver Hecke algebras, and continuing with the resulting monoidal categories, which have a geometric description via quiver varieties, in certain cases. We present basic properties of 2-representations and describe simple 2-representations, via cyclotomic quiver Hecke algebras, and through microlocalized quiver varieties.

1. Introduction

This text provides an introduction and complements to some basic constructions and results in 2-representation theory of Kac-Moody algebras. We discuss quiver Hecke algebras [Rou2], which have been introduced independently by Khovanov and Lauda [KhoLau1] and [KhoLau2],

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and their cyclotomic versions, which have been considered independently in the case of level 2 weights for type A, by Brundan and Stroppel [BrStr]. We discuss the 2-categories associated with Kac-Moody algebras and their 2-representations: this has been introduced in joint work with Joe Chuang [ChRou] for $\mathfrak{sl}_2$ and implicitly for type A (finite or affine). While the general philosophy of categorifications was older (cf for example [BeFrKho]), the new idea in [ChRou] was to introduce some structure at the level of natural transformations: an endomorphism of E and an endomorphism of E^2 satisfying Hecke-type relations. The generalization to other types is based on quiver Hecke algebras, which account for a half Kac-Moody algebra. We discuss the geometrical construction of the quiver Hecke algebras via quiver varieties, which was our starting point for the definition of quiver Hecke algebras, and that of cyclotomic quiver Hecke algebras.

The first chapter gives a gentle introduction to nil (affine) Hecke algebras of type A. We recall basic properties of Hecke algebras of symmetric groups and provide the construction via BGG-Demazure operators of the nil Hecke algebras. We also construct symmetrizing forms.

The second chapter is devoted to quiver Hecke algebras. We explain that the more complicated relation involved in the definition is actually a consequence of the other ones, up to polynomial torsion: this leads to a new, simpler, definition of quiver Hecke algebras. We construct next the faithful polynomial representation. This generalizes the constructions of the first chapter, that correspond to a quiver with one vertex. We explain the relation, for type A quivers, with affine Hecke algebras. Finally, we explain how to put together all quiver Hecke algebras associated with a quiver to obtain a monoidal category that categorifies a half Kac-Moody algebra (and its quantum version).

The third chapter introduces 2-categories associated with Kac-Moody algebras and discusses their integrable representations. We provide various results that reduce the amount of conditions to check that a category is endowed with a structure of an integrable 2-representation, once the quiver Hecke relations hold: for example, the $\mathfrak{sl}_2$-relations imply all other relations, and it can be enough to check them on K_0. We explain the universal construction of "simple" 2-representations, and give a detailed description for $\mathfrak{sl}_2$. We present a Jordan-Hölder type result. We move next to cyclotomic quiver Hecke algebras, and present Kang-Kashiwara and Webster's construction of 2-representations on cyclotomic quiver Hecke algebras. We prove that the 2-representation is equivalent to the universal simple 2-representation. Finally, we explain the construction of Fock spaces from representations of symmetric groups in this framework.

The last chapter brings in geometrical methods available in the case of symmetric Kac-Moody algebras. We start with a brief recollection of Ringel's construction of quantum groups via Hall algebras and Lusztig's construction of enveloping algebras via constructible functions. We move next to the construction of nil affine Hecke algebras in the cohomology of flag varieties. We introduce Lusztig's category of perverse sheaves on the moduli space of representations of a quiver and show that it is equivalent to the monoidal category of quiver Hecke algebras (a result obtained independently by Varagnolo and Vasserot). As a consequence, the indecomposable projective modules for quiver Hecke algebras over a field of characteristic 0, and for "geometric" parameters, correspond to the canonical basis. Finally, we show that Zheng's microlocalized categories of sheaves can be endowed with a structure of 2-representation isomorphic to the universal simple 2-representation. As a consequence, the indecomposable projective modules for cyclotomic quiver Hecke algebras over a field of characteristic 0, and for "geometric" parameters, correspond to the canonical basis of simple representations.

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2. One vertex quiver Hecke algebras

The results of this section are classical (cf for example [Rou2, §3]).

2.1. **Nil Hecke algebras.**

2.1.1. *The symmetric group as a Weyl group.* Let $n \geq 1$. Given $i \in \{1, \ldots, n-1\}$, we put $s_i = (i, i+1) \in \mathfrak{S}_n$.

We define a function $r : \mathfrak{S}_n \to \mathbf{Z}_{\geq 0}$. Given $w \in \mathfrak{S}_n$, let $R_w = \{(i,j) | i < j \text{ and } w(i) > w(j)\}$ and let $r(w) = |R(w)|$ be the number of inversions.

The *length* $l(w)$ of $w \in \mathfrak{S}_n$ is the minimal integer r such that there exists $i_1, \ldots, i_r$ with $w = s_{i_1} \cdots s_{i_r}$. Such an expression is called a *reduced decomposition* of w. Proposition 2.1 says that since $s_1, \ldots, s_{n-1}$ generate $\mathfrak{S}_n$, these notions make sense.

Note that reduced decompositions are not unique: we have for example $(13) = s_1 s_2 s_1 = s_2 s_1 s_2$. Simpler is $s_1 s_3 = s_3 s_1$.

Proposition 2.1. *The set $\{s_1, \ldots, s_{n-1}\}$ generates $\mathfrak{S}_n$. Given $w \in \mathfrak{S}_n$, we have $r(w) = l(w)$.*

Proof. Let $w \in \mathfrak{S}_n$, $w \neq 1$. Note that $R_w \neq \emptyset$. Consider $(i,j) \in R_w$ such that $j - i$ is minimal. Assume $j \neq i+1$. By the minimality assumption, $(i, i+1) \notin R_w$ and $(i+1, j) \notin R_w$, so $w(j) > w(i+1) > w(i)$, a contradiction. So, $j = i+1$. Let $w' = ws_i$. We have $R_{w'} = R_w - \{(i, i+1)\}$, hence $r(w') = r(w) - 1$. We deduce by induction that there exist $i_1, \ldots, i_{r(w)} \in \{1, \ldots, n-1\}$ such that $w = s_{i_{r(w)}} \cdots s_{i_1}$. In particular, the set $\{s_1, \ldots, s_{n-1}\}$ generates $\mathfrak{S}_n$ and $l(g) \leq r(g)$ for all $g \in \mathfrak{S}_n$.

Let $j \in \{1, \ldots, n\}$ and $v = ws_j$. Assume $(j, j+1) \notin R_w$. Then, $R_v = R_w \cup \{(j, j+1)\}$. It follows that $r(v) = r(w) + 1$. If $(j, j+1) \in R_w$, then $r(v) = r(w) - 1$. We deduce by induction that $l(g) \geq r(g)$ for all $g \in \mathfrak{S}_n$. $\square$

Proposition 2.2. *The element $w[1,n] = (1,n)(2,n-1)(3,n-2)\cdots$ is the unique element of $\mathfrak{S}_n$ with maximal length. We have $l(w[1,n]) = \frac{n(n-1)}{2}$.*

Proof. Note that $R_{w[1,n]} = \{(i,j) | i < j\}$ and this contains any set R_w for $w \in \mathfrak{S}_n$, with equality if and only if $w = w[1,n]$. The result follows from Proposition 2.1. $\square$

The set

$$C_n = \{1, s_{n-1} = (n-1, n), s_{n-2}s_{n-1} = (n-2, n-1, n), \ldots, s_1 \cdots s_{n-2}s_{n-1} = (1, 2, \ldots, n)\}$$

is a complete set of representatives for left cosets $\mathfrak{S}_n/\mathfrak{S}_{n-1}$. Let $w \in \mathfrak{S}_{n-1}$ and $g \in C_n$. We have $R(gw) = R(w) \coprod R(g)$, so $l(gw) = l(g) + l(w)$. Consider now $w \in \mathfrak{S}_n$. There is a unique decomposition $w = c_n c_{n-1} \cdots c_2$ where $c_i \in C_i$ and we have $l(w) = l(c_n) + \cdots + l(c_2)$. Each c_i has a unique reduced decomposition and that provides us with a canonical reduced decomposition of w:

$$w = (s_{j_1} s_{j_1+1} \cdots s_{i_1})(s_{j_2} s_{j_2+1} \cdots s_{i_2}) \cdots (s_{j_r} s_{j_r+1} \cdots s_{i_r})$$

where $i_1 > i_2 > \cdots > i_r$ and $1 \leq j_r < i_r$.

In the case of the longest element, we obtain

$$w[1,n] = (s_1 \cdots s_{n-1})(s_1 \cdots s_{n-2}) \cdots s_1 = (s_1 \cdots s_{n-1}) w[1, n-1].$$

Using canonical reduced decompositions, we can count the number of elements with a given length and deduce the following result.

Proposition 2.3. *We have* $\sum_{w \in \mathfrak{S}_n} q^{l(w)} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$.

Lemma 2.4. *Let* $w \in \mathfrak{S}_n$. *Then,* $l(w^{-1}) = l(w)$ *and* $l(w[1,n]w^{-1}) = l(w[1,n]) - l(w)$.

Proof. The first statement is clear, since $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression if and only if $w^{-1} = s_{i_r} \cdots s_{i_1}$ is a reduced expression.

We have $R_{w[1,n]w} = \{(i,j) | (i < j) \text{ and } w(i) < w(j)\}$. The second statement follows. □

We recall the following classical result.

Proposition 2.5. *The group* $\mathfrak{S}_n$ *has a presentation with generators* $s_1, \ldots, s_{n-1}$ *and relations*

$$s_i s_j = s_j s_i \textit{ if } |i-j| > 1 \textit{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

2.1.2. *Finite Hecke algebras.* Let us recall some classical results about Hecke algebras of symmetric groups.

Let $R = \mathbf{Z}[q_1, q_2]$. Let H_n^f be the *Hecke algebra* of $\mathfrak{S}_n$: this is the R-algebra generated by $T_1, \ldots, T_{n-1}$, with relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ if } |i-j| > 1 \text{ and } (T_i - q_1)(T_i - q_2) = 0.$$

There is an isomorphism of algebras

$$H_n^f \otimes_R R/(q_1 - 1, q_2 + 1) \xrightarrow{\sim} \mathbf{Z}[\mathfrak{S}_n], \ T_i \mapsto s_i.$$

Let $w \in \mathfrak{S}_n$ with a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$. We put $T_w = T_{i_1} \cdots T_{i_r} \in H_n^f$. One shows that T_w is independent of the choice of a reduced decomposition of w and that $\{T_w\}_{w \in \mathfrak{S}_n}$ is an R-basis of the free R-module H_n^f.

Given $w, w' \in \mathfrak{S}_n$ with $l(ww') = l(w) + l(w')$, we have $T_w T_{w'} = T_{ww'}$.

The algebra H_n^f is a *deformation* of $\mathbf{Z}[\mathfrak{S}_n]$. At the specialization $q_1 = 1$, $q_2 = -1$, the element T_w becomes the group element w.

2.1.3. *Nil Hecke algebras of type A.* Let ${}^0H_n^f = H_n^f \otimes_R R/(q_1, q_2)$. Given $w, w' \in \mathfrak{S}_n$, we have

$$T_w T_{w'} = \begin{cases} T_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

So, the algebra ${}^0H_n^f$ is graded with $\deg T_w = -2l(w)$. The choice of a negative sign will become clear soon. The factor 2 comes from the cohomological interpretation.

Given $M = \bigoplus_{i \in \mathbf{Z}} M_i$ a graded $\mathbf{Z}$-module and $r \in \mathbf{Z}$, we denote by $M\langle r \rangle$ the graded module given by $(M\langle r \rangle)_i = M_{i+r}$.

We have $({}^0H_n^f)_i = \bigoplus_{w \in \mathfrak{S}_n, l(w) = -i/2} \mathbf{Z} T_w$. So, $({}^0H_n^f)_i = 0$ unless $i \in \{0, -2, \ldots, -n(n-1)\}$.

Let k be a field and $k^0H_n^f = {}^0H_n^f \otimes_{\mathbf{Z}} k$.

Proposition 2.6. *The Jacobson radical of $k^0H_n^f$ is* $\mathrm{rad}(k^0H_n^f)=\bigoplus_{w\neq 1}kT_w$ *and $k^0H_n^f$ has a unique minimal non-zero two-sided ideal* $\mathrm{soc}(k^0H_n^f)=kT_{w[1,n]}$. *The trivial module k, with 0-action of the T_i's, is the unique simple $k^0H_n^f$-module.*

Proof. Let $A=k^0H_n^f$. Let $J=A_{<0}=\bigoplus_{w\neq 1}kT_w$. We have $J^{n(n-1)+1}=0$. So, J is a nilpotent two-sided ideal of A and $A/J\simeq k$. It follows that $J=\mathrm{rad}(A)$: the algebra A is local and k is the unique simple module.

Let M be a non-zero left ideal of A. Let $m=\sum_w \alpha_w T_w\in M$ be a non-zero element. Consider $w\in\mathfrak{S}_n$ of minimal length such that $\alpha_w\neq 0$. We have $T_{w[1,n]w^{-1}}m=\alpha_w T_{w[1,n]}$ because $T_{w[1,n]w^{-1}}T_{w'}=0$ if $l(w')\geq l(w)$ and $w'\neq w$. It follows that $kT_{w[1,n]}\subset M$. That shows that $kT_{w[1,n]}$ is the unique minimal non-zero left ideal of A. A similar proof shows it is also the unique minimal non-zero right ideal. □

Remark 2.7. Let k be a field and A be a finite dimensional graded k-algebra. Assume $A_0=k$ and $A_i=0$ for $i>0$. Then $\mathrm{rad}(A)=A_{<0}$.

2.1.4. *BGG-Demazure operators.* We refer to [Hi, Chapter IV] for a general discussion of the results below.

Let $P_n=\mathbf{Z}[X_1,\ldots,X_n]$. We let $\mathfrak{S}_n$ act on P_n by permutation of the X_i's. We define an endomorphism of abelian groups $\partial_i\in\mathrm{End}_{\mathbf{Z}}(P_n)$ by

$$\partial_i(P)=\frac{P-s_i(P)}{X_{i+1}-X_i}.$$

Note that the operators ∂_w are $P_n^{\mathfrak{S}_n}$-linear. Note also that $\mathrm{im}\,\partial_i\subset P_n^{s_i}=\ker\partial_i$.

The following lemma follows from easy calculations.

Lemma 2.8. *We have $\partial_i^2=0$, $\partial_i\partial_j=\partial_j\partial_i$ for $|i-j|>1$ and $\partial_i\partial_{i+1}\partial_i=\partial_{i+1}\partial_i\partial_{i+1}$.*

We deduce that we have obtained a representation of the nil Hecke algebra.

Proposition 2.9. *The assignment $T_i\mapsto\partial_i$ defines a representation of ${}^0H_n^f$ on P_n.*

Define a grading of the algebra P_n by $\deg X_i=2$. Then, the representation above is compatible with the gradings.

Given $w\in\mathfrak{S}_n$, we denote by ∂_w the image of T_w.

The following result is clear.

Lemma 2.10. *Let $P\in P_n$. We have $\partial_i(P)=0$ for all i if and only if $P\in P_n^{\mathfrak{S}_n}$.*

If M is a free graded module over a commutative ring k with $\dim_k M_i<\infty$ for all $i\in\mathbf{Z}$, we put $\mathrm{grdim}(M)=\sum_{i\in\mathbf{Z}}q^{i/2}\dim(M_i)$.

Theorem 2.11. *The set $\{\partial_w(X_2X_3^2\cdots X_n^{n-1})\}_{w\in\mathfrak{S}_n}$ is a basis of P_n over $P_n^{\mathfrak{S}_n}$.*

Proof. Let us show by induction on n that

$$\partial_{w[1,n]}(X_2X_3^2\cdots X_n^{n-1})=1.$$

We have $w[1,n]=s_{n-1}\cdots s_1w[2,n]$ and $l(w[1,n])=l(w[2,n])+n-1$. By induction,

$$\partial_{w[2,n]}(X_2X_3^2\cdots X_n^{n-1})=X_2\cdots X_n\cdot\partial_{w[2,n]}(X_3\cdots X_n^{n-2})=X_2\cdots X_n.$$

On the other hand, $\partial_{n-1}\cdots\partial_1(X_2\cdots X_n)=1$ and we deduce that $\partial_{w[1,n]}(X_2X_3^2\cdots X_n^{n-1})=1$.

Let M be a free $P_n^{\mathfrak{S}_n}$-module with basis $\{b_w\}_{w\in\mathfrak{S}_n}$, with $\deg b_w=2l(w[1,n]w^{-1})=n(n-1)-2l(w)$. Define a morphism of $P_n^{\mathfrak{S}_n}$-modules

$$\phi : M\to P_n,\ b_w\mapsto \partial_w(X_2X_3^2\cdots X_n^{n-1}).$$

This is a graded morphism.

Let k be a field. Let $a=\sum_w Q_wb_w\in\ker(\phi\otimes k)$, where $Q_w\in k[X_1,\ldots,X_n]^{\mathfrak{S}_n}$. Assume $a\neq 0$. Consider $v\in\mathfrak{S}_n$ with $Q_v\neq 0$ and such that $l(v)$ is minimal with this property. We have $\partial_{w[1,n]v^{-1}}(\phi(a))=Q_v$, hence we have a contradiction. It follows that $\phi\otimes k$ is injective.

We have $\mathrm{grdim}P_n=(1-q)^{-n}$. On the other hand, we have $P_n^{\mathfrak{S}_n}=\mathbf{Z}[e_1,\ldots,e_n]$, where $e_r=e_r(X_1,\ldots,X_n)=\sum_{1\le i_1<\cdots<i_r\le n}X_{i_1}\cdots X_{i_r}$. So, $\mathrm{grdim}P_n^{\mathfrak{S}_n}=(1-q)^{-1}(1-q^2)^{-1}\cdots(1-q^n)^{-1}$. We deduce that

$$\mathrm{grdim}M=(1-q)^{-1}\cdots(1-q^n)^{-1}\sum_{w\in\mathfrak{S}_n}q^{l(w)}.$$

The formula of Proposition 2.3 shows that $\mathrm{grdim}M=\mathrm{grdim}P_n$. Lemma 2.13 shows that $\phi_i\otimes k$ is an isomorphism and then Lemma 2.12 shows that ϕ_i is an isomorphism for all i. □

The following two lemmas are clear.

Lemma 2.12. *Let $f:M\to N$ be a morphism between free finitely generated $\mathbf{Z}$-modules. If $f\otimes_{\mathbf{Z}}(\mathbf{Z}/p)$ is surjective for all prime p, then f is surjective.*

Lemma 2.13. *Let k be a field and M,N be two graded k-modules with $\dim M_i=\dim N_i$ finite for all i. If $f:M\to N$ is an injective morphism of graded k-modules, then f is an isomorphism.*

Remark 2.14. Note that $\{X_2^{a_2}\cdots X_n^{a_n}\}_{0\le a_i\le i-1}$ is the more classical basis of P_n over $P_n^{\mathfrak{S}_n}$.

2.2. Nil affine Hecke algebras.

2.2.1. *Definition.* Let ${}^0H_n$ be the *nil affine Hecke algebra* of GL_n: this is the $\mathbf{Z}$-algebra with generators $X_1,\ldots,X_n,T_1,\ldots,T_{n-1}$ and relations

$$X_iX_j=X_jX_i,\ T_i^2=0,\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\ T_iT_j=T_jT_i \text{ if } |i-j|>1,$$

$$T_iX_j=X_jT_i \text{ if } j-i\neq 0,1,\ T_iX_{i+1}-X_iT_i=1 \text{ and } T_iX_i-X_{i+1}T_i=-1.$$

It is a graded algebra, with $\deg X_i=2$ and $\deg T_i=-2$.

The following lemma is easy.

Lemma 2.15. *Given $P,Q\in P_n$, we have $\partial_i(PQ)=\partial_i(P)Q+s_i(P)\partial_i(Q)$.*

Lemma 2.15 is the key ingredient to prove the following lemma.

Lemma 2.16. *We have a representation ρ of ${}^0H_n$ on P_n given by*

$$\rho(T_i)(P)=\partial_i(P) \text{ and } \rho(X_i)(P)=X_iP.$$

Proposition 2.17. *We have a decomposition ${}^0H_n=P_n\otimes{}^0H_n^f$ as a $\mathbf{Z}$-module and the representation of ${}^0H_n$ on P_n is faithful.*

Proof. Let $\{P_w\}_{w\in\mathfrak{S}_n}$ be a family of elements of P_n. Let $a=\sum_w P_wT_w$. If $a\neq 0$, there is $w\in\mathfrak{S}_n$ of minimal length such that $P_w\neq 0$. We have $aT_{w^{-1}w[1,n]}=P_wT_{w[1,n]}$ and

$$\rho(a)(\partial_{w^{-1}w[1,n]}(X_2\cdots X_n^{n-1}))=P_w\partial_{w[1,n]}(X_2\cdots X_n^{n-1})=P_w$$

(cf Proof of Proposition 2.11). We deduce that $\rho(a)\neq 0$. Consequently, the multiplication map $P_n\otimes {}^0H_n^f\to {}^0H_n$ is injective and the representation is faithful. On the other hand, the multiplication map is easily seen to be surjective. $\square$

Note that P_n and ${}^0H_n^f$ are subalgebras of ${}^0H_n$.

The module P_n is an induced module: we have an isomorphism of ${}^0H_n$-modules

$$P_n\xrightarrow{\sim}{}^0H_n\otimes_{{}^0H_n^f}\mathbf{Z},\ P\mapsto P\otimes 1.$$

Remark 2.18. Given $P\in P_n$, one shows that $T_iP-s_i(P)T_i=PT_i-T_is_i(P)=\partial_i(P)$.

2.2.2. *Description as a matrix ring.* Let $b_n=T_{w[1,n]}X_2X_3^2\cdots X_n^{n-1}$.

Lemma 2.19. *We have $b_n^2=b_n$ and ${}^0H_n={}^0H_nb_n{}^0H_n$.*

Proof. Note that $T_{w[1,n]}$ is the unique element of $\mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n)$ that sends $X_2X_3^2\cdots X_n^{n-1}$ to 1 and $\partial_w(X_2X_3^2\cdots X_n^{n-1})$ to 0 for $w\neq 1$ (cf Proof of Theorem 2.11 for the first fact). We have

$$\rho(T_{w[1,n]}X_2X_3^2\cdots X_n^{n-1}T_{w[1,n]})(\partial_w(X_2X_3^2\cdots X_n^{n-1}))=0$$

for $w\neq 1$ and

$$\rho(T_{w[1,n]}X_2X_3^2\cdots X_n^{n-1}T_{w[1,n]})(X_2X_3^2\cdots X_n^{n-1})=\partial_{w[1,n]}(X_2X_3^2\cdots X_n^{n-1})=1.$$

It follows that $b_nT_{w[1,n]}=T_{w[1,n]}$.

We show now by induction on n that $1\in{}^0H_nT_{w[1,n]}{}^0H_n$. Given $1\le r\le n-1$, we have

$$T_r\cdots T_{n-1}T_{w[1,n-1]}X_n-X_rT_r\cdots T_{n-1}T_{w[1,n-1]}=T_{r+1}\cdots T_{n-1}T_{w[1,n-1]},$$

where we use the convention that $T_{r+1}\cdots T_{n-1}=\prod_{r+1\le j\le n-1}T_j=1$ if $r=n-1$. By induction on r, we deduce that $T_{w[1,n-1]}\in{}^0H_nT_{w[1,n]}{}^0H_n$, since $T_{w[1,n]}=T_1\cdots T_{n-1}T_{w[1,n-1]}$. By induction on n, it follows that $1\in{}^0H_nT_{w[1,n]}{}^0H_n={}^0H_nb_n{}^0H_n$. $\square$

Remark 2.20. Given $w\in\mathfrak{S}_n$ and $P\in P_n$, one shows that $T_wPT_{w[1,n]}=\partial_w(P)T_{w[1,n]}$ (a particular case was obtained in the proof of Lemma 2.19).

We have an isomorphism of ${}^0H_n$-modules

$$P_n\xrightarrow{\sim}{}^0H_nb_n,\ P\mapsto Pb_n$$

This shows that P_n is a progenerator as a ${}^0H_n$-module: it is a finitely generated projective ${}^0H_n$-module and ${}^0H_n$ is a direct summand of a multiple of P_n, as a ${}^0H_n$-module.

Given A a ring, we denote by A^{opp} the opposite ring: it is A as an abelian group, but the multiplication of a and b in A^{opp} is the product ba computed in A.

Proposition 2.21. *The action of ${}^0H_n$ on P_n induces an isomorphism of $P_n^{\mathfrak{S}_n}$-algebras*

$${}^0H_n\xrightarrow{\sim}\mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n)^{\mathrm{opp}}.$$

Since P_n is a free $P_n^{\mathfrak{S}_n}$-module of rank $n!$, the algebra ${}^0H_n$ is isomorphic to a $(n!\times n!)$-matrix algebra over $P_n^{\mathfrak{S}_n}$.

Proof. Since P_n is a progenerator for ${}^0H_n$, we deduce that the canonical map ${}^0H_n \to \mathrm{End}_{P_n^{\mathfrak{S}_n}}(P_n)$ is a split injection of $P_n^{\mathfrak{S}_n}$-modules (Lemma 2.22). The proposition follows from the fact that ${}^0H_n$ is a free $P_n^{\mathfrak{S}_n}$-module of rank $(n!)^2$. □

Lemma 2.22. *Let R be a commutative ring and A an R-algebra, projective and finitely generated as an R-module. Let M be a progenerator for A. Then, the canonical map $A \to \mathrm{End}_R(M)^{\mathrm{opp}}$ is a split injection of R-modules.*

Proof. Let $f : A \to \mathrm{End}_R(M)$ be the canonical map and L its cokernel. The composition of morphisms of R-modules

$$M \xrightarrow{m\mapsto \mathrm{id}\otimes m} \mathrm{End}_R(M)\otimes_A M \xrightarrow{\alpha\otimes m\mapsto\alpha(m)} M$$

is the identity. So, $f\otimes_A 1_M$ is a split injection of R-modules, hence $L\otimes_A M$ is a projective R-module, since $\mathrm{End}_R(M)$ is a projective R-module and M is a projective A-module.

By Morita theory, there is N an $(\mathrm{End}_A(M), A)$-bimodule that is projective as an $\mathrm{End}_A(M)$-module and such that $M\otimes_{\mathrm{End}_A(M)} N \simeq A$ as (A,A)-bimodules. The R-module $L \simeq (L\otimes_A M)\otimes_{\mathrm{End}_A(M)} N$ is projective, since N is a projective $\mathrm{End}_A(M)$-module. Since L is a projective R-module, we deduce that f is a split injection of R-modules. □

We give now a second proof of Proposition 2.21. The proof of the faithfulness of the representation P_n of ${}^0H_n$ works also to show that $P_n\otimes_{P_n^{\mathfrak{S}_n}}(P_n^{\mathfrak{S}_n}/\mathfrak{m})$ is a faithful representation of ${}^0H_n\otimes_{P_n^{\mathfrak{S}_n}}(P_n^{\mathfrak{S}_n}/\mathfrak{m})$, for any maximal ideal $\mathfrak{m}$ of $P_n^{\mathfrak{S}_n}$. Proposition 2.21 follows now from Lemma 2.23.

Lemma 2.23. *Let R be a commutative ring, $f : M \to N$ a morphism between free R-modules of the same finite rank. If $f\otimes_R 1_{R/\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m}$ of R, then f is an isomorphism.*

Proof. Fix bases of M and N and let d be the determinant of f with respect to those bases. Let I be the ideal of R generated by d. Assume $d\notin R^\times$. There is a maximal ideal $\mathfrak{m}$ of R containing I. Since $f\otimes_R 1_{R/\mathfrak{m}}$ is an injective map between vector spaces of the same finite rank, it is an isomorphism, so we have $d\cdot 1_{R/\mathfrak{m}}\neq 0$, a contradiction. □

2.3. **Symmetrizing forms.**

2.3.1. *Definition and basic properties.* Let R be a commutative ring and A an R-algebra, finitely generated and projective as an R-module. A *symmetrizing form* for A is an R-linear map $t\in\mathrm{Hom}_R(A,R)$ such that

- t is a trace, *i.e.*, $t(ab)=t(ba)$ for all $a,b\in A$
- the morphism of (A,A)-bimodules

  $$\hat{t} : A\to\mathrm{Hom}_R(A,R),\ a\mapsto(b\mapsto t(ab))$$

  is an isomorphism.

Consider now a commutative ring R' such that R is an R'-algebra, finitely generated and projective as an R'-module. Consider $t\in\mathrm{Hom}_R(A,R)$ a trace and $t'\in\mathrm{Hom}_{R'}(R,R')$. We have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\hat{t}"] \arrow[dr, "\widehat{t't}"'] & \mathrm{Hom}_R(A,R) \arrow[r, "\mathrm{Hom}_R(A,\hat{t}')"] & \mathrm{Hom}_R(A,\mathrm{Hom}_{R'}(R,R')) \\
& \mathrm{Hom}_{R'}(A,R') \arrow[ur, "\sim", "\text{adjunction}"'] &
\end{tikzcd}$$

We deduce the following result.

Lemma 2.24. *If two of the forms t, t' and $t't$ are symmetrizing, then so is the third one.*

Let now B be another symmetric R-algebra and M an (A,B)-bimodule, finitely generated and projective as an A-module and as a right B-module. We have isomorphisms of functors

$$\mathrm{Hom}_A(M,-) \xleftarrow[\sim]{\text{can}} \mathrm{Hom}_A(M,A)\otimes_A - \xrightarrow[\sim]{f\mapsto tf} \mathrm{Hom}_R(M,R)\otimes_A -$$

and similarly $\mathrm{Hom}_B(\mathrm{Hom}_R(M,R),-) \xrightarrow{\sim} M\otimes_B -$. We deduce that $M\otimes_B -$ is left and right adjoint to $\mathrm{Hom}_A(M,-)$.

2.3.2. *Polynomials.*

Proposition 2.25. *The linear form $\partial_{w[1,n]}$ is a symmetrizing form for the $P_n^{\mathfrak{S}_n}$-algebra P_n.*

We will prove this proposition in §2.3.4: it will be deduced from a corresponding statement for the nil affine Hecke algebra, that is easier to prove.

Together with Lemma 2.24, Proposition 2.25 provides more general symmetrizing forms.

Corollary 2.26. *Given $1\le i\le n$, then the linear form $\partial_{w[1,n]}\partial_{w[1,i]}\partial_{w[i+1,n]}$ is a symmetrizing form for the $P_n^{\mathfrak{S}_n}$-algebra $P_n^{\mathfrak{S}\{1,\ldots,i\}\times\mathfrak{S}\{i+1,\ldots,n\}}$.*

2.3.3. *Nil Hecke algebras.* Define the $\mathbf{Z}$-linear form $t_0 : {}^0H_n^f \to \mathbf{Z}$ by $t_0(T_w)=\delta_{w,w[1,n]}$.

Define an algebra automorphism σ (the Nakayama automorphism) of ${}^0H_n^f$ by $\sigma(T_i)=T_{n-i}$. Note that $\sigma(T_w)=T_{w[1,n]ww[1,n]}$.

The form t_0 is not symmetric, it nevertheless gives rise to a Frobenius algebra structure.

Proposition 2.27. *Given $a,b\in {}^0H_n^f$, we have $t_0(ab)=t_0(\sigma(b)a)$. The form t_0 induces an isomorphism of right ${}^0H_n^f$-modules*

$$\hat{t}_0 : {}^0H_n^f \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}({}^0H_n^f,\mathbf{Z}),\, a\mapsto (b\mapsto t_0(ab)).$$

Proof. Let $w,w'\in\mathfrak{S}_n$. We have $t_0(T_wT_{w'})=0$ unless $w'=w^{-1}w[1,n]$, in which case we have $t_0(T_wT_{w^{-1}w[1,n]})=1$. We have $t_0(\sigma(T_{w'})T_w)=t_0(T_{w[1,n]w'w[1,n]}T_w)$. This is 0, unless $w=(w[1,n]w'w[1,n])^{-1}w[1,n]$, or equivalently, unless $w=w[1,n]w'^{-1}$. In that case, we get $t_0(\sigma(T_{w^{-1}w[1,n]},T_w)=1$. This shows that $t_0(T_wT_{w'})=t_0(\sigma(T_{w'})T_w)$ for all $w,w'\in\mathfrak{S}_n$.

Let p be a prime number. The kernel I of $\hat{t}_0\otimes_{\mathbf{Z}}\mathbf{F}_p$ is a two-sided ideal of $\mathbf{F}_p{}^0H_n^f$. On the other hand, $\hat{t}_0(T_{w[1,n]})(1)=t_0(T_{w[1,n]})=1$, hence $T_{w[1,n]}\not\in I$. It follows from Proposition 2.6 that $I=0$. Lemma 2.23 shows now that $\hat{t}_0$ is an isomorphism. $\square$

2.3.4. *Nil affine Hecke algebras.* We define a $P_n^{\mathfrak{S}_n}$-linear form t on ${}^0H_n$

$$t : {}^0H_n \to P_n^{\mathfrak{S}_n},\ t(PT_w) = \delta_{w,w[1,n]}\partial_{w[1,n]}(P) \text{ for } P \in P_n \text{ and } w \in \mathfrak{S}_n.$$

Let γ be the $\mathbf{Z}$-algebra automorphism of ${}^0H_n$ defined by

$$\gamma(X_i) = X_{n-i+1} \text{ and } \gamma(T_i) = -T_{n-i}.$$

Lemma 2.28. *We have $t(ab) = t(\gamma(b)a)$ for $a, b \in {}^0H_n$.*

Proof. Let $i \in \{1, \ldots, n\}$. By induction on $l(w)$, one shows that

$$T_wX_i - X_{w(i)}T_w \in \bigoplus_{w' \in \mathfrak{S}_n, l(w')<l(w)} P_nT_{w'}.$$

It follows that

$$T_{w[1,n]}X_i - X_{n-i+1}T_{w[1,n]} \in \bigoplus_{w \neq w[1,n]} P_nT_w.$$

We deduce that $t(PT_wX_i) = t(PX_{n-i}T_w)$ for $w \in \mathfrak{S}_n$ and $P \in P_n$.

Let $i \in \{1, \ldots, n-1\}$ and $P \in P_n$. Remark 2.18 shows that

$$t(T_{n-i}PT_w) = t(s_{n-i}(P)T_{n-i}T_w) + t(\partial_{n-i}(P)T_w).$$

We have $\partial_{w[1,n]}(\partial_{n-i}(P)) = 0$, so $t(\partial_{n-i}(P)T_w) = 0$. We have $\partial_{w[1,n]}(P + s_{n-i}(P)) = 0$, hence $\partial_{w[1,n]}(s_{n-i}(P)) = -\partial_{w[1,n]}(P)$. Since $s_{n-i}w = w[1,n]$ if and only if $ws_i = w[1,n]$, we deduce that $t(s_{n-i}(P)T_{n-i}T_w) = -t(PT_wT_i)$. This shows that $t(PT_wT_i) = t(-T_{n-i}PT_w)$ for $w \in \mathfrak{S}_n$ and $P \in P_n$. The lemma follows. $\square$

When viewed as a subalgebra of $\mathrm{End}_{\mathbf{Z}}(P_n)$, then ${}^0H_n$ contains $\mathfrak{S}_n$, since the action of $\mathfrak{S}_n$ is trivial on $P_n^{\mathfrak{S}_n}$ (Proposition 2.21). The injection of $\mathfrak{S}_n$ in ${}^0H_n$ is given explicitly by $s_i \mapsto (X_i - X_{i+1})T_i + 1$.

The following lemma is an immediate calculation involving endomorphisms of P_n.

Lemma 2.29. *We have $w[1,n] \cdot a \cdot w[1,n] = \gamma(a)$ for all $a \in {}^0H_n$.*

Let t' be the linear form on ${}^0H_n$ defined by $t'(a) = t(aw[1,n])$.

Proposition 2.30. *The form t' is symmetrizing for the $P_n^{\mathfrak{S}_n}$-algebra ${}^0H_n$.*

Proof. Lemmas 2.28 and 2.29 show that $t'(ab) = t'(ba)$ for all $a, b \in {}^0H_n$.

Let $\mathfrak{m}$ be a maximal ideal of $P_n^{\mathfrak{S}_n}$ and $k = P_n^{\mathfrak{S}_n}/\mathfrak{m}$. We have $k^0H_n \simeq M_{n!}(k)$ by Proposition 2.21. We have $t'(X_2 \cdots X_n^{n-1}T_{w[1,n]}w[1,n]) = 1$ (cf the proof of Theorem 2.11), hence the form $t' \otimes_{P_n^{\mathfrak{S}_n}} 1_k$ is not zero. As a consequence, it is a symmetrizing form, since $k^0H_n \simeq M_{n!}(k)$ by Proposition 2.21. We deduce that $\hat{t}' \otimes_{P_n^{\mathfrak{S}_n}} k$ is an isomorphism, so $\hat{t}'$ is an isomorphism by Lemma 2.23. $\square$

Proof of Proposition 2.25. Let $\mathfrak{m}$ be a maximal ideal of $P_n^{\mathfrak{S}_n}$ and $k = P_n^{\mathfrak{S}_n}/\mathfrak{m}$. Let P be a non-zero element of $P_n \otimes_{P_n^{\mathfrak{S}_n}} k$. By Proposition 2.30, there is $a \in k^0H_n$ such that $t'(PT_{w[1,n]}a) \neq 0$. So, $t(PT_{w[1,n]}aw[1,n]) \neq 0$. There are elements $Q_w \in P_n \otimes_{P_n^{\mathfrak{S}_n}} k$ such that $aw[1,n] = \sum_w T_wQ_w$. Then

$$t(PT_{w[1,n]}aw[1,n]) = t(PT_{w[1,n]}Q_1) = t(\gamma(Q_1)PT_{w[1,n]}) = \partial_{w[1,n]}(\gamma(Q_1)P) \neq 0.$$

We deduce that $\hat{\partial}_{w[1,n]} \otimes_{P_n^{\mathfrak{S}_n}} (P_n^{\mathfrak{S}_n}/\mathfrak{m})$ is injective for any maximal ideal $\mathfrak{m}$ of $P_n^{\mathfrak{S}_n}$. It follows from Lemma 2.23 that $\hat{\partial}_{w[1,n]}$ is an isomorphism. $\square$

Remark 2.31. One can show that the automorphism σ of ${}^0H_n^f$ is not inner for $n \geq 3$.

3. Quiver Hecke algebras

3.1. **Representations of quivers.** We refer to [GaRoi] for a general discussion of quivers and their representations.

3.1.1. *Quivers.* Let Q be a *quiver* (= an oriented graph), *i.e.*,

- a finite set Q_0 (the vertices)
- a finite set Q_1 (the arrows)
- maps $p, q : Q_1 \to Q_0$ (tail=source and head=target of an arrow).

Let k be a commutative ring. A *representation* of Q over k is the data of $(V_s, \phi_a)_{s \in Q_0, a \in Q_1}$ where V_s is a k-module and $\phi_a \in \mathrm{Hom}_k(V_{p(a)}, V_{q(a)})$.

A morphism from $(V_s, \phi_a)_{s,a}$ to $(V'_s, \phi'_a)_{s,a}$ is the data of a family $(f_s)_{s \in Q_0}$, where $f_s \in \mathrm{Hom}_k(V_s, V'_s)$, such that for all $a \in Q_1$, the following diagram commutes:

$$\begin{tikzcd} V_{p(a)} \arrow[r, "\phi_a"] \arrow[d, "f_{p(a)}"'] & V_{q(a)} \arrow[d, "f_{q(a)}"] \\ V'_{p(a)} \arrow[r, "\phi'_a"'] & V'_{q(a)} \end{tikzcd}$$

The *quiver algebra* $k(Q)$ associated to Q is the k-algebra generated by the set $Q_0 \cup Q_1$ with relations

$$sa = \delta_{q(a),s} a, \quad as = \delta_{p(a),s} a, \quad ss' = \delta_{s,s'} s \text{ for } s, s' \in Q_0 \text{ and } a \in Q_1 \text{ and } 1 = \sum_{t \in Q_0} t$$

Let $\gamma = (s_1, a_1, s_2, a_2, \ldots, s_n)$ be a *path* in Q, *i.e.*, a sequence of vertices $s_i \in Q_0$ and arrows $a_i \in Q_1$ such that $p(a_i) = s_i$ and $q(a_i) = s_{i+1}$. We put $\tilde{\gamma} = s_n a_{n-1} \cdots a_2 s_2 a_1 s_1 \in k(Q)$.

The following proposition is clear.

Proposition 3.1. *The set of $\tilde{\gamma}$, where γ runs over the set of paths of Q, is a basis of $k(Q)$.*

Note that $k(Q)$ is a graded algebra, with Q_0 in degree 0 and Q_1 in degree 1. In general, a path of length n is homogeneous of degree n.

There is an equivalence from the category of representations of Q to the category of (left) $k(Q)$-modules: given (V_s, ϕ_a) a representation of Q, let $M = \bigoplus_s V_s$. We define a structure of $k(Q)$-module as follows: $s \in Q_0$ is the projection onto V_s. An element $a \in Q_1$ acts by zero on $\bigoplus_{s \neq p(a)} V_s$ and sends $V_{p(a)}$ to $V_{q(a)}$ via ϕ_a.

Assume k is a field. Given $s \in Q_0$, there is a simple representation $S = S(s)$ of Q given by $S_t = 0$ for $t \neq s$, $S_s = k$ and $\phi_a = 0$ for all $a \in Q_1$. When $k(Q)$ is finite dimensional, we obtain all simple representations of Q, up to isomorphism.

Example 3.2. For each of the following quivers, we give the list of finite dimensional indecomposable representations (up to isomorphism) and we indicate the isomorphism type of the quiver algebra. We assume k is a field.

(i) $\bullet$: (k). The quiver algebra is k.

(ii) $1 \longrightarrow 2$: $S(1) = (S_1 = k, S_2 = 0, \phi = 0)$, $S(2) = (S_1 = 0, S_2 = k, \phi = 0)$ and $M = (M_1 = k, M_2 = k, \phi = 1)$. The quiver algebra is isomorphic to the algebra of 2×2 upper triangular matrices.

(iii) $\circlearrowleft \bullet$: $(k^n, \phi(n,\lambda))_{n \geq 1, \lambda \in k}$ with $\phi(n,\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$, assuming k is an algebraically closed. The quiver algebra is isomorphic to $k[x]$.

3.1.2. *Quivers with relations.* Let Q be a quiver and k a commutative ring. A set R of relations for Q over k is a finite set of elements of $k(Q)_{\geq 2}$. We denote by $I = (R)$ the two-sided ideal of $k(Q)$ generated by R and we put $A = k(Q)/I$.

Remark 3.3. Assume k is an algebraically closed field. Let A be a basic finite dimensional k-algebra (*i.e.*, all simple A-modules have dimension 1). One shows that there is a quiver Q with relations R such that $A \simeq k(Q)/(R)$. The vertices of Q are in bijection with the set of simple A-modules, up to isomorphism, while the set of arrows is in bijection with a basis of $\mathrm{rad}(A)/\mathrm{rad}(A)^2$.

3.2. **Quiver Hecke algebras.** We review some constructions and results of [Rou2, §3.2] and provide some complements. We will give three definitions of quiver Hecke algebras:

- By generators and relations, modulo polynomial torsion
- By (more complicated) generators and relations
- As a subalgebra of a ring of endomorphisms of a polynomial ring (over a quiver)

3.2.1. *Wreath and nil-wreath products.* Let k be a commutative ring. Let I be a finite set and $n \geq 0$. We define a quiver $\Psi_{I,n}$ with vertex set I^n. We will use the action of the symmetric group $\mathfrak{S}_n$ on I^n. The arrows are

- $s_i = s_{i,v} : v \to s_i(v)$ for $1 \leq i \leq n-1$ and $v \in I^n$
- $x_i = x_{i,v} : v \to v$ for $1 \leq i \leq n$ and $v \in I^n$.

We define the quiver algebra $A = A(\Psi_{I,n}, R_1)$ over k with the quiver above and relations R_1:

$$s_i^2 = 1,\ s_i s_j = s_j s_i \text{ if } |i-j| > 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$
$$x_i x_j = x_j x_i,\ s_i x_j = x_j s_i \text{ if } j \neq i, i+1 \text{ and } s_i x_i = x_{i+1} s_i.$$

When $|I| = 1$, we have $A = k[x_1, \ldots, x_n] \rtimes \mathfrak{S}_n = k[x] \wr \mathfrak{S}_n$. In general, $A = k[x]^I \wr \mathfrak{S}_n$.

We can construct a similar algebra, based on the nil Hecke algebra instead of $k\mathfrak{S}_n$. We use T_i to denote the arrow called s_i earlier. We define a quiver algebra $A' = A(\Psi_{I,n}, R_0)$ with relations R_0:

$$T_i^2 = 0,\ T_i T_j = T_j T_i \text{ if } |i-j| > 1,\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$
$$x_i x_j = x_j x_i,\ T_i x_j = x_j T_i \text{ if } j \neq i, i+1, T_i x_i = x_{i+1} T_i \text{ and } T_i x_{i+1} = x_i T_i.$$

Remark 3.4. Let B be a k-algebra and $n \geq 0$. There is a (unique) k-algebra structure on $(B^{\otimes_k n}) \otimes ({}^0H_n^f)$ such that $B^{\otimes_k n} = \underbrace{B \otimes_k B \otimes_k \cdots \otimes_k B}_{n \text{ factors}}$ and ${}^0H_n^f$ are subalgebras and $(1 \otimes T_w) \cdot \big((a_1 \otimes \cdots \otimes a_n) \otimes 1\big) = \big((a_{w(1)} \otimes \cdots \otimes a_{w(n)}) \otimes T_w\big)$. We denote by $B \wr ({}^0H_n^f)$ the corresponding algebra.

We have $A' = k[x]^I \wr ({}^0H^f_n)$.

3.2.2. *Definition.* We come now to the definition of the quiver Hecke algebras [Rou2, §3.2]. Fix a matrix $Q = (Q_{ij})_{i,j\in I}$ in $k[u,u']$. Assume

- $Q_{ii} = 0$
- Q_{ij} is not a zero-divisor in $k[u,u']$ for $i \neq j$ and
- $Q_{ij}(u,u') = Q_{ji}(u',u)$.

We define the algebra $H'_n(Q) = A(\Psi_{I,n}, R'_Q)$ with relations R'_Q (we use τ_i to denote the arrow called s_i earlier):

$$\tau_{i,s_i(v)}\tau_{i,v} = Q_{v_i,v_{i+1}}(x_{i,v},x_{i+1,v}),\ \tau_i\tau_j = \tau_j\tau_i \text{ if } |i-j|>1$$

$$\tau_{i+1,s_is_{i+1}(v)}\tau_{i,s_{i+1}(v)}\tau_{i+1,v} = \tau_{i,s_{i+1}s_i(v)}\tau_{i+1,s_i(v)}\tau_{i,v} \text{ if } v_i \neq v_{i+2} \text{ or } v_i = v_{i+1} = v_{i+2}$$

$$x_ix_j = x_jx_i,\ \tau_{i,v}x_{a,v} - x_{s_i(a),s_i(v)}\tau_{i,v} = \begin{cases} -1_v & \text{if } a = i \text{ and } v_i = v_{i+1} \\ 1_v & \text{if } a = i+1 \text{ and } v_i = v_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

We define $H_n(Q) = A(\Psi_{I,n}, R_Q)$ where R_Q consist of the relations R'_Q together with

$$\tau_{i+1,s_is_{i+1}(v)}\tau_{i,s_{i+1}(v)}\tau_{i+1,v} - \tau_{i,s_{i+1}s_i(v)}\tau_{i+1,s_i(v)}\tau_{i,v} = \\ (x_{i+2,v}-x_{i,v})^{-1}\big(Q_{v_i,v_{i+1}}(x_{i+2,v},x_{i+1,v}) - Q_{v_i,v_{i+1}}(x_{i,v},x_{i+1,v})\big)$$

when $v_i = v_{i+2} \neq v_{i+1}$

Note that the relation is written using $(x_{i+2,v}-x_{i,v})^{-1}$ to simplify the expression: the fraction is actually a polynomial in $x_{i,v}$, $x_{i+1,v}$ and $x_{i+2,v}$.

The following lemma shows that, up to multiplication by a polynomial, these extra relations follow from the ones in R'_Q. Since $H_n(Q)$ has no polynomial torsion, it follows that $H_n(Q)$ is the quotient of $H'_n(Q)$ by the polynomial torsion.

Lemma 3.5. *The kernel of the canonical surjective morphism of quiver algebras $H'_n(Q) \twoheadrightarrow H_n(Q)$ is the subspace of elements a such that there is $P \in k[x_1,\ldots,x_n]$ that is not a zero-divisor and such that $Pa = 0$.*

Proof. The property of $H_n(Q)$ to have no polynomial torsion is a consequence of Proposition 3.7 below. We show here that the kernel of the canonical map $H'_n(Q) \to H_n(Q)$ is made of torsion elements, *i.e.*, that suitable polynomial multiples of the relations in R_Q but no in R'_Q come from relations in R'_Q.

Consider $v \in I^n$ with $v_i = v_{i+2} \neq v_{i+1}$. We have the following equalities in $H'_n(Q)$:

$$(\tau_{i,v}\tau_{i+1,s_{i+1}(v)}\tau_{i,s_{i+1}(v)})\tau_{i+1,v} = (\tau_{i+1,s_i(v)}\tau_{i,v}\tau_{i+1,s_{i+1}(v)})\tau_{i+1,v} = \tau_{i+1,s_i(v)}\tau_{i,v}Q_{v_{i+1},v_i}(x_{i+1,v},x_{i+2,v})$$

$$(\tau_{i,v}\tau_{i,s_i(v)})\tau_{i+1,s_i(v)}\tau_{i,v} = Q_{v_{i+1},v_i}(x_{i,s_i(v)},x_{i+1,s_i(v)})\tau_{i+1,s_i(v)}\tau_{i,v} = \\ \tau_{i+1,s_i(v)}\tau_{i,v}Q_{v_{i+1},v_i}(x_{i+1,v},x_{i+2,v}) + \tau_{i,v}(x_{i+2,v}-x_{i+1,v})^{-1}(Q_{v_i,v_{i+1}}(x_{i+2,v},x_{i+1,v}) - Q_{v_i,v_{i+1}}(x_{i,v},x_{i+1,v}))$$

Let

$$a = \tau_{i+1,s_is_{i+1}(v)}\tau_{i,s_{i+1}(v)}\tau_{i+1,v} - \tau_{i,s_{i+1}s_i(v)}\tau_{i+1,s_i(v)}\tau_{i,v} - \\ - (x_{i+2,v}-x_{i,v})^{-1}(Q_{v_i,v_{i+1}}(x_{i+2,v},x_{i+1,v}) - Q_{v_i,v_{i+1}}(x_{i,v},x_{i+1,v}))$$

We have shown that $\tau_{i,v}a = 0$, hence

$$0 = \tau_{i,s_i(v)}\tau_{i,v}a = Q_{v_i,v_{i+1}}(x_{i,v}, x_{i+1,v})a$$

and the lemma follows. ☐

When $|I| = 1$, we have $A = k^0H_n$.

Let J be a set of finite sequences of elements of $\{1,\ldots,n-1\}$ such that $\{s_{i_r}\cdots s_{i_1}\}_{(i_1,\ldots,i_r)\in J}$ is a set of minimal length representatives of elements of $\mathfrak{S}_n$.

The following lemma is straightforward.

Lemma 3.6. *The set*

$$S = \{\tau_{i_r,s_{i_{r-1}}\cdots s_{i_1}(v)}\cdots\tau_{i_1,v}x_{1,v}^{a_1}\cdots x_{n,v}^{a_n}\}_{(i_1,\ldots,i_r)\in J,(a_1,\ldots,a_n)\in\mathbf{Z}^n_{\geq 0},v\in I^n}$$

generates $H_n(Q)$ as a k-module.

Proof. Given a product a of generators τ_i and x_i, one shows by induction on the number of τ_i's in the product, then on the number of pairs of an x_i to the left of a τ_j, that a is a linear combination of elements in S. ☐

Note that the generating set is compatible with the quiver algebra structure, as it is made of paths. Given $v, v' \in I^n$, the set

$$\{\tau_{i_r,s_{i_{r-1}}\cdots s_{i_1}(v)}\cdots\tau_{i_1,v}x_{1,v}^{a_1}\cdots x_{n,v}^{a_n}\}_{(i_1,\ldots,i_r)\in J,(a_1,\ldots,a_n)\in\mathbf{Z}^n_{\geq 0},s_{i_r}\cdots s_{i_1}(v)=v'}$$

generates $1_{v'}H_n(Q)1_v$ as a k-module.

The algebra $H_n(Q)$ is filtered with 1_v and $x_{i,v}$ in degree 0 and $\tau_{i,v}$ in degree 1. The relations R_Q become the relations R_0 after neglecting terms of lower degree, *i.e.*, the morphism $A(\Psi_{I,n}) \to H_n(Q)$ gives rise to a surjective algebra morphism $f : A' = k[x]^I \wr {}^0H_n^f \to \mathrm{gr}H_n(Q)$, where $\mathrm{gr}H_n(Q) = \bigoplus_{i\geq 0} F^iH_n(Q)/F^{i-1}H_n(Q)$ is the graded algebra associated with the filtration.

Proposition 3.7. *The algebra $H_n(Q)$ satisfies the Poincaré-Birkhoff-Witt property,* i.e., *the morphism f is an isomorphism. Furthermore, $H_n(Q)$ is a free k-module with basis S*

We will prove this proposition by constructing a faithful polynomial representation. This is similar to the case of the nil affine Hecke algebra (case $|I| = 1$).

Assume there is $d \in \mathbf{Z}^I$ such that $Q_{ij}(u^{d_i}, v^{d_j})$ is a homogeneous polynomial for all $i \neq j$. We denote by p_{ij} the degree of $Q_{ij}(u^{d_i}, v^{d_j})$. Then, the algebra $H_n(Q)$ is a graded k-algebra with $\deg x_i = 2d_{\nu_i}$ and $\deg \tau_{ij} = p_{\nu_i\nu_j}$.

The quiver Hecke algebras have been introduced and studied independently by Khovanov and Lauda [KhoLau1, KhoLau2] for particular Q's.

3.2.3. *Polynomial realization.* Let $P = (P_{ij})_{i,j\in I}$ be a matrix in $k[u,u']$ with $P_{ii} = 0$ for all $i \in I$ and such that P_{ij} is not a zero-divisor for $i \neq j$. Let $Q_{i,j}(u,u') = P_{i,j}(u,u')P_{j,i}(u',u)$.

We consider the following representation $M = (M_v)_{v\in I^n}$ of our quiver algebra. We put $M_v = k[x_1,\ldots,x_n]$. We let x_i act by multiplication by x_i and

$$\tau_{i,v} : M_v \to M_{s_i(v)} \text{ acts by } \begin{cases} (x_i - x_{i+1})^{-1}(s_i - 1) & \text{if } s_i(v) = v \\ P_{v_i,v_{i+1}}(x_{i+1}, x_i)s_i & \text{otherwise.} \end{cases}$$

Proposition 3.8. *The construction above defines a faithful representation of $H_n(Q)$ on M.*

Proof of Propositions 3.7 and 3.8. Let $\tau'_{i,v} = \begin{cases} (x_i - x_{i+1})^{-1}(s_i - 1) & \text{if } v_i = v_{i+1} \\ P_{v_i,v_{i+1}}(x_{i+1}, x_i)s_i & \text{otherwise.} \end{cases}$

We have $\tau'_{i,s_{i+1}(v)}\tau'_{i+1,v} =$

$$\begin{cases} (x_i - x_{i+1})^{-1}\left((x_i - x_{i+2})^{-1}(s_is_{i+1} - s_i) - (x_{i+1} - x_{i+2})^{-1}(s_{i+1} - 1)\right) & \text{if } v_i = v_{i+1} = v_{i+2} \\ P_{v,v_{i+1}}(x_{i+1}, x_i)(x_i - x_{i+2})^{-1}(s_is_{i+1} - s_i) & \text{if } v_{i+1} = v_{i+2} \neq v_i \\ (x_i - x_{i+1})^{-1}\left(P_{v_{i+1},v_{i+2}}(x_{i+2}, x_i)s_is_{i+1} - P_{v_{i+1},v_{i+2}}(x_{i+2}, x_{i+1})s_{i+1}\right) & \text{if } v_i = v_{i+2} \neq v_{i+1} \\ P_{v_i,v_{i+2}}(x_{i+1}, x_i)P_{v_{i+1},v_{i+2}}(x_{i+2}, x_i)s_is_{i+1} & \text{if } v_{i+2} \notin \{v_i, v_{i+1}\}. \end{cases}$$

One checks then easily that the defining relations of $H_n(Q)$ hold.

It is easy to check that the image of S in $\mathrm{End}_{k^{I^n}}(M)$ is linearly independent over k: this can be done by extending scalars to $k(x_1, \ldots, x_n)$ and relating by a triangular base change the bases $\{\partial_w\}_{w\in\mathfrak{S}_n}$ and $\{w\}_{w\in\mathfrak{S}_n}$ of $\mathrm{End}_{k(x_1,\ldots,x_n)^{\mathfrak{S}_n}}(k(x_1, \ldots, x_n))$ as a left $k(x_1, \ldots, x_n)$-module. It follows that the canonical map $H_n(Q) \to \mathrm{End}_{k^{I^n}}(M)$ is injective and that S is a basis of $H_n(Q)$ over k. Also, the image of S in $\mathrm{gr}H_n(Q)$ lifts to a basis of A'.

Note finally that given (Q_{ij}), we can construct a matrix (P_{ij}) as follows: for $i \neq j$, choose an order. When $i < j$, we define $P_{ij} = Q_{ij}$ and $P_{ji} = 1$. □

Let $\nu \in I^n$. We put $1_{|\nu|} = \sum_{\sigma\in\mathfrak{S}_n/\operatorname{Stab}(\nu)} 1_{\sigma(\nu)}$ and $H(|\nu|) = 1_{|\nu|}H_n(Q)1_{|\nu|}$. The next proposition shows that $H(|\nu|)$ doesn't have "non-obvious" quotients that remain torsion-free over polynomials.

Let $n_i = \#\{r|\nu_r = i\}$ and let $\gamma_i : \{1, \ldots, n_i\} \to \{1, \ldots, n\}$ be the increasing map such that $\nu_{\gamma_i(r)} = i$ for all r.

For every $\sigma \in \mathfrak{S}_n$, we have a morphism of algebras

$$\bigotimes_i k[X_{i,1}, \ldots, X_{i,n_i}] \to 1_{\sigma(\nu)}H_n(\Gamma)1_{\sigma(\nu)},\ X_{i,r} \mapsto x_{i,\sigma(\gamma_i(r))}.$$

The diagonal map restricts to an algebra morphism $\bigotimes_i k[X_{i,1}, \ldots, X_{i,n_i}]^{\mathfrak{S}_{n_i}} \to Z(H(|\nu|))$ (this is actually an isomorphism by [Rou2, Proposition 3.9]).

Proposition 3.9. *Let J be a non-zero two-sided ideal of $H(|\nu|)$. Then, there is a non-zero $P \in \bigotimes_i k[X_{i,1}, \ldots, X_{i,n_i}]^{\mathfrak{S}_{n_i}}$ such that $P \cdot \mathrm{id}_M \in J$.*

Proof. Consider the algebra $A = k^I[x] \wr \mathfrak{S}_n = \left(\bigoplus_{\mu\in I^n} k[x_1, \ldots, x_n]1_\mu\right) \rtimes \mathfrak{S}_n$. Let $\mathcal{O} = \bigoplus_\mu k[x_1, \ldots, x_n][\{(x_i-x_j)^{-1}\}]_{i\neq j,\mu_i=\mu_j}1_\mu$ and $A' = \mathcal{O}\otimes_{(k^I[x])^{\otimes n}}A$. Let $B = \bigoplus_{\sigma,\sigma'\in\mathfrak{S}_n} 1_{\sigma(\nu)}A'1_{\sigma'(\nu)}$. The algebra B is Morita-equivalent to its center which is isomorphic to

$$\bigotimes_{i\in I}\left(k[x_1, \ldots, x_{n_i}][\{(x_a - x_b)^{-1}\}]_{a\neq b}\right)^{\mathfrak{S}_{n_i}}.$$

It follows that any non-zero ideal of B intersects non-trivially $Z(B)$. The proposition follows now from the embedding of $\mathrm{End}_{\mathcal{B}}(M)$ in B and the properties of that embedding [Rou2, Proposition 3.12]. □

3.2.4. *Cartan matrices and quivers.* A generalized Cartan matrix is a matrix $C = (a_{ij})_{i,j\in I}$ such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$. The matrix C is symmetrizable if in addition there is a diagonal matrix D with diagonal coefficients in $\mathbf{Z}_{>0}$ such that DC is symmetric.

Consider now a graph with vertex set I and with no loop. We define a symmetric Cartan matrix by putting $a_{ij} = -m_{ij}$ for $i \neq j$, where m_{ij} is the number of edges between i and j. This correspondence gives a bijection between graphs with no loops and symmetric Cartan matrices.

Let Γ be a quiver (with no loops) and I its vertex set. This defines a graph by forgetting the orientation, hence a symmetric Cartan matrix. Let d_{ij} be the number of arrows $i \to j$, so that $m_{ij} = d_{ij} + d_{ji}$. Let $Q_{ij} = (-1)^{d_{ij}}(u-v)^{m_{ij}}$ for $i \neq j$. We put $H_n(\Gamma) = H_n(Q)$, where $k = \mathbf{Z}$.
This is a graded algebra, with $\deg x_i = 2$ and $\deg \tau_{v,i} = -a_{v_i,v_{i+1}}$.

Let $v, v' \in I^n$. Let $n_i = \#\{r | v_r = i\}$ and $n'_i = \#\{r | v'_r = i\}$. We have $1_{v'} H_n(\Gamma) 1_v = 0$ unless $n_i = n'_i$ for all i. Assume this holds. Define $\gamma_i, \gamma'_i : \{1, \ldots, n_i\} \to \{1, \ldots, n\}$ to be the increasing maps such that $v_{\gamma_i(r)} = v'_{\gamma'_i(r)} = i$ for all r. Let $W = \prod_i \mathfrak{S}_{n_i}$.

Lemma 3.10. *The left (resp. right)* $\mathbf{Z}[x_1, \ldots, x_n]$*-module* $1_{v'} H_n(\Gamma) 1_v$ *is free: there is a graded* $\mathbf{Z}$*-module* L *such that*

- $1_{v'} H_n(\Gamma) 1_v \simeq \mathbf{Z}[x_1, \ldots, x_n] \otimes_{\mathbf{Z}} L$ *as graded left* $\mathbf{Z}[x_1, \ldots, x_n]$*-modules and*
- $1_{v'} H_n(\Gamma) 1_v \simeq L \otimes_{\mathbf{Z}} \mathbf{Z}[x_1, \ldots, x_n]$ *as graded right* $\mathbf{Z}[x_1, \ldots, x_n]$*-modules.*

We have
$$\mathrm{grdim} L = \sum_{w \in W} q^{\frac{1}{2}\sum_{s,t\in I} a_{st}\cdot\#\{a,b | \gamma_s(a) < \gamma_t(b) \text{ and } \gamma'_s(w_s(a)) > \gamma'_t(w_t(b))\}}.$$

Proof. The first part of the lemma follows from Proposition 3.7 (and its right counterpart). We have
$$\mathrm{grdim} L = \sum_{\substack{(i_1,\ldots,i_r)\in J \\ s_{i_r}\cdots s_{i_1}(v) = v'}} q^{\frac{1}{2}\deg(\tau_{i_r, s_{i_{r-1}}\cdots s_{i_1}(v)}\cdots\tau_{i_1,v})}.$$
The set E of elements $h \in \mathfrak{S}_n$ such that $h(v) = v'$ is a left (and right) principal homogeneous set under the action of W, via the maps γ_i and γ'_i. Denote by $g \in \mathfrak{S}_n$ the unique element of minimal length such that $g(v) = v'$. Then, we obtain a bijection $W \xrightarrow{\sim} E$, $w \mapsto w \circ g$. The formula follows from a variant of Proposition 2.1. $\square$

3.2.5. *Relation with (degenerate) affine Hecke algebras.* We show in this section that quiver Hecke algebras associated with quivers of type A (finite or affine) are connected with (degenerate) affine Hecke algebras for GL_n.

Let $\bar{R} = \mathbf{Z}[q^{\pm 1}] = R[q^{\pm 1}]/(q_1 - q, q_2 + 1)$. Let H_n be the *affine Hecke algebra*: it is the $\bar{R}$-algebra generated by elements $X_1, \ldots, X_n$, $T_1, \ldots, T_{n-1}$ where the X_i are invertible and the relations are
$$(T_i - q)(T_i + 1) = 0,\ T_iT_j = T_jT_i \text{ if } |i-j| > 1, T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$
$$X_iX_j = X_jX_i,\ T_iX_j = X_jT_i \text{ if } j - i \neq 0, 1 \text{ and } T_iX_{i+1} - X_iT_i = (q-1)X_{i+1}.$$
As in the nil affine Hecke case (Proposition 2.17), we have a decomposition as $\bar{R}$-modules
$$H_n = \bar{R}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes_{\bar{R}} \bar{R}H_n^f$$
and $\bar{R}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and $\bar{R}H_n^f$ are subalgebras.

Let k be field endowed with an $\bar{R}$-algebra structure and assume $q1_k \neq 1_k$.

Given M a k-vector space and x an endomorphism of M, we say that x is *locally nilpotent* on M if M is the union of subspaces on which x is nilpotent or, equivalently, if for every $m \in M$, there is $N > 0$ such that $x^N m = 0$.

Let M be a kH_n-module. Given $v \in (k^\times)^n$, we denote by M_v the subspace of M on which $X_i - v_i$ acts locally nilpotently for $1 \leq i \leq n$.

Let I be a subset of $k^\times$ and let $\mathcal{C}_I$ be the category of kH_n-modules M such that $M = \bigoplus_{v \in I^n} M_v$. Note that a finite dimensional kH_n-module is in $\mathcal{C}_I$ if and only if the eigenvalues of the X_i acting on M are in I.

We define a quiver Γ with vertex set I and arrows $i \to qi$. Assume $q \neq 1$. Denote by e the multiplicative order of q. When Γ is connected, the possible types of the underlying graph are

- A_n if $|I| = n < e$.
- $\tilde{A}_{e-1}$ if $|I| = e$.
- A_∞ if I is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if I is unbounded in both directions.

We denote by $\mathcal{C}_\Gamma^0$ the category of $kH_n(\Gamma)$-modules M such that for every $v \in I^n$, then $x_{i,v}$ acts locally nilpotently on M_v.

The proof of the following Theorem (and the next one) relies on checking relations and writing formulas for an inverse functor.

Theorem 3.11 ([Rou2, Theorem 3.20])**.** *There is an equivalence of categories $\mathcal{C}_\Gamma^0 \xrightarrow{\sim} \mathcal{C}_I$ given by $(M_v)_v \mapsto \bigoplus_v M_v$ and where X_i acts on M_v by $(x_i + v_i)$ and T_i acts on M_v by*

- $(qx_i - x_{i+1})\tau_i + q$ *if* $v_i = v_{i+1}$
- $(q^{-1}x_i - x_{i+1})^{-1}(\tau_i + (1-q)x_{i+1})$ *if* $v_{i+1} = qv_i$
- $(v_i x_i - v_{i+1}x_{i+1})^{-1}\left((qv_i x_i - v_{i+1}x_{i+1})\tau_i + (1-q)v_{i+1}x_{i+1}\right)$ *otherwise.*

There is yet another version of affine Hecke algebras: the *degenerate affine Hecke algebra $\bar{H}_n$*, a $\mathbf{Z}$-algebra generated by $X_1, \ldots, X_n$ and $s_1, \ldots, s_{n-1}$ with relations

$$s_i^2 = 1, s_i s_j = s_j s_i \text{ if } |i-j| > 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$X_i X_j = X_j X_i,\ s_i X_j = X_j s_i \text{ if } j - i \neq 0, 1 \text{ and } s_i X_{i+1} - X_i s_i = 1.$$

We have a $\mathbf{Z}$-module decomposition

$$\bar{H}_n = \mathbf{Z}[X_1, \ldots, X_n] \otimes \mathbf{Z}\mathfrak{S}_n$$

and $\mathbf{Z}[X_1, \ldots, X_n]$ and $\mathbf{Z}\mathfrak{S}_n$ are subalgebras.

Let k be a field.

Let I a subset of k. We denote by Γ the quiver with set of vertices I and with arrows $i \to i+1$.

When Γ is connected, the possible types of the underlying graph are

- A_n if $|I| = n$ and k has characteristic 0 or $p > n$.
- $\tilde{A}_{p-1}$ if $|I| = p$ is the characteristic of k.
- A_∞ if I is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if I is unbounded in both directions.

Given M a $k\bar{H}_n$-module and $v \in k^n$, we denote by M_v the subspace of M where $X_i - v_i$ acts locally nilpotently for all i. Let I be a subset of $k^\times$ and let $\bar{\mathcal{C}}_I$ be the category of $k\bar{H}_n$-modules M such that $M = \bigoplus_{v \in I^n} M_v$.

Theorem 3.12 ([Rou2, Theorem 3.17]). *There is an equivalence of categories $\mathcal{C}^0_\Gamma \xrightarrow{\sim} \bar{\mathcal{C}}_I$ given by $(M_v)_v \mapsto \bigoplus_v M_v$ and where X_i acts on M_v by $(x_i + v_i)$ and s_i acts on M_v by*

- $(x_i - x_{i+1} + 1)\tau_i + 1$ *if* $v_i = v_{i+1}$
- $(x_i - x_{i+1} - 1)^{-1}(\tau_i - 1)$ *if* $v_{i+1} = v_i + 1$
- $(x_i - x_{i+1} + v_{i+1} - v_i + 1)(x_i - x_{i+1} + v_{i+1} - v_i)^{-1}(\tau_i - 1) + 1$ *otherwise.*

3.3. **Half 2-Kac-Moody algebras.**

3.3.1. *Monoidal categories.* Recall that a *strict monoidal category* is a category equipped with a tensor product with a unit and satisfying $(V \otimes W) \otimes X = V \otimes (W \otimes X)$. We will write XY for $X \otimes Y$. We will also denote by X the identity endomorphism of an object X.

We have a canonical map $\mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(W_1, W_2) \to \mathrm{Hom}(V_1 \otimes W_1, V_2 \otimes W_2)$. Given $f : V_1 \to V_2$ and $g : W_1 \to W_2$, there is a commutative diagram

$$\begin{tikzcd} V_1 \otimes W_1 \arrow[r, "f\otimes W_1"] \arrow[d, "V_1\otimes g"'] \arrow[dr, "f\otimes g"] & V_2 \otimes W_1 \arrow[d, "V_2\otimes g"] \\ V_1 \otimes W_2 \arrow[r, "f\otimes W_2"'] & V_2 \otimes W_2 \end{tikzcd}$$

A typical example of a monoidal category is the category of vector spaces over a field (or more generally, modules over a commutative algebra).

A more interesting example is the following. Let $\mathcal{A}$ be a category and $\mathcal{C}$ be the category of functors $\mathcal{A} \to \mathcal{A}$. Then, $\mathcal{C}$ is a strict monoidal category where the product is the composition of functors. The Hom-spaces are given by natural transformations of functors.

Let $\mathcal{C}$ be an additive category. We define the *idempotent completion* $\mathcal{C}^i$ of $\mathcal{C}$ as the additive category obtained from $\mathcal{C}$ by adding images of idempotents: its objects are pairs (M, e) where M is an object of $\mathcal{C}$ and e is an idempotent of $\mathrm{End}_{\mathcal{C}}(M)$. We put $\mathrm{Hom}_{\mathcal{C}^i}((M, e), (N, f)) = f \,\mathrm{Hom}_{\mathcal{C}}(M, N)e$. We have a fully faithful functor $\mathcal{C} \to \mathcal{C}^i$ given by $M \mapsto (M, \mathrm{id}_M)$. If A is an algebra, the idempotent completion of the category of free A-modules is equivalent to the category of projective A-modules.

We say that $\mathcal{C}$ is *idempotent complete* if the canonical functor $\mathcal{C} \to \mathcal{C}^i$ is an equivalence, *i.e.*, if every idempotent has an image.

3.3.2. *Symmetric groups.* Let us start with an example of monoidal category based on symmetric groups. We define $\mathcal{C}$ to be the strict monoidal $\mathbf{Z}$-linear category generated by an object E and by an arrow $s : E^2 \to E^2$ subject to the relations

$$s^2 = E^2,\ (Es) \circ (sE) \circ (Es) = (sE) \circ (Es) \circ (sE).$$

This category is easy to describe: its objects are direct sums of copies of E^n for various n's. We have $\mathrm{Hom}(E^n, E^m) = 0$ if $m \neq n$ and $\mathrm{End}(E^n) = \mathbf{Z}[\mathfrak{S}_n]$: this is given by $s_i \mapsto E^{n-i-1}sE^{i-1}$.

Note that, as a monoidal category, $\mathcal{C}$ is equipped with maps $\mathrm{End}(E^m)\times\mathrm{End}(E^n) \to \mathrm{End}(E^{m+n})$. They correspond to the embedding $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$, where $\mathfrak{S}_n$ goes to $\mathfrak{S}\{1,\dots,n\}$ and $\mathfrak{S}_m$ goes to $\mathfrak{S}\{n+1,\dots,n+m\}$.

Remark 3.13. The category $\mathcal{C}$ can also be defined as the free symmetric monoidal $\mathbf{Z}$-linear category on one object E.

3.3.3. *Half.* Let us follow now [Rou2, §4.1.1]. Let $C = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix. We construct a matrix Q satisfying the conditions of §3.2.2.

Let $\{t_{i,j,r,s}\}$ be a family of indeterminates with $i \neq j \in I$, $0 \le r < -a_{ij}$ and $0 \le s < -a_{ji}$ and such that $t_{j,i,s,r} = t_{i,j,r,s}$. Let $\{t_{ij}\}_{i\neq j}$ be a family of indeterminates with $t_{ij} = t_{ji}$ if $a_{ij} = 0$.

Let $k = k^C = \mathbf{Z}[\{t_{i,j,r,s}\} \cup \{t_{ij}^{\pm 1}\}]$. Given $i, j \in I$, we put

$$Q_{ij} = \begin{cases} 0 & \text{if } i = j \\ t_{ij} & \text{if } i \neq j \text{ and } a_{ij} = 0 \\ t_{ij}u^{-a_{ij}} + \sum_{\substack{0\le r<-a_{ij}\\ 0\le s<-a_{ji}}} t_{i,j,r,s}u^r v^s + t_{ji}v^{-a_{ji}} & \text{if } i \neq j \text{ and } a_{ij} \neq 0. \end{cases}$$

We define $\mathcal{B} = \mathcal{B}(C)$ as the strict monoidal k-linear category generated by objects F_s for $s \in I$ and by arrows

$$x_s : F_s \to F_s \text{ and } \tau_{st} : F_sF_t \to F_tF_s \text{ for } s,t \in I$$

with relations

(1) $\tau_{st} \circ \tau_{ts} = Q_{st}(F_t x_s, x_t F_s)$

(2) $\tau_{tu}F_s \circ F_t\tau_{su} \circ \tau_{st}F_u - F_u\tau_{st} \circ \tau_{su}F_t \circ F_s\tau_{tu} = \begin{cases} \frac{Q_{st}(x_sF_t, F_sx_t)F_s - F_sQ_{st}(F_tx_s, x_tF_s)}{x_sF_tF_s - F_sF_tx_s} F_s & \text{if } s = u \\ 0 & \text{otherwise.} \end{cases}$

(3) $\tau_{st} \circ x_sF_t - F_tx_s \circ \tau_{st} = \delta_{st}$

(4) $\tau_{st} \circ F_sx_t - x_tF_s \circ \tau_{st} = -\delta_{st}$

Given $n \ge 0$, we denote by $\mathcal{B}_n$ the full subcategory of $\mathcal{B}$ whose objects are direct sums of objects of the form $F_{v_n} \cdots F_{v_1}$ for $v_1, \dots, v_n \in I$. We have $\mathcal{B} = \bigoplus_n \mathcal{B}_n$ (as k-linear categories).

The relations state that the maps x_s and τ_{st} give an action of the quiver Hecke algebra associated with Q on sum of products of F_s. More precisely, we have an isomorphism of algebras

$$\begin{aligned} H_n(Q) &\xrightarrow{\sim} \bigoplus_{v,v'\in I^n} \mathrm{Hom}_{\mathcal{B}}(F_{v_n}\cdots F_{v_1}, F_{v'_n}\cdots F_{v'_1}) \\ 1_v &\mapsto \mathrm{id}_{F_{v_n}\cdots F_{v_1}} \\ x_{i,v} &\mapsto F_{v_n}\cdots F_{v_{i+1}} x_{v_i} F_{v_{i-1}} \cdots F_{v_1} \\ \tau_{i,v} &\mapsto F_{v_n}\cdots F_{v_{i+2}} \tau_{v_{i+1},v_i} F_{v_{i-1}} \cdots F_{v_1} \end{aligned}$$

Note that divided powers can be defined in $\mathcal{B}^i$, following [ChRou, §5.2.1]. We have an isomorphism $k \otimes_{\mathbf{Z}} {}^0H_n \xrightarrow{\sim} \mathrm{End}(E_i^n)$. The endomorphism of E_i^n induced by $T_{w[1,n]}$ has an image $E_i^{(n)} \in \mathcal{B}^i$. We have $E_i^n \simeq E_i^{(n)} \otimes_k k^{n!}$.

Assume now C is symmetrizable, with $D = \mathrm{diag}(d_i)_{i\in I}$. We define $k^{\mathrm{gr}} = k^C/(\{t_{i,j,r,s}\}_{d_ir+d_js\neq -d_ia_{ij}})$ and $\mathcal{B}^{\mathrm{gr}} = \mathcal{B} \otimes_k k^{\mathrm{gr}}$. This category can be graded by setting $\deg x_s = 2d_s$, $\deg \tau_{st} = d_s a_{st}$ and $\deg \varepsilon_{s,\lambda} = d_s(1 - \langle \lambda, \alpha_s^\vee \rangle)$.

Let Γ be quiver with no loops whose underlying graph corresponds to C (which is then symmetric). Let J be the ideal of k generated by the coefficients of the polynomials $Q_{ij} - (-1)^{d_{ij}}(u-v)^{-a_{ij}}$ (cf §3.2.4) and $k^\Gamma = k^C/J = \mathbf{Z}$. We put $\mathcal{B}(\Gamma) = \mathcal{B} \otimes_k k^\Gamma$. This is again graded, as above.

Similar monoidal categories have been constructed independently by Khovanov and Lauda [KhoLau1, KhoLau2], who have shown that, over a field, they provide a categorification of $U_{\mathbf{Z}}(\mathfrak{n}^-)$ and $U_{\mathbf{Z}[q^{\pm 1/2}]}(\mathfrak{n}^-)$, where $\mathfrak{n}^-$ is the half Kac-Moody algebra associated to C (cf §4.1.1).

Given an additive category $\mathcal{C}$, we denote by $K_0(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$. Assume $\mathcal{C}$ is enriched in graded $\mathbf{Z}$-modules. We can define a new additive category $\mathcal{C}$-gr with objects families $\{M_i\}_{i\in\mathbf{Z}}$ of objects of $\mathcal{C}$ with $M_i = 0$ for almost all i. We put $\mathrm{Hom}_{\mathcal{C}\text{-gr}}(\{M_i\},\{N_i\}) = \bigoplus_{m,n} \mathrm{Hom}_{\mathcal{C}}(M_m, N_n)_{n-m}$. The category $\mathcal{C}$-gr is $\mathbf{Z}$-graded, *i.e.*, it is equipped with an automorphism T given by $T(\{M_i\})_n = M_{n+1}$. The action of T on $K_0(\mathcal{C}\text{-gr})$ endows it with a structure of $\mathbf{Z}[q^{\pm 1/2}]$-module, where $q^{1/2}$ acts by $[T]$.

Theorem 3.14 ([KhoLau2, Corollary 7 and Theorem 8]). *Given $s \neq t \in I$, there are isomorphisms in $\mathcal{B}^i$*

$$\bigoplus_{i \text{ even}} F_s^{(-a_{st}-i+1)} F_t F_s^{(i)} \simeq \bigoplus_{i \text{ odd}} F_s^{(-a_{st}-i+1)} F_t F_s^{(i)}.$$

Let K be a field that is a k-algebra. The relations above provide isomorphisms of rings

$$U_{\mathbf{Z}}(\mathfrak{n}^-) \xrightarrow{\sim} K_0(\mathcal{B}^i \otimes_k K).$$

When C is symmetrizable and K is in addition a k^{gr}-algebra, this gives an isomorphism of $\mathbf{Z}[q^{\pm 1/2}]$*-algebras*

$$U_{\mathbf{Z}[q^{\pm 1/2}]}(\mathfrak{n}^-) \xrightarrow{\sim} K_0((\mathcal{B}(\Gamma)^i \otimes_k K)\text{-gr}).$$

4. 2-Kac-Moody algebras

4.1. **Kac-Moody algebras.**

We recall some basic facts on Kac-Moody algebras and their representations [Kac] and quantum counterparts [Lu1].

Given an algebra A, we denote by A-Mod the category of A-modules, by A-mod the category of finitely generated A-modules and by A-proj the category of finitely generated projective A-modules.

4.1.1. *Data.* Let $C = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix. Let

$$(X, Y, \langle -,-\rangle, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$$

be a root datum of type C, *i.e.*,

- X and Y are finitely generated free abelian groups and $\langle -,-\rangle : X \times Y \to \mathbf{Z}$ is a perfect pairing
- $\{\alpha_i\}$ is a linearly independent set in X and $\{\alpha_i^\vee\}$ is a linearly independent set in Y
- $\langle \alpha_j, \alpha_i^\vee\rangle = a_{ij}$.

Associated with this data, there is a Kac-Moody algebra $\mathfrak{g}$ (over $\mathbf{C}$) generated by elements e_i, f_i and h_ζ for $i \in I$ and $\zeta \in Y$. When C is symmetrizable, there is also a quantum enveloping algebra $U_q(\mathfrak{g})$. When C corresponds to a Dynkin diagram, we recover complex reductive Lie algebras and their corresponding quantum groups.

We denote by $\mathfrak{n}^-$ the Lie subalgebra of $\mathfrak{g}$ generated by the f_i, $i \in I$. We denote by $U_{\mathbf{Z}}(\mathfrak{n}^-)$ the subring of $U(\mathfrak{n}^-)$ generated by the elements $\frac{f_i^n}{n!}$ for $i \in I$ and $n \geq 1$. Assume C is symmetrizable. The quantum algebra $U_q(\mathfrak{n}^-)$ is the $\mathbf{C}(q^{1/2})$-algebra with generators f_i, $i \in I$, and relations

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_q f_i^r f_j f_i^{1-a_{ij}-r} = 0$$

for $i \neq j$, where

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad \begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} \text{ and } [n]_q! = [2]_q \cdots [n]_q.$$

In the symmetrizable case, we denote by $U_{\mathbf{Z}[q^{\pm 1/2}]}(\mathfrak{n}^-)$ the $\mathbf{Z}[q^{\pm 1/2}]$-subalgebra of $U_q(\mathfrak{n}^-)$ generated by the $\frac{f_i^n}{[n]_q}$.

Example 4.1. Let $C = (2)$, $X = Y = \mathbf{Z}$, $\alpha = 2$ and $\alpha^\vee = 1$. Then $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$.

4.1.2. *Integrable representations.* Let M be a representation of $\mathfrak{g}$. We say that it is *integrable* if

- $M = \bigoplus_{\lambda \in X} M_\lambda$, where $M_\lambda = \{m \in M | h_\zeta \cdot m = \langle \lambda, \zeta \rangle m, \ \forall \zeta \in Y\}$
- e_i and f_i are locally nilpotent on M for every i, *i.e.*, given $m \in M$, there is an integer n such that $f^l(m) = e^l(m) = 0$ for $l \geq n$.

Define a quiver with vertex set X and arrows $e_i = e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ and $f_i = f_{i,\lambda} : \lambda \to \lambda - \alpha_i$. Let $A(\mathfrak{g})$ be its quiver algebra over $\mathbf{C}$, subject to the relations

$$e_{i,\lambda-\alpha_j} f_{j,\lambda} - f_{j,\lambda+\alpha_i} e_{i,\lambda} = \delta_{ij} \langle \lambda, \alpha_i^\vee \rangle 1_\lambda.$$

The following proposition has a proof based on the representation theory of $\mathfrak{sl}_2$. It says that the Serre relations are automatically satisfied under integrability conditions.

Proposition 4.2. *The functor $M \mapsto (M_\lambda)_{\lambda \in X}$ is an equivalence from the category of integrable representations of $\mathfrak{g}$ to the category of representations of the quiver algebra $A(\mathfrak{g})$ on which the e_i's and f_i's are locally nilpotent.*

We define a category $\mathcal{U}(\mathfrak{g})$ with $\mathbf{C}$-linear Hom-spaces. Its set of objects is X. The morphisms are generated by $e_i : \lambda \to \lambda + \alpha_i$, $f_i : \lambda \to \lambda - \alpha_i$, subject to the relations

$$[e_i, f_j]_{|\lambda} = \delta_{ij} \langle \lambda, \alpha_i^\vee \rangle.$$

Then, a $\mathbf{C}$-linear functor $\mathcal{U}(\mathfrak{g}) \to \mathbf{C}$-Mod is the same as a representation of the quiver algebra $A(\mathfrak{g})$.

Remark 4.3. A representation of a k-algebra A is the same as a functor compatible with the k-linear structure $\mathcal{C} \to k$-Mod, where $\mathcal{C}$ is the category with one object $*$ and $\mathrm{End}(*) = A$.

4.1.3. *Quantum counterpart.* Assume C is symmetrizable. One can proceed similarly and show that the category of integrable representations of $U_q(\mathfrak{g})$ is equivalent to the category of representations of the quiver algebra $A_q(\mathfrak{g})$, defined as the $\mathbf{C}(\sqrt{q})$-algebra with the same quiver as above and relations

$$e_{i,\lambda-\alpha_j} f_{j,\lambda} - f_{j,\lambda+\alpha_i} e_{i,\lambda} = \delta_{ij} [\langle \lambda, \alpha_i^\vee \rangle]_q 1_\lambda. \tag{1}$$

4.1.4. *Category* $\mathcal{O}^{\mathrm{int}}$. Define the set of integral dominant weights $X^+ = \{\lambda \in X | \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I\}$. We denote by $\mathcal{O}_{\mathfrak{g}}^{\mathrm{int}}$ the category of integrable highest weight modules M of $\mathfrak{g}$, *i.e.*, $\mathfrak{g}$-modules such that

- $M = \bigoplus_{\lambda \in X} M_\lambda$ and $\dim M_\lambda < \infty$ for all λ
- e_i and f_i are locally nilpotent on M for $i \in I$
- there is a finite set $K \subset X$ such that $\{\lambda \in X | M_\lambda \neq 0\} \subset \bigcup_{\mu \in K} (\mu + \sum_{i \in I} \mathbf{Z}_{\leq 0} \alpha_i)$.

Let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the elements e_i, $i \in I$ and h_ζ, $\zeta \in Y$. Let $\lambda \in X^+$. We denote by $\mathbf{C}v_\lambda$ the one-dimensional representation of $\mathfrak{b}$ where e_i acts by 0 and h_i acts by $\langle \lambda, \alpha_i^\vee \rangle$. We define the Verma module

$$\Delta(\lambda) = \mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbf{C}v_\lambda \simeq U(\mathfrak{n}^-)$$

and

$$L^{\max}(\lambda) = \Delta(\lambda) / \Big(\sum_{i \in I} U(\mathfrak{n}^-) f_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} v_\lambda \Big).$$

This is the largest quotient of $\Delta(\lambda)$ in $\mathcal{O}_{\mathfrak{g}}^{\mathrm{int}}$. The Verma module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$ and there is a surjection $L^{\max}(\lambda) \twoheadrightarrow L(\lambda)$. When C is symmetrizable, $L^{\max}(\lambda) = L(\lambda)$. The set $\{L(\lambda)\}_{\lambda \in X^+}$ is a complete set of representatives of isomorphism classes of simple integrable highest weight modules. These are the finite-dimensional simple $U(\mathfrak{g})$-modules when $\mathfrak{g}$ is finite-dimensional. When C is symmetrizable, integrable highest weight modules are semi-simple.

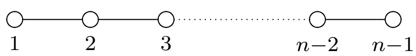
Note that $L^{\max}(\lambda)$ is characterized by the fact that it represents the functor

$$\mathcal{O}_{\mathfrak{g}}^{\mathrm{int}} \to \mathbf{C}\text{-Mod}, \ M \mapsto M_\lambda^{\mathrm{hw}} := \{m \in M_\lambda \mid e_i(m) = 0 \ \forall i \in I\},$$

i.e.,

$$\mathrm{Hom}_{\mathfrak{g}}(L^{\max}(\lambda), M) \xrightarrow{\sim} M_\lambda^{\mathrm{hw}}, \ f \mapsto f(v_\lambda).$$

4.1.5. $\hat{\mathfrak{sl}}_n$. Consider the complex simple Lie algebra $\mathfrak{sl}_n(\mathbf{C})$. This is a Kac-Moody algebra associated with the following graph:

$A_{n-1} =$ 

The Lie algebra $\mathfrak{sl}_n(\mathbf{C})$ is generated by the elements $e_i = e_{i,i+1}$, $f_i = e_{i+1,i}$ and $h_i = e_{i,i} - e_{i+1,i+1}$ for $1 \leq i \leq n-1$. We have $Y_{\mathfrak{sl}_n} = \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_{n-1}$, $X_{\mathfrak{sl}_n} = \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1}$ where $\langle \Lambda_i, h_j \rangle = \delta_{ij}$. We have $\alpha_i = e_{i,i}^* - e_{i+1,i+1}^*$ and $\alpha_i^\vee = h_i$ for $1 \leq i \leq n-1$.

Consider now the Lie algebra $\mathfrak{g}_l = \mathfrak{sl}_n(\mathbf{C}) \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$. We define $\mathfrak{g}' = \mathfrak{g}_l \oplus \mathbf{C}c$, a central extension of $\mathfrak{g}_l$ given by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m,-n} \mathrm{tr}(ab) c \text{ for } a, b \in \mathfrak{sl}_n(\mathbf{C}).$$

Finally, we define $\mathfrak{g} = \mathfrak{g}' \oplus \mathbf{C}d$. We endow it with a Lie algebra structure where

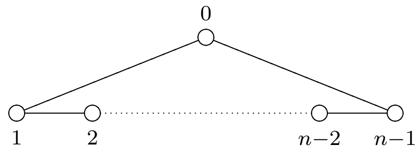
- $\mathfrak{g}'$ is a Lie subalgebra
- $[d, at^n] = nat^n$ for $a \in \mathfrak{sl}_n(\mathbf{C})$
- $[d, c] = 0$.

Note that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Let $e_0 = e_{n,1} \otimes t$, $f_0 = e_{1,n} \otimes t^{-1}$ and $h_0 = (e_{n,n} - e_{1,1}) + c$.

Let $Y = \mathbf{Z}h_0 \oplus \cdots \oplus \mathbf{Z}h_{n-1} \oplus \mathbf{Z}d = Y_{\mathfrak{sl}_n} \oplus \mathbf{Z}c \oplus \mathbf{Z}d$ and $X = \mathrm{Hom}_{\mathbf{Z}}(Y, \mathbf{Z})$, with $(\Lambda_0, \ldots, \Lambda_{n-1}, \partial)$ the basis dual to $(h_0, \ldots, h_{n-1}, d)$.

Let $\alpha_0 = \partial - (\alpha_1 + \cdots + \alpha_{n-1})$ and $\alpha_0^\vee = h_0$. This provides an identification of $\mathfrak{g}$ with a Kac-Moody algebra of graph

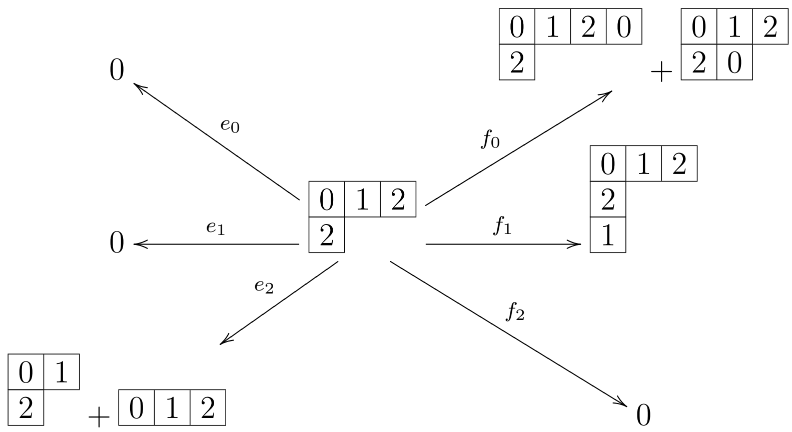
$\hat{A}_{n-1} =$ 

4.1.6. *Fock spaces.* We recall in this section some classical results on representations of symmetric groups and related Hecke algebras, and the relation with Fock spaces [Ari, Gr, Kl, Ma].

Let $\mathcal{F}$ be the complex vector space with basis all partitions. Let $p \geq 2$ be an integer.

Let us construct an action of $\hat{\mathfrak{sl}}_p$ on $\mathcal{F}$. Let λ be a partition. We consider the associated Young diagram, whose boxes we number modulo p. We define $e_i(\lambda)$ (resp. $f_i(\lambda)$) as the sum of the partitions obtained by removing (resp. adding) an i-node to λ. We put $d(\lambda) = N_0(\lambda)\lambda$, where $N_0(\lambda)$ is the number of 0-nodes of λ.

Example 4.4. Let us consider for example $p = 3$ and $\lambda = (3, 1)$.



The construction above defines an action of $\hat{\mathfrak{sl}}_p$ on $\mathcal{F}$, where c acts by 1 (*i.e.*, the level is 1). This defines an object of $\mathcal{O}^{\mathrm{int}}$.

Let $K_0(\mathbf{Q}\mathfrak{S}_n)$ be the Grothendieck group of the category $\mathbf{Q}\mathfrak{S}_n$-mod. It is a free abelian group with basis the isomorphism classes of irreducible representations of $\mathfrak{S}_n$ over $\mathbf{Q}$.

There is an isomorphism

$$\mathcal{F} \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathbf{C} \otimes K_0(\mathbf{Q}\mathfrak{S}_n)$$

It sends a partition λ to the class of the corresponding simple module S_λ of $\mathbf{Q}\mathfrak{S}_n$. We identify $\mathcal{F}$ with the sum of Grothendieck groups via this isomorphism.

We consider now a prime number p and $\bar{\mathcal{F}} = \bigoplus_{n \geq 0} \mathbf{C} \otimes K_0(\mathbf{F}_p\mathfrak{S}_n)$. The decomposition map defines a surjective morphism of abelian groups $\mathrm{dec} : \mathcal{F} \to \bar{\mathcal{F}}$. There is a $\mathbf{Z}\mathfrak{S}_n$-module $\tilde{S}_\lambda$, free over $\mathbf{Z}$, such that $S_\lambda \simeq \mathbf{Q} \otimes_{\mathbf{Z}} \tilde{S}_\lambda$. We have $\mathrm{dec}([S_\lambda]) = [\mathbf{F}_p \otimes_{\mathbf{Z}} \tilde{S}_\lambda]$.

The action of $\hat{\mathfrak{sl}}_p$ on $\mathcal{F}$ stabilizes the kernel of the decomposition map: this provides us with an action on $\bar{\mathcal{F}}$. The action of $\hat{\mathfrak{sl}}_p$ on $\bar{\mathcal{F}}$ is irreducible and $\bar{\mathcal{F}} \simeq L(\Lambda_0)$.

Let $\mathcal{V} = \bigoplus_{n\geq 0} \mathbf{F}_p\mathfrak{S}_n\text{-mod}$. Define $E = \bigoplus_{n\geq 0} \mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$ and $F = \bigoplus_{n\geq 0} \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$, two exact endofunctors of $\mathcal{V}$. They are left and right adjoint. We have $\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} = \mathbf{F}_p\mathfrak{S}_{n+1} \otimes_{\mathbf{F}_p\mathfrak{S}_n} -$.

Left multiplication by $(1,n+1)+\cdots+(n,n+1)$ defines an endomorphism of the $(\mathbf{F}_p\mathfrak{S}_{n+1}, \mathbf{F}_p\mathfrak{S}_n)$-bimodule $\mathbf{F}_p\mathfrak{S}_{n+1}$, hence an endomorphism of the functor $\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$. We denote by X the corresponding endomorphism of F.

Given M an $\mathbf{F}_p\mathfrak{S}_n$-module, all eigenvalues of X acting on $F(M) = \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} M$ are in $\mathbf{F}_p$. We denote by $F_i(M)$ the generalized i-eigenspace of X, for $i \in \mathbf{F}_p$. This gives us a decomposition $F = \bigoplus_{i\in\mathbf{F}_p} F_i$. Similarly, we have a decomposition $E = \bigoplus_{i\in\mathbf{F}_p} E_i$, where E_i is left and right adjoint to F_i.

The following proposition shows that the action of $\hat{\mathfrak{sl}}'_n$ on $\bar{\mathcal{F}}$ comes from the i-induction and i-restriction functors.

Proposition 4.5. *Given $M \in \mathbf{F}_p\mathfrak{S}_n\text{-mod}$, we have $[E_i(M)] = e_i([M])$ and $[F_i(M)] = f_i([M])$.*

We denote by $\hat{\mathfrak{S}}_p$ the affine symmetric group, a Coxeter group of type $\hat{A}_{p-1}$.

Proposition 4.6. *The decomposition of $\bar{\mathcal{F}}$ into weight spaces corresponds to the decomposition into blocks. Two blocks are in the same orbit under the adjoint action of $\hat{\mathfrak{S}}_p$ if and only if they have the same defect.*

Let k be a field and $q \in k^\times$ be an element with finite order $p \geq 2$ (p needs not be a prime). The construction above extends with $\mathbf{Q}\mathfrak{S}_n$ replaced by $H_n^f \otimes_{k[q_1,q_2]} \big(k[q_2]/(q_2+1)\big)(q_1)$ and $\mathbf{F}_p\mathfrak{S}_n$ replaced by $H_n^f \otimes_{k[q_1,q_2]} k[q_1,q_2]/(q_2+1, q_1-q)$. This provides a realization of $L(\lambda_0)$ as $\bigoplus_{n\geq 0} \mathbf{C} \otimes K_0(H_n^f \otimes_{k[q_1,q_2]} k[q_1,q_2]/(q_2+1,q_1-q))$, via i-induction and i-restriction functors.

4.2. **2-categories.**

4.2.1. *Duals.* Let $\mathcal{C}$ be a strict monoidal category and $V \in \mathcal{C}$. A *right dual* to V is an object $V^* \in \mathcal{C}$ together with maps $\varepsilon_V : V \otimes V^* \to 1$ and $\eta_V : 1 \to V^* \otimes V$ such that the following two compositions are the identity maps:

$$V \xrightarrow{V\otimes\eta_V} V \otimes V^* \otimes V \xrightarrow{\varepsilon_V\otimes V} V \text{ and } V^* \xrightarrow{\eta_V\otimes V^*} V^* \otimes V \otimes V^* \xrightarrow{V^*\otimes\varepsilon_V} V^*.$$

When $\mathcal{C}$ is the category of finite dimensional vector spaces, we obtain the usual dual.

When $\mathcal{C}$ is the category of endofunctors of a category, the notion of right dual coincides with that of right adjoint.

4.2.2. *2-categories.* A strict 2-category $\mathfrak{C}$ is a category enriched in categories: *i.e.*, it is the data of a set of objects, and given M, N two objects, the data of a category $\mathcal{H}om(M,N)$ together with composition functors $\mathcal{H}om(L,M) \times \mathcal{H}om(M,N) \to \mathcal{H}om(L,N)$ satisfying associativity conditions. We also require that $\mathcal{E}nd(M)$ comes with an identity object for the composition, which makes it into a strict monoidal category.

We can think of this as the data of objects, 1-arrows (the objects of the categories $\mathcal{H}om(M,N)$) and 2-arrows (the arrows of the categories $\mathcal{H}om(M,N)$).

A strict monoidal category $\mathcal{C}$ is the same data as a strict 2-category with one object $*$ and $\mathcal{E}nd(*) = \mathcal{C}$.

While the typical example of a category is the category of sets, the typical example of a strict 2-category is the 2-category $\mathfrak{Cat}$ of categories: its objects are categories, and $\mathcal{H}om(\mathcal{C}, \mathcal{C}')$ is the category of functors $\mathcal{C} \to \mathcal{C}'$.

A related example of a 2-category is that $\mathfrak{B}imod$ of bimodules: its objects are algebras over a fixed commutative ring k and $\mathcal{H}om(A, B)$ is the category of (B, A)-bimodules. Composition is given by tensor product.

4.2.3. *2-Kac Moody algebras.* We come now to the definition of 2-Kac-Moody algebras [Rou2, §4.1.3]. Our aim now is to add E_s's to $\mathcal{B}$ and construct maps which we will make formally invertible to enforce the relations $[e_i, f_i]_{|\lambda} = \langle \lambda, \alpha_i^\vee \rangle$. In order to make sense of this, we will need to add formally "idempotents" corresponding to weights $\lambda \in X$: this requires moving from a monoidal category to a 2-category.

Let C be a generalized Cartan matrix and let $\mathcal{B}'$ be the strict monoidal k-linear category obtained from $\mathcal{B} = \mathcal{B}(C)$ by adding E_s right dual to F_s for every $s \in I$. Define

$$\varepsilon_s = \varepsilon_{F_s} : F_s E_s \to \mathbf{1} \text{ and } \eta_s = \eta_{F_s} : \mathbf{1} \to E_s F_s.$$

Consider now a root datum $(X, Y, \langle -, - \rangle, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ of type C.

Consider the strict 2-category $\mathfrak{A}'$ with set of objects X and where $\mathcal{H}om(\lambda, \lambda')$ is the full k-linear subcategory of $\mathcal{B}'$ with objects direct sums of objects of the form $E_{s_n}^{a_n} F_{t_n}^{b_n} \cdots E_{s_1}^{a_1} F_{t_1}^{b_1}$ where $a_l, b_l \geq 0$, $s_l, t_l \in I$ and $\lambda' - \lambda = \sum_l (a_l \alpha_{s_l} - b_l \alpha_{t_l})$.

Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{g})$ be the k-linear strict 2-category deduced from $\mathfrak{A}'$ by inverting the following 2-arrows:

- when $\langle \lambda, \alpha_s^\vee \rangle \leq 0$,

$$\rho_{s,\lambda} = \sigma_{ss} + \sum_{i=0}^{-\langle \lambda, \alpha_s^\vee \rangle + 1} \varepsilon_s \circ (x_s^i E_s) : F_s E_s \mathbf{1}_\lambda \to E_s F_s \mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{-\langle \lambda, \alpha_s^\vee \rangle}$$

- when $\langle \lambda, \alpha_s^\vee \rangle \geq 0$,

$$\rho_{s,\lambda} = \sigma_{ss} + \sum_{i=0}^{-1+\langle \lambda, \alpha_s^\vee \rangle} (E_s x_s^i) \circ \eta_s : F_s E_s \mathbf{1}_\lambda \oplus \mathbf{1}_\lambda^{\langle \lambda, \alpha_s^\vee \rangle} \to E_s F_s \mathbf{1}_\lambda$$

- $\sigma_{st} : F_s E_t \mathbf{1}_\lambda \to E_t F_s \mathbf{1}_\lambda$ for all $s \neq t$ and all λ

where we define

$$\sigma_{st} = (E_t F_s \varepsilon_t) \circ (E_t \tau_{ts} E_s) \circ (\eta_t F_s E_t) : F_s E_t \to E_t F_s.$$

We put $\mathfrak{A}^{\mathrm{gr}} = \mathfrak{A} \otimes_k k^{\mathrm{gr}}$ as in §3.3.3. The grading defined on $\mathcal{B}^{\mathrm{gr}}$ extends to a grading on $\mathfrak{A}^{\mathrm{gr}}$ with $\deg \varepsilon_{s,\lambda} = d_s(1 + \langle \lambda, \alpha_s^\vee \rangle)$ and $\deg \eta_{s,\lambda} = d_s(1 - \langle \lambda, \alpha_s^\vee \rangle)$.

Given a quiver Γ with no loops corresponding to C, we put $\mathfrak{A}^\Gamma = \mathfrak{A} \otimes_k k^\Gamma$, a 2-category with graded spaces of 2-arrows.

Remark 4.7. One shows easily by passing to the Grothendieck groups, the graded category $\mathfrak{A}^{\mathrm{gr}}$-gr gives rise to the relations (1) of §4.1. Khovanov and Lauda have constructed a related

2-category and shown that the canonical morphism from the integral form of $U_q(\mathfrak{g})$ to the K_0 is an isomorphism in type A_n [Lau1, KhoLau3].

It can be shown that the isomorphisms $\rho_{i,\lambda}$ give rise to commutation isomorphisms between E_i^m and F_i^n (cf [Rou2, Lemma 4.12] for a version with explicit isomorphisms).

Lemma 4.8 ([Rou2, [Lemma 4.12]). *Let $m, n \geq 0$, $\lambda \in X$ and $i \in I$. Let $r = m - n + \langle \lambda, \alpha_i^\vee \rangle$. There are isomorphisms*

$$F_i^n E_i^m \mathbf{1}_\lambda \xrightarrow{\sim} \bigoplus_{l=0}^{\min(m,n)} E_i^{m-l} F_i^{n-l} \mathbf{1}_\lambda \otimes_k k^{\frac{m!n!}{(m-l)!(n-l)!}\binom{-r}{l}} \text{ if } r \leq 0$$

$$\bigoplus_{l=0}^{\min(m,n)} F_i^{n-l} E_i^{m-l} \mathbf{1}_\lambda \otimes_k k^{\frac{m!n!}{(m-l)!(n-l)!}\binom{r}{l}} \xrightarrow{\sim} E_i^m F_i^n \mathbf{1}_\lambda \text{ if } r \geq 0.$$

Remark 4.9. We have chosen to switch the roles of E and F, compared to [Rou2], as we will deal here with highest weight representations, while in [Rou2] we dealt with lowest weight representations. The two definitions are equivalent, as there is a strict equivalence of 2-categories

$$I : \mathfrak{A}^{\mathrm{opp}} \xrightarrow{\sim} \mathfrak{A},\ \mathbf{1}_\lambda \mapsto \mathbf{1}_{-\lambda},\ E_s \mapsto F_s,\ F_s \mapsto E_s,\ \tau_{st} \mapsto -\tau_{ts},\ x_s \mapsto x_s.$$

Given a 2-category $\mathfrak{C}$, we have denoted by $\mathfrak{C}^{\mathrm{opp}}$ the 2-category with the same set of objects as $\mathfrak{C}$ and with $\mathcal{H}om_{\mathfrak{C}^{\mathrm{opp}}}(c, c') = \mathcal{H}om_{\mathfrak{C}}(c, c')^{\mathrm{opp}}$.

4.3. **2-representation theory.** We are now reaching our main object of study. We review [Rou2, §5] (based in part on [ChRou]) and provide some complements.

4.3.1. *Integrable 2-representations.* A representation of $\mathfrak{A}$ on k-linear categories is defined to be a strict 2-functor from $\mathfrak{A}$ to the strict 2-category of k-linear categories. This is the same thing as the data of

- a k-linear category $\mathcal{V}_\lambda$ for $\lambda \in X$
- a k-linear functor $F_i : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_i}$ for $\lambda \in X$ and $i \in I$ admitting a right adjoint E_i
- $x_i \in \mathrm{End}(F_i)$ and $\tau_{ij} \in \mathrm{Hom}(F_i F_j, F_j F_i)$

such that

- the quiver Hecke algebra relations for x_i and τ_{ij}
- the maps $\rho_{i,\lambda}$ and σ_{ij} $(i \neq j)$ are isomorphisms.

A representation of $\mathfrak{A}$ such that E_i and F_i are locally nilpotent for all i will be called an *integrable 2-representation* of $\mathfrak{A}$ (or of $\mathfrak{g}$).

In the definition on an integrable 2-representations, the condition that the maps σ_{ij} are isomorphisms for $i \neq j$ is a consequence of the other conditions [Rou2, Theorem 5.25].

Remark 4.10. One can equivalently start with the functors E_i's, and natural transformations x_i and τ_{ij} between products of E's.

The definition provides E_i as a right adjoint of F_i, but the next result shows that E_i will actually also be a left adjoint (cf [Rou1, §4.1.4] for the explicit units and counits).

Theorem 4.11 ([Rou2, Theorem 5.16]). *Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}$. Then E_i is a left adjoint of F_i, for all i.*

It is often unreasonable to check directly that the maps $\rho_{i,\lambda}$ and σ_{ij} are isomorphisms in examples. It turns out that, under finiteness assumptions, it is enough to check that the $\mathfrak{sl}_2$-relations hold on the Grothendieck group (the crucial part is [ChRou, Theorem 5.27]).

Theorem 4.12 ([Rou2, Theorem 5.27]). *Let K be a field that is a K-algebra and let $\mathcal{V} = \bigoplus_{\lambda\in X}\mathcal{V}_\lambda$ be a K-linear abelian category such that all objects have finite composition series and simple objects have endomorphism ring K. Assume given*

- *a K-linear exact functor $F_i : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_i}$ for $\lambda \in X$ and $i \in I$ with an exact right adjoint E_i*
- *$x_i \in \mathrm{End}(F_i)$ and $\tau_{ij} \in \mathrm{Hom}(F_iF_j, F_jF_i)$*

such that

- *E_i and F_i are locally nilpotent*
- *E_i is left adjoint to F_i*
- *the quiver Hecke algebra relations for x_i and τ_{ij}*
- *the endomorphisms $[E_i]$ and $[F_i]$ define an integrable representation of $\mathfrak{sl}_2$ on $\mathbf{C}\otimes K_0(\mathcal{V})$, and $[E_i][F_i] - [F_i][E_i]$ acts by $\langle\lambda, \alpha_i^\vee\rangle$ on $K_0(\mathcal{V}_\lambda)$, for all i and λ.*

Then, the data above defines an integrable 2-representation of $\mathfrak{g}$ on $\mathcal{V}$.

Let us give a variant, based on “abstract” $\mathfrak{sl}_2$-relations between functors.

Corollary 4.13. *Let k' be a commutative k-algebra. Let $\{\mathcal{V}_\lambda\}_{\lambda\in X}$ be a family of k'-linear categories whose* Hom*'s are finitely generated k'-modules.*

Assume given

- *$F_s : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_s}$ with a right adjoint E_s for $s \in I$*
- *$x_s \in \mathrm{End}(F_s)$ and $\tau_{st} \in \mathrm{Hom}(F_sF_t, F_tF_s)$ for every $s,t \in I$.*

We assume that

- *E_s is a left adjoint of F_s*
- *E_s and F_s are locally nilpotent*
- *given $\lambda \in X$, there are isomorphisms of functors*

$$(E_sF_s)_{|\mathcal{V}_\lambda} \simeq (F_sE_s)_{|\mathcal{V}_\lambda} \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{\langle\lambda,\alpha_s^\vee\rangle} \quad \text{if } \langle\lambda,\alpha_s^\vee\rangle \geq 0$$

$$(F_sE_s)_{|\mathcal{V}_\lambda} \simeq (E_sF_s)_{|\mathcal{V}_\lambda} \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{-\langle\lambda,\alpha_s^\vee\rangle} \quad \text{if } \langle\lambda,\alpha_s^\vee\rangle \leq 0$$

- *the quiver Hecke algebra relations hold.*

Then, the data above defines an integrable action of $\mathfrak{A}(\mathfrak{g})$ on $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$.

Proof. Let K be an algebraically closed field that is a k'-algebra. Let $\mathcal{W} = \mathcal{V}\text{-mod}_K$ be the category of k'-linear functors $\mathcal{V}^{\mathrm{opp}} \to K\text{-mod}$. The functors F_s and E_s induce adjoint exact functors on $\mathcal{W}$ satisfying the conditions of Theorem 4.12. Consequently, the maps $\rho_{s,\lambda}$ and σ_{st} (for $s \neq t$), taken in $\mathcal{V}$, are isomorphisms after applying $-\otimes_{k'} K$. Since this holds for all K, a variant of Lemma 2.23 shows that those maps are isomorphisms. $\square$

4.3.2. *Some 2-representations of $\mathfrak{sl}_2$.* We assume here $|I| = 1$, $X = \mathbf{Z}$ and $\alpha = 2$. We have $k = \mathbf{Z}$. Fix a field K.

The most obvious example of a 2-representation is $\bar{\mathcal{L}}(0)$ defined by $\bar{\mathcal{L}}(0)_\lambda = 0$ for $\lambda \neq 0$ and $\bar{\mathcal{L}}(0)_0 = K\text{-mod}$. All the extra data vanishes.

Consider now $\bar{\mathcal{L}}(1)$, a categorification of the simple 2-dimensional representation of $\mathfrak{sl}_2$. We put

$$\bar{\mathcal{L}}(1)_\lambda = 0 \text{ for } \lambda \neq \pm 1,\ \bar{\mathcal{L}}(1)_1 = K\text{-mod and } \bar{\mathcal{L}}(1)_{-1} = K\text{-mod}.$$

We define E and F to be the identity functors between $\bar{\mathcal{L}}(1)_1$ and $\bar{\mathcal{L}}(1)_{-1}$ and we set $x = \tau = 0$.

A categorification of the simple 3-dimensional representation is given by

$$\bar{\mathcal{L}}(2)_\lambda = 0 \text{ for } \lambda \neq -2, 0, 2,\ \bar{\mathcal{L}}(2)_{-2} = \bar{\mathcal{L}}(2)_2 = K\text{-mod and } \bar{\mathcal{L}}(2)_0 = (K[y]/y^2)\text{-mod}.$$

The functors E and F are induction and restriction functors. We define x as multiplication by $-y$ on $F = \mathrm{Ind} : \bar{\mathcal{L}}(2)_2 \to \bar{\mathcal{L}}(2)_0$ and as multiplication by y on $F = \mathrm{Res} : \bar{\mathcal{L}}(2)_0 \to \bar{\mathcal{L}}(2)_{-2}$. We define $\tau \in \mathrm{End}_K(K[y]/y^2)$ by $\tau(1) = 0$ and $\tau(y) = 1$.

Let us construct more generally $\bar{\mathcal{L}}(n)$. Let $H_{i,n}$ be the subalgebra of ${}^0H_n$ generated by ${}^0H_i$ and $P_n^{\mathfrak{S}_n}$. We have $H_{i,n} = {}^0H_i^f \otimes_{\mathbf{Z}} P_n^{\mathfrak{S}\{i+1,\ldots,n\}}$ as $\mathbf{Z}$-modules and $H_{i,n} = {}^0H_i \otimes_{\mathbf{Z}} \mathbf{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}\{i+1,\ldots,n\}}$ as algebras. By Proposition 2.21, we have a Morita equivalence between the $P_n^{\mathfrak{S}_n}$-algebras $H_{i,n}$ and $P_n^{\mathfrak{S}\{1,\ldots,i\}\times\mathfrak{S}\{i+1,\ldots,n\}}$. Since ${}^0H_i$ is a symmetric algebra over $P_i^{\mathfrak{S}_i}$ (Proposition 2.30) and $P_n^{\mathfrak{S}\{1,\ldots,i\}\times\mathfrak{S}\{i+1,\ldots,n\}}$ is symmetric over $P_n^{\mathfrak{S}_n}$ (Corollary 2.26), we deduce from Lemma 2.24 that $H_{i,n}$ is a symmetric algebra over $P_n^{\mathfrak{S}_n}$.

We have a chain of algebras

$$H_{0,n} = P_n^{\mathfrak{S}_n} \subset H_{1,n} \subset \cdots \subset H_{n,n} = {}^0H_n$$

and $H_{i+1,n}$ is a free left (and right) $H_{i,n}$-module of rank $(i+1)(n-i)$.

Let $\bar{H}_{i,n} = H_{i,n} \otimes_{P_n^{\mathfrak{S}_n}} K$, where the morphism of rings $P_n^{\mathfrak{S}_n} \to K$ is given by sending homogeneous polynomials of positive degree to 0. This is a finite-dimensional K-algebra Morita-equivalent to its center $P_n^{\mathfrak{S}\{1,\ldots,i\}\times\mathfrak{S}\{i+1,\ldots,n\}} \otimes_{P_n^{\mathfrak{S}_n}} K$. That center is $\mathbf{Z}_{\geq 0}$-graded, with degree 0 part of dimension 1, hence it is local. It follows that $\bar{H}_{i,n}$ has a unique simple module, of dimension $i!$.

We put $\bar{\mathcal{L}}(n)_\lambda = \bar{H}_{(n-\lambda)/2,n}$-mod for $\lambda \in \{n, n-2, \ldots, 2-n, -n\}$ and $\bar{\mathcal{L}}(n)_\lambda = 0$ otherwise.

We denote by E the restriction functor and F the induction functor: they are both exact functors. Since the algebras $\bar{H}_{i,n}$ are symmetric over K, we deduce that E is both right and left adjoint to F. It is immediate to check that $[E]$ and $[F]$ induce an action of $\mathfrak{sl}_2(\mathbf{C})$ on $\mathbf{C} \otimes K_0(\bar{\mathcal{L}}(n)) = \mathbf{C}^{n+1}$.

We denote by x the endomorphism of the $(\bar{H}_{i+1,n}, \bar{H}_{i,n})$-bimodule $\bar{H}_{i+1,n}$ given by right multiplication by X_{i+1}: this provides a corresponding endomorphism of the functor F. Similarly, we define an endomorphism τ of F^2 corresponding to the right multiplication by T_i on the $(\bar{H}_{i+2,n}, \bar{H}_{i,n})$-bimodule $\bar{H}_{i+2,n}$.

Theorem 4.12 provides the following result. These are the "minimal categorifications" of [ChRou, §5.3].

Proposition 4.14. *The data above defines an action of* $\mathfrak{A}$ *on* $\bar{\mathcal{L}}(n)$.

Let us now consider a deformed additive version $\mathcal{L}(n)$. They are necessary to have Jordan-Hölder type decompositions in the additive setting. We put $\mathcal{L}(n)_\lambda = H_{(n-\lambda)/2,n}$-proj, and we define E, F, X and T as above. Proposition 4.14 shows that the morphisms of bimodules corresponding to the maps ρ_λ become isomorphisms after applying $\otimes_{P_n^{\mathfrak{S}_n}} K$, for any field K. A graded version of Lemma 2.23 enables us to deduce that the maps ρ_λ are isomorphisms. We obtain consequently the following proposition.

Proposition 4.15. *The data above defines an action of $\mathfrak{A}$ on $\mathcal{L}(n)$.*

The categories $\mathcal{L}(n)$ and $\bar{\mathcal{L}}(n)$ are enriched in graded vector spaces, and the actions are compatible with the gradings.

4.3.3. *Simple 2-representations $\mathcal{L}(\lambda)$.* Let $\lambda \in X$. Given $\mathcal{V}$ a 2-representation, we put

$$\mathcal{V}_\lambda^{\mathrm{hw}} = \{M \in \mathcal{V}_\lambda | E_i(M) = 0 \ \forall i\}.$$

Note that $\mathcal{V}_\lambda^{\mathrm{hw}} = 0$ if $\lambda \notin X^+$.

A consequence of the relations in Lemma 4.8 is the following description of highest weight objects.

Lemma 4.16. *Let $\lambda \in X^+$ and $i \in I$. Let $d = \langle \lambda, \alpha_i^\vee \rangle + 1$. Then*

- *$E_i \mathbf{1}_\lambda$ is a direct summand of $E_i^{d+1} F_i^d \mathbf{1}_\lambda$*
- *$F_i^d \mathbf{1}_\lambda$ is a direct summand of $F_i^{d+1} E_i \mathbf{1}_\lambda$.*

As a consequence, given $\mathcal{V}$ a 2-representation and $M \in \mathcal{V}_\lambda$, we have $M \in \mathcal{V}_\lambda^{hw}$ if and only if $F_i^{\langle \lambda, \alpha_i^\vee \rangle + 1}(M) = 0$ for all $i \in I$.

Assume $\lambda \in X^+$. There is a 2-representation $\mathcal{L}(\lambda)$ with an object $v_\lambda \in \mathcal{L}(\lambda)_\lambda$ that has the following property: given $\mathcal{V}$ a 2-representation, there is an equivalence

$$\mathcal{H}om_{\mathfrak{A}}(\mathcal{L}(\lambda), \mathcal{V}) \xrightarrow{\sim} \mathcal{V}_\lambda^{\mathrm{hw}}, \ \Phi \mapsto \Phi(v_\lambda).$$

So, $\mathcal{L}(\lambda)$ represents the 2-functor $\mathcal{V} \mapsto \mathcal{V}_\lambda^{\mathrm{hw}}$ and it thus unique up to an equivalence unique up to a unique isomorphism.

Let us provide a construction of $\mathcal{L}(\lambda)$. We define a 2-representation $\mathcal{M}(\lambda)$ by setting $\mathcal{M}(\lambda)_\mu = \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$. The composition in $\mathfrak{A}$ provides a natural action of $\mathfrak{A}$ on $\mathcal{M}(\lambda)$. Define now $\mathcal{N}(\lambda)$, a sub-2-representation, by setting $\mathcal{N}(\lambda)_\mu$ as the full additive subcategory of $\mathcal{M}(\lambda)_\mu$ generated by objects of the form RE_i, where R is a 1-arrow in $\mathfrak{A}$ from $\lambda + \alpha_i$ to μ and $i \in I$.

We put now $\mathcal{L}(\lambda) = \mathcal{M}(\lambda)/\mathcal{N}(\lambda)$ (quotient as additive categories) and we denote by v_λ the image of $\mathbf{1}_\lambda$. We put $Z_\lambda = \mathrm{End}_{\mathcal{L}(\lambda)}(v_\lambda)$.

Remark 4.17. A important fact is that this construction provides a higher version of $L^{\max}(\lambda)$ ($= L(\lambda)$ in the symmetrizable case), not of $\Delta(\lambda)$: this is a consequence of Lemma 4.16.

4.3.4. *Isotypic 2-representations.* An isotypic representation is a multiple of a simple representation, or equivalently, the tensor product of a simple representation by a multiplicity vector space. The 2-categorical version of that requires to take a tensor product by a multiplicity category.

Let A be a commutative ring and $\mathcal{C}$ be an A-linear category. Let B a commutative A-algebra. We denote by $\mathcal{C} \otimes_A B$ the B-linear category with same objects as $\mathcal{C}$ and with $\mathrm{Hom}_{\mathcal{C} \otimes_A B}(M, N) = \mathrm{Hom}_{\mathcal{C}}(M, N) \otimes_A B$. Let now $\mathcal{C}'$ be another A-linear category. We denote by $\mathcal{C} \otimes_A \mathcal{C}'$ the A-linear category with objects finite families $((M_1, M_1'), \ldots, (M_m, M_m'))$ where $M_i \in \mathcal{C}$ and $M_i' \in \mathcal{C}'$. We put

$$\mathrm{Hom}(((M_1, M_1'), \ldots, (M_m, M_m')), ((N_1, N_1'), \ldots, (N_n, N_n'))) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(M_i, N_j) \otimes_A \mathrm{Hom}_{\mathcal{C}'}(M_i', N_j').$$

Let $\mathcal{V}$ be a k-linear 2-representation of $\mathfrak{A}$ and let $\lambda \in X^+$. There is a canonical fully faithful functor compatible with the $\mathfrak{A}$-action

$$R_\lambda : \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{V}_\lambda^{\mathrm{hw}} \to \mathcal{V},\ M \otimes N \mapsto M(N).$$

If $\mathcal{V}$ is idempotent-closed and every object of $\mathcal{V}$ is a direct summand of an object of $\mathfrak{A}(\mathcal{V}_\lambda^{\mathrm{hw}})$, then R_λ induces an equivalence $\big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{V}_\lambda^{\mathrm{hw}}\big)^i \xrightarrow{\sim} \mathcal{V}$.

The only full sub-2-representations of isotypic 2-representations are the obvious ones.

Proposition 4.18. *Let $\lambda \in X^+$, let $\mathcal{M}$ be a Z_λ-linear category, and let $\mathcal{W}$ be an idempotent-complete full k-linear sub-2-representation of $\big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}\big)^i$. Let $\mathcal{N}$ be the subcategory of $\mathcal{M}^i$ image of $\mathcal{W}_\lambda$ under the canonical equivalence $\big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}\big)^i_\lambda \xrightarrow{\sim} \mathcal{M}^i$*

Then $\mathcal{W} = \big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{N}\big)^i$.

Proof. Let $\mathcal{V} = \big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}\big)^i$. Every object M of $\mathcal{V}$ is a direct summand of a direct sum of objects of the form $F_{i_r} \cdots F_{i_1}(N)$, where $N \in \mathcal{V}_\lambda$. Since $E_{i_1} \cdots E_{i_r}$ is right adjoint to $F_{i_r} \cdots F_{i_1}$, we deduce that $F_{i_r} \cdots F_{i_1}$ is a direct summand of $F_{i_r} \cdots F_{i_1} E_{i_1} \cdots E_{i_r} F_{i_r} \cdots F_{i_1}$ (in $\mathfrak{A}$). As a consequence, any $M \in \mathcal{V}$ is a direct summand of a direct sum of objects of the form $F_{i_r} \cdots F_{i_1} E_{i_1} \cdots E_{i_r}(M)$, where $E_{i_1} \cdots E_{i_r}(M) \in \mathcal{V}_\lambda$.

This shows that every object of $\mathcal{W}$ is a direct summand of an object of $\mathfrak{A}(\mathcal{W}_\lambda)$, hence the canonical functor $\big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{W}_\lambda\big)^i \to \mathcal{W}$ is an equivalence. $\square$

4.3.5. *Structure.* We explain here a counterpart of Jordan-Hölder series. This provides a powerful tool to reduce statements to the case of $\mathcal{L}(\lambda)$'s and this is one the key ideas of [ChRou].

Let $\mathcal{V}$ be a 2-representation. Given $\xi \in X/(\bigoplus_i \mathbf{Z}\alpha_i)$, let $\mathcal{V}_\xi = \bigoplus_{\lambda \in \xi} \mathcal{V}_\lambda$. Then $\mathcal{V} = \bigoplus_\xi \mathcal{V}_\xi$ is a decomposition as a direct sum of 2-representations. This gives a direct sum decomposition of the 2-category of 2-representations.

Theorem 4.19 ([Rou2, Theorem 5.8])**.** *Let $\mathcal{V}$ be a k-linear category acted on by $\mathfrak{A}$. Assume that given $\lambda \in X$ and $M \in \mathcal{V}_\lambda$, there is $r > 0$ such that $E_{i_1} \cdots E_{i_r}(M) = 0$ for all $i_1, \ldots, i_r \in I$.*

Then $\mathcal{V}$ is integrable and there are Z_λ-linear categories $\mathcal{M}_{\lambda,r}$ for $\lambda \in X^+$ and $r \in \mathbf{Z}_{\geq 1}$, there is a filtration by full k-linear sub-2-representations closed under taking direct summands

$$0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \subset \cdots \subset \mathcal{V}$$

with $\bigcup_r \mathcal{V}\{r\} = \mathcal{V}$, and there are isomorphisms of 2-representations

$$\big(\mathcal{V}\{r+1\}/\mathcal{V}\{r\}\big)^i \xrightarrow{\sim} \bigoplus_{\lambda \in X^+} \left(\mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}_{\lambda,r}\right)^i.$$

Note that the assumption of the theorem is automatically satisfied if $\mathfrak{g}$ is a finite-dimensional Lie algebra.

Proposition 4.20. *Assume C is a finite Cartan matrix,* i.e., *$\mathfrak{g}$ is a finite-dimensional Lie algebra. Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}$. Then given $\lambda \in X$ and $M \in \mathcal{V}_\lambda$, there is $r > 0$ such that $E_{i_1} \cdots E_{i_r}(M) = 0$ for all $i_1, \ldots, i_r \in I$.*

Proof. We switch the roles of E's and F's in the proof. There are positive integers n_i, such that $F_i^{n_i}(M) = 0$. It follows that the canonical $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{V},\ L \mapsto L(M)$, factors through the additive quotient $\bar{\mathcal{B}}$ of $\mathcal{B}$ by its full additive subcategory generated by the $\mathcal{B}F_i^{n_i}$ for $i \in I$. We will be done by showing that there is $r > 0$ such that $\bar{\mathcal{B}}_r = 0$

Note that the corresponding property holds in $\mathbf{C} \otimes K_0(\bar{\mathcal{B}})$, which is a quotient of the vector space $U(\mathfrak{n}^-)/(\sum_i U(\mathfrak{n}^-)f_i^{n_i})$ by Theorem 3.14: this is an $L(\mu)$ for some $\mu \in X^+$, hence it is finite-dimensional (cf §4.1.4). We deduce that $K_0(\bar{\mathcal{B}}_r) = 0$ for some r. The Hom-spaces in $\mathcal{B}_r$ are modules of finite rank modules over $Z(\mathcal{B}_r)$, which is a noetherian ring [Rou2, Proposition 3.10], and the same holds for $\bar{\mathcal{B}}_r$. So, the vanishing of K_0 forces $\bar{\mathcal{B}}_r = 0$. ☐

Extensions between $\mathcal{L}(\lambda)$'s can occur only in one direction.

Lemma 4.21. *Let $\mathcal{V}$ be a k-linear 2-representation of $\mathfrak{A}$ and $\mathcal{W}$ a full k-linear sub-2-representation closed under taking direct summands. Let $\lambda \in X^+$ and let $\mathcal{M}$ be a Z_λ-linear category.*

Assume there is a morphism $\Phi : \mathcal{V} \to \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}$ of 2-representations of $\mathfrak{A}$ with $\Phi(\mathcal{W}) = 0$ and inducing an isomorphism $\mathcal{V}/\mathcal{W} \xrightarrow{\sim} \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}$.

If $\mathcal{W}_{\lambda+\alpha} = 0$ for all $\alpha \in \bigoplus_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$, then there is a morphism of 2-representations $\Psi : \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M} \to \mathcal{V}$ that is a right inverse to Φ. As a consequence, $\mathcal{V} \simeq \mathcal{W} \oplus \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M}$ as 2-representations of $\mathfrak{A}$.

Proof. By assumption, $\mathcal{V}_{\lambda+\alpha_i} = 0$ for all i. It follows that the restriction of Φ to $\mathcal{V}_\lambda$ is an equivalence $\Phi' : \mathcal{V}_\lambda^{\mathrm{hw}} \xrightarrow{\sim} \mathcal{M}$. The functor Φ'^{-1} induces a fully faithful functor $\Psi : \mathcal{L}(\lambda) \otimes_{Z_\lambda} \mathcal{M} \to \mathcal{V}$ that is a right inverse to Φ. ☐

As a consequence, we can order terms and obtain a Jordan-Hölder filtration under stronger finiteness assumptions from Theorem 4.19.

Theorem 4.22. *Let $\mathcal{V}$ be an integrable k-linear 2-representation of $\mathfrak{A}$. Assume there is a finite set $K \subset X$ such that $\{\lambda \in X | \mathcal{V}_\lambda \neq 0\} \subset \bigcup_{\mu \in K}(\mu + \sum_{i \in I} \mathbf{Z}_{\leq 0}\alpha_i)$.*

Then there are

- *$\lambda_1, \lambda_2, \ldots \in X^+$ such that $\lambda_a - \lambda_b \in \sum_i \mathbf{Z}_{\geq 0}\alpha_i$ implies $a < b$*
- *Z_{λ_r}-linear categories $\mathcal{M}_r$*
- *and a filtration by full k-linear sub-2-representations closed under taking direct summands*

$$0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \subset \cdots \subset \mathcal{V}$$

such that $\bigcup_r \mathcal{V}\{r\} = \mathcal{V}$ and $\left(\mathcal{V}\{r+1\}/\mathcal{V}\{r\}\right)^i \xrightarrow{\sim} \left(\mathcal{L}(\lambda_r) \otimes_{Z_{\lambda_r}} \mathcal{M}_r\right)^i$ as 2-representations of $\mathfrak{A}$.

4.4. **Cyclotomic quiver Hecke algebras.**

4.4.1. *Construction of $\mathcal{B}(\lambda)$.* Let $\lambda \in X^+$. Let $n_i = \langle \lambda, \alpha_i^\vee \rangle$, let $A_\lambda = \mathbf{Z}[\{z_{i,r}\}_{i \in I, 1 \leq r \leq n_i}]$ and let $k_\lambda = k \otimes_{\mathbf{Z}} A_\lambda$. We define the additive category quotient

$$\mathcal{B}(\lambda) = (\mathcal{B} \otimes_{\mathbf{Z}} A_\lambda)/(\sum_{r=0}^{n_i} x_i^{n_i - r} \otimes z_{i,r})_{i \in I}$$

where we put $z_{i,0} = 1$ for $i \in I$.

We define now the *cyclotomic quiver Hecke algebras*

$$H_n(\lambda) = \mathrm{End}_{\mathcal{B}(\lambda)}(\bigoplus_{(i_1,\ldots,i_n) \in I^n} F_{i_1} \cdots F_{i_n}).$$

We have

$$H_n(\lambda) = (H_n(Q) \otimes_{\mathbf{Z}} A_\lambda)/(\sum_{r=0}^{n_i} x_{1,i}^{n_i-r} \otimes z_{i,r})_{i\in I}$$

and in particular $H_0(\lambda) = k_\lambda$.

One can also consider the *reduced cyclotomic quiver Hecke algebras* $\bar{H}_n(\lambda) = H_n(\lambda) \otimes_{A_\lambda} \mathbf{Z}$, where $z_{i,r}$ acts by 0 on $\mathbf{Z}$ for $r \neq 0$.

Note that these constructions depend only on $\{n_i\}_{i\in I}$, not on λ.

4.4.2. *Fock spaces.* We explain how to construct a 2-representation of affine Lie algebras of type A, following [ChRou], and we explain the relation with $\mathcal{B}(\Lambda_0)$. We consider the setting of §4.1.6.

Similarly to the construction of X, we construct an endomorphism T of $F^2 = \bigoplus_{n\geq 0} \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+2}}$ by left multiplication by $(n+1, n+2)$ on the $(\mathbf{F}_p\mathfrak{S}_{n+2}, \mathbf{F}_p\mathfrak{S}_n)$-bimodule $\mathbf{F}_p\mathfrak{S}_{n+2}$.

We have a morphism of algebras

$$\bar{H}_n \to \mathrm{End}(F^n),\ X_i \mapsto F^{n-i}XF^{i-1},\ s_i \mapsto F^{n-i-1}TF^{i-1}.$$

Let Γ be the quiver with vertex set $I = \mathbf{F}_p$ and with arrows $i \to i+1$. Theorem 3.12 shows how to deduce a morphism of algebras $H_n(\Gamma) \to \mathrm{End}(\bigoplus_{\nu\in I^n} F_{\nu_1}\cdots F_{\nu_n})$.

Theorem 4.23. *The constructions above endow $\mathcal{V}$ with a 2-representation of $\mathfrak{A}(\hat{\mathfrak{sl}}_p)$. We have an equivalence of 2-representations*

$$\left(\mathbf{F}_p \otimes_{Z_{\Lambda_0}} \mathcal{L}(\Lambda_0)\right)^i \simeq \bigoplus_{n\geq 0} \mathbf{F}_p\mathfrak{S}_n\text{-proj}$$

Proof. We need to show that the maps $\rho_{i,\lambda}$ and are isomorphisms. The corresponding relations hold at the level of the Grothendieck groups. It follows from Theorem 4.12 that the required maps are isomorphisms. The equivalence follows from the fact that $\mathcal{V}$ is a highest weight 2-representation, with highest weight Λ_0. □

Since the left side of the equivalence is graded, this provides us with gradings of group algebras of symmetric groups over $\mathbf{F}_p$. This can be made explicit. Indeed, Theorem 3.12 induces an isomorphism of algebras $\mathbf{F}_p \otimes_{\mathbf{Z}} \bar{H}_n(\Gamma) \xrightarrow{\sim} \mathbf{F}_p\mathfrak{S}_n$, and the right hand side has homogeneous generators.

The constructions above extend to arbitrary $p \geq 2$, with the group algebra of the symmetric group replaced by its Hecke algebra, as explained in §4.1.6.

The isomorphism and the gradings above have been constructed and studied independently by Brundan and Kleshchev [BrKl1, BrKl2]. They have built a new approach to the representation theory of symmetric groups and their Hecke algebras using these gradings.

Such gradings had been shown to exist earlier (using derived equivalences and good blocks) for blocks with abelian defect [Rou1, Remark 3.11]. Leonard Scott had raised the question in the mid-nineties to construct gradings for group algebras of symmetric groups and more generally Hecke algebras of finite Coxeter groups.

4.4.3. *Simple 2-representations.* We explain here how cyclotomic quiver Hecke algebras provide a vast generalization of the constructions of §4.1.6 and §4.4.2, as conjectured by Khovanov-Lauda and ourselves.

The left action of $\mathcal{B}$ on itself induces an action of $\mathcal{B}$ on $\mathcal{B}(\lambda)^i = \bigoplus_n H_n(\lambda)$-proj.

Theorem 4.24 (Kang-Kashiwara, Webster)**.** *Given $s \in I$, the functor $F_s : \mathcal{B}(\lambda)^i \to \mathcal{B}(\lambda)^i$ has a right adjoint. This provides an action of $\mathfrak{A}$ on $\mathcal{B}(\lambda)^i$, with highest weight $(\mathcal{B}(\lambda)^i)_\lambda = k_\lambda$-proj.*

There is an isomorphism of $\mathfrak{g}$-modules $\mathbf{C} \otimes_{\mathbf{Z}} K_0(\mathcal{B}(\lambda)^i) \xrightarrow{\sim} L(\lambda)$.

Kang-Kashiwara's result [KanKas, Theorem 4.6] is given in the case of symmetrizable Cartan matrices and in a graded setting, but it extends with no change to our setting. Webster's result [We1] is also in a graded setting.

Note that the algebras $H_n(\lambda)$ are finitely generated projective k_λ-modules [KanKas, Remark 4.20(ii)].

There is a canonical morphism of k-algebras $Z_\lambda \to k_\lambda$ and an equivalence of additive $\mathfrak{A}$-categories (cf §4.3.4)

$$\Psi : \big(\mathcal{L}(\lambda) \otimes_{Z_\lambda} k_\lambda\big)^i \xrightarrow{\sim} \mathcal{B}(\lambda)^i$$

Theorem 4.25. *The canonical map gives an isomorphism $Z_\lambda \xrightarrow{\sim} k_\lambda$. In particular, there is an equivalence of categories $\mathcal{L}(\lambda)^i \xrightarrow{\sim} \mathcal{B}(\lambda)^i$ compatible with the action of $\mathfrak{A}$.*

Proof. The proof is similar to that of [Rou2, Proposition 5.15]. Let $i \in I$. We have $F_i^{n_i}(v_\lambda) \neq 0$, while $F_i^{n_i+1}(v_\lambda) = 0$. It follows that the canonical map $F_i^{(n_i)} E_i^{(n_i)}(v_\lambda) \to v_\lambda$ is an isomorphism [Rou2, Lemma 4.12]. The action of $\mathbf{Z}[x_{1,i}, \ldots, x_{n_i,i}]^{\mathfrak{S}_{n_i}}$ on $E^{(n_i)}$ gives then an action on v_λ. We let $z_{i,r}$ act on v_λ as $(-1)^r e_r(x_{1,i}, \ldots, x_{n_i,i})$ for $1 \leq r \leq n_i$. This provides a morphism of k-algebras $k_\lambda \to \operatorname{End}(v_\lambda) = Z_\lambda$.

We have a canonical functor $\mathcal{B} \otimes_k k_\lambda \to \mathcal{L}(\lambda)$ given by $M \mapsto M(v_\lambda)$. It induces a functor $\Phi : \mathcal{B}(\lambda) \to \mathcal{L}(\lambda)$ compatible with the $\mathcal{B}$-action: there are canonical isomorphisms $\Phi F_i \xrightarrow{\sim} F_i \Phi$ compatible with the x_i's and τ_{ij}'s. Note that the functor $\mathcal{L}(\lambda) \xrightarrow{\text{can}} \mathcal{L}(\lambda) \otimes_{Z_\lambda} k_\lambda \xrightarrow{\Psi} \mathcal{B}(\lambda)$ is a left inverse to Ψ.

Let us show that Φ can be endowed with a compatibility for the $\mathfrak{A}$-action. We need to show that the composition

$$\Phi E_i \xrightarrow{\eta_i \Phi E_i} E_i F_i \Phi E_i \xrightarrow{E_i(\text{can}^{-1})E_i} E_i \Phi F_i E_i \xrightarrow{E_i \Phi \varepsilon_i} E_i \Phi$$

is an isomorphism for all $i \in I$ (cf [ChRou, §5.1.2]). Let us show that the composition above is an isomorphism when applied to $L \in \mathcal{B}(\lambda)$. Consider M, γ, α as in Lemma 4.26 below. Since $\Phi\Psi(\alpha)$ is an isomorphism, we deduce that the composition

$$\Phi(M) \xrightarrow{\Phi(\eta_i \bullet)} \Phi(E_i F_i M) \xrightarrow{\Phi(E_i \gamma)} \Phi(E_i L)$$

is an isomorphism. To clarify the exposition, we identify ΦF_i with $F_i\Phi$. There is a commutative diagram

$$\begin{tikzcd}
\Phi(E_iL) \arrow[r, "\eta_i\bullet"] & E_iF_i\Phi(E_iL) \arrow[r, "E_i\Phi(\varepsilon_i\bullet)"] & E_i\Phi(L) \\
\Phi(E_iF_iM) \arrow[u, "\Phi(E_i\gamma)"'] \arrow[r, "\eta_i\bullet"] & E_iF_i\Phi(E_iF_iM) \arrow[u, "\bullet\Phi(E_i\gamma)"'] \arrow[r, "E_i\Phi(\varepsilon_i\bullet)"] & E_i\Phi(F_iM) \arrow[u, "E_i\Phi(\gamma)"'] \\
\Phi(M) \arrow[u, "\Phi(\eta_i\bullet)"'] \arrow[r, "\eta_i\bullet"] \arrow[uu, bend left=60, "\sim"] \arrow[urr, bend right=40, "\sim"'] & E_iF_i\Phi(M) \arrow[u, "\bullet\Phi(\eta_iM)"'] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

This completes the proof that Φ is compatible with the action of $\mathfrak{A}$. Since $\Phi(k_\lambda) = v_\lambda$, we obtain a morphism of Z_λ-algebras $k_\lambda \to Z_\lambda$ that is a left inverse to the canonical morphism $Z_\lambda \to k_\lambda$. Consequently, these morphisms are isomorphisms. $\square$

Lemma 4.26. *Given i and $L \in \mathcal{B}(\lambda)$, there are $M \in \mathcal{B}(\lambda)$ and $\gamma : F_iM \to L$ such that the composition*

$$\alpha : \Phi(M) \xrightarrow{\eta_i\bullet} E_iF_i\Phi(M) = E_i\Phi(F_iM) \xrightarrow{E_i\Phi(\gamma)} E_i\Phi(L)$$

is an isomorphism.

Proof. It is enough to prove the lemma for $L = F_{i_r}\cdots F_{i_1}(k_\lambda)$ for any $i_1,\ldots,i_r \in I$. We prove this by induction on r. Assume the lemma holds for r. Consider $i_1,\ldots,i_r \in I$. Let M, α and γ be provided by the lemma.

Consider now $j = i_{r+1} \in I$. Let $M' = F_jM$. Let

$$\gamma' = (F_iF_jM \xrightarrow{\tau_{ij}\bullet} F_jF_iM \xrightarrow{F_j\gamma} F_jL).$$

There is a commutative diagram

$$\begin{tikzcd}
\Phi(F_jM) \arrow[r, "\eta_i\bullet"] \arrow[d, "F_j\eta_i\bullet"'] \arrow[dd, bend right=60, "\sim"'] & E_iF_i\Phi(F_jM) \arrow[r, "E_i\tau_{ij}\bullet"] \arrow[d, "\bullet\eta_i\Phi(M)"'] & E_i\Phi(F_jF_iM) \arrow[r, "E_i\Phi(F_j\gamma)"] \arrow[d, "\bullet\eta_i\Phi(M)"'] \arrow[dd, bend left=60, "\sim"] & E_i\Phi(F_jL) \arrow[ddd, phantom, ""] \arrow[dddlll, bend left=20, "\mathrm{id}"] \\
F_jE_iF_i\Phi(M) \arrow[r, "\eta_i\bullet"] \arrow[d, "\bullet\Phi(\gamma)"'] & E_iF_iF_jE_iF_i\Phi(M) \arrow[r, "E_i\tau_{ij}\bullet"] \arrow[d, "\bullet\Phi(\gamma)"'] & E_iF_jF_iE_iF_i\Phi(M) \arrow[dd, "E_iF_j\varepsilon_i\bullet"'] \arrow[ddl, "\bullet\Phi(\gamma)"'] & \\
F_jE_i\Phi(L) \arrow[r, "\eta_i\bullet"] \arrow[d, "\sigma_{ji}\bullet"'] & E_iF_iF_jE_i\Phi(L) \arrow[d, "E_i\tau_{ij}\bullet"'] & & \\
E_iF_j\Phi(L) & E_iF_jF_iE_i\Phi(L) \arrow[l, "\bullet\varepsilon_i\Phi(L)"'] & E_iF_jF_i\Phi(M) \arrow[ll, bend left=15, "\bullet\Phi(\gamma)"] &
\end{tikzcd}$$

If $j \neq i$, then σ_{ij} is an isomorphism, hence (M', γ') satisfies the requirements. We assume now $i = j$. Let $n = \langle \lambda + \alpha_{i_1} + \cdots + \alpha_{i_r}, \alpha_i^\vee \rangle$.

Assume $n \geq 0$. Let $M'' = M' \oplus L^{\oplus n}$ and $\gamma'' = \gamma' + \sum_{a=0}^{n-1}(x_i^a \bullet) : F_i M'' \to F_i L$. Then, (M'', γ'') satisfies the required properties.

Assume finally $n \leq 0$. Consider $g = \sum_{l=0}^{1-n} \gamma \circ (x_i^l M) : F_i M \to L^{\oplus -n}$. The map $\Phi(g)$ is equal to the composition

$$\begin{tikzcd} F_i\Phi(M) \arrow[r, "F_i\eta_i\bullet"] \arrow[rr, bend right, "\sim"'] & F_iE_iF_i\Phi(M) \arrow[r, "F_iE_i\Phi(\gamma)"] & F_iE_i\Phi(L) \arrow[r, "\sum_l x_i^l\bullet"] & (F_iE_i\Phi(L))^{\oplus -n} \arrow[r, "\varepsilon_i\bullet"] & \Phi(L)^{\oplus -n} \end{tikzcd}$$

which is a split surjection. As a consequence, $g = \Psi\Phi(g)$ is a split surjection. Let M'' be its kernel and $\gamma'' = \gamma'_{|F_iM'}$. The composition

$$\Phi(M'') \hookrightarrow \Phi(F_iM) \xrightarrow{F_i\eta_i\bullet} F_iE_iF_i\Phi(M) \xrightarrow{\bullet\Phi(\gamma)} F_iE_i\Phi(L) \xrightarrow{\sigma_{ji}\bullet} E_iF_i\Phi(L)$$

is an isomorphism and we deduce that (M'', γ'') satisfies the requirements. $\square$

Note that Lauda and Vazirani had shown earlier that $\mathcal{B}(\lambda)$ gives rise to the crystal graph of $L(\lambda)$, in the symmetrizable case [LauVa].

4.4.4. *Cyclotomic Hecke algebras for* $\mathfrak{sl}_2$. Let $n \in \mathbf{Z}_{\geq 0}$. We have $H_n(Q) = {}^0H_n$. One can deduce from Theorem 4.25 that the 2-representations $\mathcal{L}(n)$ of §4.3.2 and §4.3.3 are equivalent. One can also show this directly without using $\mathcal{B}(n)$, by using the same method as in the proof of Theorem 4.25. Let us prove the more concrete fact that $\mathcal{B}(n)$ coincides with the category $\mathcal{L}(n)$ of §4.3.2.

Lemma 4.27. *Given $i \leq n$, then there is an isomorphism of rings*

$$\phi : H_i(n) \xrightarrow{\sim} H_{i,n},\ T_j \mapsto T_j,\ X_{j'} \mapsto X_{j'} \text{ and } z_l \mapsto (-1)^l e_l(x_1, \ldots, x_n)$$

for $1 \leq j \leq i-1$, $1 \leq j' \leq i$ and $1 \leq l \leq n$. If $i > n$, then $H_i(n) = 0$.

Proof. Assume $i \leq n$. In order to prove that the map ϕ of the lemma is well defined, it is enough to consider the case $i = n$. We have

$$x_1^n - e_1(x_1, \ldots, x_n) + \cdots + (-1)^n e_n(x_1, \ldots, x_n) = 0,$$

hence the map is well defined. It is clear that the map is surjective.

Let $A_i = \mathbf{Z}[z_1, \ldots, z_n] \otimes_{\mathbf{Z}} {}^0H_i$ and $V_i = \mathbf{Z}[z_1, \ldots, z_n] \otimes_{\mathbf{Z}} P_i$, a faithful A_i-module. Let M_i be the A_i-submodule of V_i generated by $X_1^n + X_1^{n-1}z_1 + \cdots + X_1 z_{n-1} + z_n$ and let $L_i = \sum_{a_j \leq n-j} \mathbf{Z}[z_1, \ldots, z_n] x_1^{a_1} \cdots x_i^{a_i}$. Let $V_i' = L_i + M_i$. Let us show by induction on i that $V_i = V_i'$. This is true for $n = 1$. Assume $V_i' = V_i$. Note that V_{i+1}' is stable under the action of A_i and the action of T_i. It follows that $\partial_i(X_i^{n-i+1}) \in V_{i+1}'$, hence $X_{i+1}^{n-i} \in V_{i+1}'$. This shows that V_{i+1}' is stable under multiplication by X_{i+1}, hence under the action of A_{i+1}. Since V_{i+1} is generated by $1 \in V_{i+1}'$ under the action of A_{i+1}, we deduce that $V_{i+1}' = V_{i+1}$.

We deduce that $H_i(n)$ has a faithful module of rank $\leq \frac{n!}{(n-i)!}$ over $\mathbf{Z}[z_1, \ldots, z_n]$. As a consequence, the image of $P_i \otimes \mathbf{Z}[z_1, \ldots, z_n]$ in $H_i(n)$ has rank $\leq \frac{n!}{(n-i)!}$ over $\mathbf{Z}[z_1, \ldots, z_n]$, hence $H_i(n)$ has rank $\leq \frac{i!n!}{(n-i)!}$ over $\mathbf{Z}[z_1, \ldots, z_n]$. That is the rank of $H_{i,n}$ over $P_n^{\mathfrak{S}_n}$: it follows that ϕ is an isomorphism. $\square$

5. Geometry

5.1. **Hall algebras.** We refer to [Sch] for a general text on Hall algebras.

5.1.1. *Definition.* Let $\mathcal{A}$ be an abelian category such that given $M, N \in \mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(M, N)$ and $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ are finite sets. One can take for example the category of finite dimensional representations of a quiver over a finite field.

Given $M, N \in \mathcal{A}$, let $F^L_{M,N}$ be the number of submodules N' of L such that $N' \simeq N$ and $L/N' \simeq M$.

Let $P^L_{M,N}$ denote the number of exact sequences $0 \to N \to L \to M \to 0$. Then,

$$F^L_{M,N} = \frac{P^L_{M,N}}{|\operatorname{Aut}(M)| \cdot |\operatorname{Aut}(N)|}. \tag{2}$$

Let $H'_{\mathcal{A}}$ be the free abelian group with basis the isomorphism classes of objects of $\mathcal{A}$

$$H'_{\mathcal{A}} = \bigoplus_{L \in \mathcal{A}/\sim} \mathbf{Z}[L].$$

We define a product in $H'_{\mathcal{A}}$ by

$$[M] * [N] = \sum_{L \in \mathcal{A}/\sim} F^L_{M,N}[L].$$

The class $[0]$ is a unit for the product. The algebra $H'_{\mathcal{A}}$ is the *Hall algebra* of $\mathcal{A}$.

One shows that the product is associative and more generally, that an iterated product counts filtrations.

Given $N_1, \ldots, N_n, L \in \mathcal{A}$, let $F^L_{N_1,\ldots,N_n}$ be the number of filtrations

$$L = L_0 \supset \cdots \supset L_n = 0$$

with $L_{i-1}/L_i \simeq N_i$.

Proposition 5.1. *We have* $[N_1] * \cdots * [N_n] = \sum_{L \in \mathcal{A}/\sim} F^L_{N_1,\ldots,N_n}[L]$.

Remark 5.2. When $\mathcal{A}$ is semi-simple, then $H'_{\mathcal{A}}$ is commutative. The "next" case is the following. Let $\mathcal{A}$ be the category of finite abelian p-groups. The algebra $H'_{\mathcal{A}}$ has a basis parametrized by partitions and $H'_{\mathcal{A}} = \mathbf{Z}[u_1, u_2, \ldots]$ is a polynomial ring in the countably many variables $u_i = [(\mathbf{Z}/p)^i]$ (Steinitz-Hall).

5.1.2. *Hall algebra for an A_2 quiver.* Let us now describe the Hall algebra for $\mathcal{A}$ the category of finite dimensional representations of the quiver $\Gamma = 1 \to 2$ over a finite field k with q elements.

The indecomposable representations of Γ are $S(1)$, $S(2)$ and M (cf Example 3.2). Let $f_1 = [S(1)]$, $f_2 = [S(2)]$ and $f_{12} = [M]$. We find $f_1 * f_2 = f_{12} + [S(1) \oplus S(2)]$ and $f_2 * f_1 = [S(1) \oplus S(2)]$. The algebra $H'_{\mathcal{A}}$ is not commutative. We have $[f_1, f_2] = f_{12}$.

We have $f_1 * f_{12} = q[M \oplus S(1)]$ and $f_{12} * f_1 = [M \oplus S(1)]$. So, $f_1 * f_{12} = q f_{12} * f_1$. If we view q as an indeterminate and specialize it to 1, then the Lie subalgebra of $H'_{\mathcal{A}}$ generated by f_1, f_2 and f_{12} is isomorphic to the Lie algebra of strictly upper triangular 3×3-matrices:

$$f_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{12} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5.1.3. *Quantum groups as Ringel-Hall algebras.* Let Γ be a quiver with vertex set I and assume Γ has no loops. Let $\mathcal{A}$ be the category of finite dimensional representations of Γ over a finite field k with q elements.

The *Euler form* is defined by

$$\langle M, N\rangle = \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}^1(M,N)$$

for $M, N \in \mathcal{A}$.

We define the *Ringel-Hall* algebra $H_{\mathcal{A}}$ as the $\mathbf{C}$-vector space $\mathbf{C} \otimes_{\mathbf{Z}} H'_{\mathcal{A}}$ with the product

$$[M]\cdot[N] = q^{\langle M,N\rangle/2}[M] * [N].$$

The graph underlying Γ encodes a symmetric Cartan matrix, hence give rise to the nilpotent part $\mathfrak{n}^-$ of a Kac-Moody algebra.

We can now state Ringel's Theorem.

Theorem 5.3 (Ringel)**.** *There is an injective morphism of* $\mathbf{C}$*-algebras* $U_q(\mathfrak{n}^-) \hookrightarrow H_{\mathcal{A}}$, $f_i \mapsto [S(i)]$. *If* Γ *corresponds to a Dynkin diagram, then this morphism is an isomorphism.*

In the next section we will explain, following Lusztig, how to construct directly the non-quantum enveloping algebra $U(\mathfrak{n}^-)$.

5.2. **Functions on moduli stacks of representations of quivers.** We refer to [ChrGi] for a general introduction to geometric representation theory.

5.2.1. *Moduli stack of representations of quivers.* Let Γ be a quiver with vertex set I. Let $\mathrm{Rep} = \mathrm{Rep}(\Gamma)$ be the moduli stack of representations of Γ over $\mathbf{C}$. This is a geometrical object whose points are isomorphism classes of finite dimensional representations of Γ over $\mathbf{C}$. The moduli stack also encodes the information of the group $\mathrm{Aut}(M)$, given M a representation of Γ. We have $\mathrm{Rep} = \coprod_{d\in \mathbf{Z}^I_{\geq 0}} \mathrm{Rep}_d$, where Rep_d corresponds to representations V with $\dim V_i = d_i$ for $i \in I$. We refer to §5.3.4 for a more precise description.

This stack can be described explicitly as a quotient. Let $d \in \mathbf{Z}^I_{\geq 0}$. The data of a representation of Γ with underlying vector spaces $\{\mathbf{C}^{d_i}\}_i$ is the same as the data of an element of $\mathcal{M}_d = \bigoplus_{a:i\to j} \operatorname{Hom}_{\mathbf{C}}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$, where a runs over the arrows of Γ. This vector space has an action by conjugation of the group $G_d = \prod_i \mathrm{GL}_{d_i}(\mathbf{C})$:

$$g\cdot (f_a)_a = (g_j f_a g_i^{-1})_a \text{ for } g = (g_i)_i.$$

Two representations are isomorphic is they correspond to elements of $\mathcal{M}_d$ in the same G_d-orbit. Given $f \in \mathcal{M}_d$, we have $\mathrm{Stab}_{G_d}(f) = \mathrm{Aut}(M)$, where M is the representation of Γ defined by f.

We have $\mathrm{Rep}_d = \mathcal{M}_d/G_d$.

5.2.2. *Convolution of functions.* Let X be a set and $\mathcal{F}(X)$ the vector space of functions $X \to \mathbf{C}$.

Consider a map $\phi : X \to Y$ between sets.

We define $\phi^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ by $\phi^*(f)(x) = f(\phi(x))$.

Assume $\phi^{-1}(y)$ is finite for all $y \in Y$. Define $\phi_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ by $\phi_*(f)(y) = \sum_{x\in\phi^{-1}(y)} f(x)$.

Now, given a diagram

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "r"] & \\ X & X & X \end{tikzcd}$$

with the fibers of r finite, we define a convolution of functions

$$\mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X),\ (f, g) \mapsto f \circ g = r_*(p^*(f) \cdot q^*(g)).$$

5.2.3. *Convolution of constructible functions.* We want now to extend the constructions of §5.2.2 to the case of varieties, or rather stacks. The main problem is to give a sense to ϕ_* when ϕ doesn't have finite fibers.

Let X be a stack over $\mathbf{C}$. We define $\mathcal{F}_c(X)$, the space of *constructible functions*, as the subspace of $\mathcal{F}(X)$ generated by the functions 1_V, where V runs over locally closed subspaces of X. Here, $1_V(x) = 1$ if $x \in V$ and $1_V(x) = 0$ otherwise.

Given X a stack, we denote by $\chi(X) = \sum_{i \geq 0}(-1)^i \dim H^i(X)$ the *Euler characteristic* of X. There is a unique extension of χ to disjoint unions of locally closed subsets of stacks that satisfies $\chi(X) = \chi(V) + \chi(X - V)$.

Given $\phi : X \to Y$ a morphism of stacks, we define ϕ_* as follows. Let $f = \sum_\alpha m_\alpha 1_{V_\alpha}$, where the V_α are locally closed subsets of X and $m_\alpha \in \mathbf{C}$. We put

$$\phi_*(f)(y) = \sum_\alpha m_\alpha \chi(V_\alpha \cap \phi^{-1}(y)).$$

5.2.4. *Realization of* $U(\mathfrak{n}^-)$. Denote by X the stack over $\mathrm{Rep} \times \mathrm{Rep}$ of pairs $(V \subset V')$. We have three morphisms $p, q, r : X \to \mathrm{Rep}$

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "r"] & \\ \mathrm{Rep} & \mathrm{Rep} & \mathrm{Rep} \end{tikzcd}$$

$$p(V \subset V') = V,\ q(V \subset V') = V'/V \text{ and } r(V \subset V') = V'.$$

Define a convolution

$$\mathcal{F}_c(\mathrm{Rep}) \times \mathcal{F}_c(\mathrm{Rep}) \to \mathcal{F}_c(\mathrm{Rep}),\ (f, g) \mapsto f \circ g = r_*(p^*(f) \cdot q^*(g)).$$

Let $a_i = 1_{\mathrm{Rep}_i}$: we have $a_i(S(i)) = 1$ and $a_i(M) = 0$ if M is a representation of Γ not isomorphic to $S(i)$.

Theorem 5.4 (Lusztig)**.** *There is an injective morphism of* $\mathbf{C}$*-algebras* $U(\mathfrak{n}^-) \to \mathcal{F}_c(\mathrm{Rep}),\ f_i \mapsto a_i$. *If* Γ *corresponds to a Dynkin diagram, this is an isomorphism.*

5.3. **Flag varieties.** We recall classical facts on affine Hecke algebra actions and flag varieties in type A (cf e.g. [ChrGi]) and then flags of representations of quivers [Lu1].

5.3.1. *Notations.* We fix a prime number l and put $\Lambda = \bar{\mathbf{Q}}_l$. By scheme, we mean a separated scheme of finite type over $\mathbf{C}$. Given X a scheme or a stack, we denote by $D(X)$ the bounded derived category of l-adic constructible sheaves on X (cf [LaOl1, LaOl2]). All quotients will be taken in the category of stacks.

Given X a smooth stack and $i : Z \to X$ a smooth closed substack, both of pure dimension, we have a Gysin morphism $i_*\Lambda_Z \to \Lambda_X[2(\dim X - \dim Z)]$. Let D be the duality functor. Via the canonical identifications $D(\Lambda_Z) \xrightarrow{\sim} \Lambda_Z[2\dim Z]$, $D(\Lambda_X) \xrightarrow{\sim} \Lambda_X[2\dim X]$ and $D \circ i_* \xrightarrow{\sim} i_* \circ D$, the Gysin morphism is the dual of the canonical map $\Lambda_X \to i_*\Lambda_Z$. Note finally that the automorphism of $\mathrm{Ext}^*(\Lambda_X, \Lambda_X)$ induced by taking α to $D(\alpha)$ is the identity.

Let $\mathcal{C}$ be a graded additive category and M, N two objects of $\mathcal{C}$. We put $\mathrm{Hom}^\bullet_{\mathcal{C}}(M, N) = \bigoplus_i \mathrm{Hom}(M, N[i])$.

Given a graded ring A, we define the graded dimension of a free finitely generated graded A-module by $\mathrm{grdim}(A[i]) = q^{-i/2}$ and $\mathrm{grdim}(M) = \mathrm{grdim}(M_1) + \mathrm{grdim}(M_2)$ if $M \simeq M_1 \oplus M_2$.

5.3.2. *Nil affine Hecke algebra and* $\mathbf{P}^1$*-bundles.* Let X be a stack, E a rank 2 vector bundle on X and $\pi : Y = \mathbf{P}(E) \to X$ the projectivized bundle.

Let $\alpha = c_1(\mathcal{O}_\pi(-1))$ and $\beta = c_1(\pi^*E/\mathcal{O}_\pi(-1))$. Let $x = \pi_*(\alpha)$ and $y = \pi_*(\beta)$, viewed in $\mathrm{Hom}(\pi_*\Lambda_Y, \pi_*\Lambda_Y[2])$.

Let $T \in \mathrm{Hom}(\pi_*\Lambda_Y, \pi_*\Lambda_Y[-2])$ be the composition

$$T : \pi_*\Lambda_Y \xrightarrow{t} \Lambda_X[-2] \xrightarrow{\mathrm{can}} \pi_*\Lambda_Y[-2]$$

where $t : \pi_*\Lambda_Y \xrightarrow{\mathrm{can}} \mathcal{H}^2(\pi_*\Lambda_Y)[-2] \underset{\sim}{\xrightarrow{\mathrm{tr}}} \Lambda_X[-2]$ is the trace map.

Proposition 5.5. *We have $T^2 = 0$ $yT - Tx = 1$ $xy = yx$ and $T(x + y) = (x + y)T$. This defines a morphism of algebras*

$$^0H_2 \to \mathrm{End}^\bullet(\pi_*\Lambda_Y),\ X_1 \mapsto x,\ X_2 \mapsto y,\ T_1 \mapsto T.$$

Proof. We have $\alpha + \beta = c_1(\pi^*E)$ and $\alpha\beta = c_2(\pi^*E)$. The composition

$$\pi_*\pi^*\Lambda_X \underset{\sim}{\xrightarrow{\mathrm{can}}} \pi_*\Lambda_Y \xrightarrow{T} \pi_*\Lambda_Y[-2] \underset{\sim}{\xrightarrow{\mathrm{can}}} \pi_*\pi^*\Lambda_X[-2]$$

comes from natural transformations of functors, hence it commutes with $\mathrm{End}^\bullet(\Lambda_X)$. It follows that T commutes with $c_1(E) = x + y$ and $c_2(E) = xy$.

Since T^2 factors through a map $\Lambda_X[-2] \to \Lambda_X[-4]$, we have $T^2 = 0$.

The composition

$$\Lambda_X \xrightarrow{\mathrm{can}} \pi_*\Lambda_Y \xrightarrow{-x} \pi_*\Lambda_Y[2] \xrightarrow{t} \Lambda_X$$

is the identity. Indeed, after taking the fiber at a point $P \in X$, the composition is

$$\Lambda \xrightarrow{c_1(\mathcal{O}(1))} H^2(\pi^{-1}(P), \Lambda) \xrightarrow{t} \Lambda$$

and $c_1(\mathcal{O}(1))$ is the class of a point.

So, we have an isomorphism

$$(\mathrm{can}, x \circ \mathrm{can}) : \Lambda_X \oplus \Lambda_X[-2] \xrightarrow{\sim} \pi_*\Lambda_Y.$$

Note that the composition

$$\Lambda_X \xrightarrow{\mathrm{can}} \pi_*\Lambda_Y \xrightarrow{y} \pi_*\Lambda_Y[2] \xrightarrow{t} \Lambda_X$$

is also the identity.

The composition
$$\Lambda_X \xrightarrow{\text{can}} \pi_*\Lambda_Y \xrightarrow{T} \pi_*\Lambda_Y[-2]$$
vanishes since it factors through a map $\Lambda_X \to \Lambda_X[-2]$. It follows that
$$\Lambda_X \xrightarrow{\text{can}} \pi_*\Lambda_Y \xrightarrow{yT-Tx} \pi_*\Lambda_Y$$
is equal to the canonical map.

We have a commutative diagram
$$\begin{tikzcd} \Lambda_X \arrow[r, "\text{can}"] \arrow[dr, "c_2(E)"'] & \pi_*\Lambda_Y \arrow[r, "xy"] & \pi_*\Lambda_Y[4] \arrow[r, "T"] & \pi_*\Lambda_Y[2] \ . \\ & \Lambda_X[4] \arrow[ur, "\text{can}"'] & & \end{tikzcd}$$
So, the composition $\Lambda_X \to \pi_*\Lambda_Y[2]$ vanishes since it factors through a map $\Lambda_X[4] \to \Lambda_X[2]$.

It follows that the composition
$$\Lambda_X \xrightarrow{\text{can}} \pi_*\Lambda_Y \xrightarrow{y} \pi_*\Lambda_Y[2] \xrightarrow{yT-Tx} \pi_*\Lambda_Y[2]$$
is equal to the composition
$$\Lambda_X \xrightarrow{\text{can}} \pi_*\Lambda_Y \xrightarrow{y} \pi_*\Lambda_Y[2].$$
So, $yT - Tx = 1$. □

5.3.3. *Nil affine Hecke algebras and flag varieties.* We recall now the construction of an action of ${}^0H_n$ on $H^*(\mathrm{Gr}_n/\mathrm{GL}_n)$ (cf [Ku]), where Gr_n is the variety of complete flags in $\mathbf{C}^n$.

Let $\psi : \mathrm{Gr}_n/\mathrm{GL}_n \to \mathrm{pt}/\mathrm{GL}_n$ be the canonical map.

Consider the first Chern class of the line bundle defined by V_d/V_{d-1} over a complete flag $(0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{C}^n)$ and let X_d be the corresponding element $X_d : \psi_*\Lambda \to \psi_*\Lambda[2]$.

Let $\mathrm{Gr}_n(d)$ the the variety of flags $(0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} = \mathbf{C}^n)$ such that $\dim V_r/V_{r-1} = 1$ for $r \neq d$ and $\dim V_d/V_{d-1} = 2$. The canonical map
$$\mathrm{Gr}_n \to \mathrm{Gr}_n(d)$$
$$(0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{C}^n) \mapsto (0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_{d+1} \subset \cdots \subset V_n = \mathbf{C}^n)$$
is the projectivization of the 2-dimensional vector bundle V_d/V_{d-1} over $\mathrm{Gr}_n(d)$. It induces a map $p_d : \mathrm{Gr}_n/\mathrm{GL}_n \to \mathrm{Gr}_n(d)/\mathrm{GL}_n$.

Let $T \in \mathrm{Hom}(p_{d*}\Lambda, p_{d*}\Lambda[-2])$ be the composition
$$T : p_{d*}\Lambda \xrightarrow{t} \Lambda[-2] \xrightarrow{\text{can}} p_{d*}\Lambda[-2]$$
where $t : p_{d*}\Lambda \xrightarrow{\text{can}} \mathcal{H}^2(p_{d*}\Lambda)[-2] \underset{\sim}{\xrightarrow{\text{tr}}} \Lambda[-2]$ is the trace map. We denote by $T_d : \psi_*\Lambda \to \psi_*\Lambda[-2]$ the induced map.

Let X be a stack. The data of a rank n vector bundle $\mathcal{L}$ on X is equivalent to the data of a morphism of stacks $X \to \mathrm{pt}/\mathrm{GL}_n$. Let Y be the stack of full flags in a rank n vector bundle $\mathcal{L}$ on X and $\phi : Y \to X$ be the associated map: this is the pullback of ψ via l.

The following Theorem follows from Proposition 5.5, together with a verification of the braid relations between T_d's.

Theorem 5.6. *The construction above provides by base change a morphism* ${}^0H_n \to \mathrm{End}^\bullet(\phi_*\Lambda)$.

Let $j : X' \to X$ be a closed immersion. Assume X and X' are smooth of pure dimension. Then, the canonical morphism $\phi_*\Lambda \to \phi'_*\Lambda$ and the morphism induced by the Gysin map $\phi'_*\Lambda \to \phi_*\Lambda[2(\dim X - \dim X')]$ commute with the action of ${}^0H_n$.

5.3.4. *Sheaves on moduli stacks of quivers.* We follow Lusztig [Lu1, §9]. Instead of working with equivariant derived categories of varieties, we work with derived categories of the corresponding quotient stacks. Given X a variety acted on by G, our perverse sheaves on X/G correspond to shifts by $\dim G$ of the G-equivariant perverse sheaves on X considered by Lusztig. The duality is similarly shifted.

Given X a scheme, we have an abelian category $\mathcal{O}_X[\Gamma]$-Mod of representations of Γ over X, *i.e.*, sheaves of $(\mathbf{Z}\Gamma \otimes_{\mathbf{Z}} \mathcal{O}_X)$-modules. Its objects can be viewed as pairs $V = (\mathcal{V}, \rho)$ where $\mathcal{V}$ is an $\mathcal{O}_X$-module and $\rho : \mathbf{Z}\Gamma \to \mathrm{End}(\mathcal{V})$ is a morphism of rings.

Given J a subset of I, we put $\mathcal{V}_J = \sum_{i\in J} \rho(i)\mathcal{V}$.

We denote by $\mathrm{Rep} = \mathrm{Rep}(\Gamma)$ the algebraic stack of representations of Γ. It is defined by assigning to a scheme X the subcategory of $\mathcal{O}_X[\Gamma]$-Mod defined as follows:

- objects are pairs $(\mathcal{V}, \rho)$ such that $\mathcal{V}$ is a vector bundle (of finite rank) over X
- maps are isomorphisms.

Given a morphism of schemes $f : X \to Y$, we have a functor $f^* : \mathrm{Rep}(\Gamma)(Y) \to \mathrm{Rep}(\Gamma)(X)$ given by base change.

We define the rank vector of $V = (\mathcal{V}, \rho)$ as $\mathrm{rk}\, V = \sum_{i\in I} (\mathrm{rk}\, \mathcal{V}_i) i \in \mathbf{N}[I]$. We have a decomposition into connected components

$$\mathrm{Rep} = \coprod_{\alpha\in\mathbf{N}[I]} \mathrm{Rep}_\alpha$$

where Rep_α is the substack of representations with rank vector α. We denote by j_α the embedding of the component Rep_α. We have $\dim \mathrm{Rep}_\alpha = -\langle\alpha, \alpha\rangle$. Note that Rep_0 is a point. We denote by $\mathcal{W}_\alpha$ the tautological vector bundle $\{V\}$ on Rep_α.

Given $i \in I$, we put $L_i = j_{i*}\Lambda[-1]$, a perverse sheaf on Rep.

Given $M \in D(\mathrm{Rep}_\mu)$ and $N \in D(\mathrm{Rep}_{\mu'})$, we put $M \circ N = r_!(p^*M \otimes q^*N)[-\langle\mu', \mu\rangle]$. This endows $D(\mathrm{Rep})$ with a structure of monoidal category (it is the reverse of the tensor structure defined by Lusztig).

5.3.5. *Flags of representations.* Given $\nu = (\nu^1, \ldots, \nu^n) \in \mathbf{N}[I]^n$, we consider the stack Rep_ν of flags $(0 = V^0 \subset V^1 \subset \cdots \subset V^n)$ of representations of Γ such that $\mathrm{rk}\, V^r/V^{r-1} = \nu^r$. We have a proper morphism

$$\pi_\nu : \mathrm{Rep}_\nu \to \mathrm{Rep}_{\sum_r \nu^r},\ (0 = V^0 \subset V^1 \subset \cdots \subset V^n) \mapsto V^n$$

and we put $\rho_\nu = j_{\sum_r \nu^r} \circ \pi_\nu : \mathrm{Rep}_\nu \to \mathrm{Rep}$.

Given $\varepsilon_1, \ldots, \varepsilon_n \in \{\mathrm{empty}, \mathrm{ss}\}$, we define $\mathrm{Rep}_{(\nu^1_{\varepsilon_1}, \ldots, \nu^n_{\varepsilon_n})}$ as the closed substack of $\mathrm{Rep}_{(\nu^1, \ldots, \nu^n)}$ of flags $(0 = V^0 \subset V^1 \subset \cdots \subset V^n)$ such that for any r such that $\varepsilon_r = \mathrm{ss}$, then $V^r/V^{r-1} \simeq \bigoplus_i (V^r/V^{r-1})_i$ as a representation of Γ.

5.3.6. *Flags and quotients.* Let $\bar{\Gamma}$ be the discrete quiver with vertex set I and let $\overline{\mathrm{Rep}} = \mathrm{Rep}(\bar{\Gamma})$. The restriction map defines a morphism $\mathrm{Rep}_\nu \to \overline{\mathrm{Rep}}_\nu$. This is a vector bundle of rank

$$\sum_{i\neq j} d_{ij} \sum_r \nu_i^r(\nu_j^1 + \cdots + \nu_j^r).$$

Let $\alpha = \sum_i \alpha_i i \in \mathbf{N}[I]$. We denote by $\widetilde{\mathrm{Rep}}_\alpha$ the variety $\prod_{h:i\to j} \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^{\alpha_i}, \mathbf{C}^{\alpha_j})$, where h runs over the set of arrows of Γ. There is an action of $G_\alpha = \prod_i \mathrm{GL}_{\alpha_i}$ on $\widetilde{\mathrm{Rep}}_\alpha$ given by $g\cdot f = (g_j f_h g_i^{-1})_{h:i\to j}$, where $g = (g_i)_{i\in I}$ and $f = (f_h)_h$. A point of $\widetilde{\mathrm{Rep}}_\alpha$ defines a representation of Γ of dimension vector α: this provides an isomorphism $\widetilde{\mathrm{Rep}}_\alpha/G_\alpha \xrightarrow{\sim} \mathrm{Rep}_\alpha$. In particular, we obtain $\mathrm{B}G_\alpha \xrightarrow{\sim} \overline{\mathrm{Rep}}_\alpha$.

Given $d_1, \ldots, d_r \geq 0$, we denote by $\mathrm{Gr}_{d_1,\ldots,d_r}$ the variety of flags $(0 = V_0 \subset V_1 \subset \cdots \subset V_r = \mathbf{C}^{\sum d_l})$ such that $\dim V_l/V_{l-1} = d_l$. Let $\nu = (\nu^1, \ldots, \nu^n) \in \mathbf{N}[I]^n$. Let $\alpha = \sum_r \nu^r$ and $n_i = \sum_{r=1}^n \nu_i^r$. We denote by $\widetilde{\mathrm{Rep}}_\nu$ the subvariety of $\prod_i \mathrm{Gr}_{\nu_i^1,\ldots,\nu_i^n} \times \widetilde{\mathrm{Rep}}_\alpha$ given by families $((0 = V_{i,0} \subset \cdots \subset V_{i,n} = \mathbf{C}^{n_i})_i, (f_h)_h)$ such that $f_h(V_{i,r}) \subset V_{j,r}$ for all $h : i \to j$ and all r. The diagonal action of $G = G_\alpha$ restricts to an action on $\widetilde{\mathrm{Rep}}_\nu$. Sending a point to the associated filtered representation of Γ defines an isomorphism $\widetilde{\mathrm{Rep}}_\nu/G \xrightarrow{\sim} \mathrm{Rep}_\nu$. Let P_i be the parabolic subgroup of GL_{n_i} stabilizing the standard flag $F_i = (V_{i,0} = 0 \subset V_{i,1} = \mathbf{C}^{\nu_i^1} \oplus 0 \subset \cdots \subset V_{i,n} = \mathbf{C}^{\nu_i^1} \oplus \cdots \oplus \mathbf{C}^{\nu_i^n})$ and let $P = \prod_i P_i$. We have a canonical isomorphism $G/P \xrightarrow{\sim} \prod_i \mathrm{Gr}_{\nu_i^1,\ldots,\nu_i^n}$ inducing an isomorphism $G \backslash G/P \xrightarrow{\sim} \overline{\mathrm{Rep}}_\nu$.

Let $\nu' = (\nu'^1, \ldots, \nu'^{n'}) \in \mathbf{N}[I]^{n'}$. We assume $\alpha = \sum_r \nu'^r$. This defines as above a parabolic subgroup P' of G. We denote by W, W_P and $W_{P'}$ the Weyl groups of G, P and P'. We have an isomorphism

$$\left(\widetilde{\mathrm{Rep}}_\nu \times_{\widetilde{\mathrm{Rep}}_\alpha} \widetilde{\mathrm{Rep}}_{\nu'}\right)/G \xrightarrow{\sim} \mathrm{Rep}_\nu \times_{\mathrm{Rep}} \mathrm{Rep}_{\nu'}\,.$$

The isomorphisms above induce an isomorphism

$$P' \backslash G/P \xrightarrow{\sim} \overline{\mathrm{Rep}}_\nu \times_{\overline{\mathrm{Rep}}} \overline{\mathrm{Rep}}_{\nu'}.$$

Its closed points are in bijection with $W_{P'}\backslash W/W_P$ and each such point w defines a locally closed closed substack X_w. This corresponds to the decomposition

$$\prod_i \left(\mathrm{Gr}_{\nu_i^1,\ldots,\nu_i^n} \times \mathrm{Gr}_{\nu_i'^1,\ldots,\nu_i'^{n'}}\right) = \coprod_w \mathcal{O}_w$$

into orbits under the action of G, *i.e.*, $\mathcal{O}_w/G \xrightarrow{\sim} X_w$.

The restriction map $V \to \{V_i\}_{i\in I}$ induces a map $\kappa : \mathrm{Rep}_\nu \times_{\mathrm{Rep}} \mathrm{Rep}_{\nu'} \to \overline{\mathrm{Rep}}_\nu \times_{\overline{\mathrm{Rep}}} \overline{\mathrm{Rep}}_{\nu'}$. The restriction of κ over each X_w is a vector bundle. Note that $H_c^*(\mathrm{Rep}_\nu \times_{\mathrm{Rep}} \mathrm{Rep}_{\nu'})$ is a free graded $H^*(BG)$-module of graded rank equal to the graded rank of $H_c^*(\widetilde{\mathrm{Rep}}_\nu \times_{\widetilde{\mathrm{Rep}}_\alpha} \widetilde{\mathrm{Rep}}_{\nu'})$ as a Λ-module. The pullback of κ is the projection map

$$\tilde{\kappa} : \widetilde{\mathrm{Rep}}_\nu \times_{\widetilde{\mathrm{Rep}}_\alpha} \widetilde{\mathrm{Rep}}_{\nu'} \to \prod_i \left(\mathrm{Gr}_{\nu_i^1,\ldots,\nu_i^n} \times \mathrm{Gr}_{\nu_i'^1,\ldots,\nu_i'^{n'}}\right).$$

Assume now $\nu^r \in I$ for all r. We have $W_P = W_{P'} = 1$. Define $\gamma_i, \gamma_i' : \{1, \ldots, n_i\} \to \{1, \ldots, n\}$ to be the increasing maps such that $\nu^{\gamma_i(r)} = \nu'^{\gamma_i'(r)} = i$ for all r.

We identify W with $\prod_i \mathfrak{S}_{n_i}$. Let $w = (w_i)_i \in W$. The fiber of $\tilde{\kappa}$ over $((F_i, w_i(F'_i)))_i \in \mathcal{O}(w)$ has dimension

$$\sum_{s\neq t} d_{st} \cdot \#\{a, b | \gamma_t(b) < \gamma_s(a) \text{ and } \gamma'_t(w_t^{-1}(b)) < \gamma'_s(w_s^{-1}(a))\}.$$

We deduce that

$$\mathrm{grdim} H_c^*(\widetilde{\mathrm{Rep}}_\nu \times_{\widetilde{\mathrm{Rep}}_\alpha} \widetilde{\mathrm{Rep}}_{\nu'}) = P(W, q) \sum_{w\in W} q^{(l(w)+\sum_{s\neq t} d_{st}\cdot\#\{a,b|\gamma_t(b)<\gamma_s(a) \text{ and } \gamma'_t(w_t^{-1}(b))<\gamma'_s(w_s^{-1}(a))\})}$$

where $P(W, q) = \prod_i \prod_{r=1}^{n_i} \frac{q^r-1}{q-1}$ is the Poincaré polynomial of W.

5.4. **Quiver Hecke algebras and geometry.**

5.4.1. *Monoidal category of semi-simple perverse sheaves.* We denote by $\mathcal{P}$ the smallest full additive monoidal subcategory of $D(\mathrm{Rep})$ closed under translations and containing the objects L_i for $i \in I$.

The following theorem gives a presentation of $\mathcal{P}$ by generators and relations. It has been proven independently by Varagnolo and Vasserot [VarVas].

Theorem 5.7. *There is an equivalence of graded monoidal categories* $R : (\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} \mathcal{B}(\Gamma))^i\text{-gr} \xrightarrow{\sim} \mathcal{P}$.

The category $\mathcal{B}(\Gamma)$ is defined by generators and relations and in §5.4.3 we define the images of the generating objects and arrows. The verification of the relations and the proof that the induced functor is an equivalence start in §5.4.4.

Theorem 5.7 shows that quiver Hecke algebras $H_n(\Gamma)$ are Ext-algebras of certain sums of shifted simple perverse sheaves on quiver varieties, as all objects of $\mathcal{P}$ are of that form.

5.4.2. *Canonical basis.* There is an isomorphism of $\mathbf{Z}[q^{\pm 1/2}]$-algebras [Lu1, §14]

$$U_{\mathbf{Z}[q^{\pm1/2}]}(\mathfrak{n}^-) \xrightarrow{\sim} K_0(\mathcal{P}),\ f_s \mapsto [L_s]. \tag{3}$$

Let B be the set of isomorphism classes of simple perverse sheaves on Rep that are contained in $\mathcal{P}$. Every object of $\mathcal{P}$ is isomorphic to a direct sum of shifts of objects of B. The canonical basis C of $U_{\mathbf{Z}[q^{\pm1/2}]}(\mathfrak{n}^-)$ corresponds, via the isomorphism (3), to $\{[L]\}_{L\in B}$.

Recall that there is a duality Δ [Rou2, §4.2.1] on $\mathcal{B}(\Gamma)$, *i.e.*, a graded equivalence of monoidal categories $\mathcal{B}(\Gamma)^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{B}(\Gamma)$ with $\Delta^2 = \mathrm{Id}$ given by

$$F_s[n] \mapsto F_s[-n],\ x_s \mapsto x_s \text{ and } \tau_{st} \mapsto \tau_{ts}.$$

Let C' be the set of classes in K_0 of indecomposable objects M of $(\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} \mathcal{B}(\Gamma))^i$-gr such that $\Delta(M) \simeq M$.

Corollary 5.8. *We have an isomorphism* $U_{\mathbf{Z}[q^{\pm1/2}]}(\mathfrak{n}^-) \xrightarrow{\sim} K_0((\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} \mathcal{B}(\Gamma))^i\text{-gr})$. *It induces a bijection* $C \xrightarrow{\sim} C'$.

5.4.3. *Hecke generators.* We set $R(F_s) = L_s$. Let us now define the value of R on the generating arrows of $\mathcal{B}(\Gamma)$.

- Let $s \in I$. We denote by $x_s \in \operatorname{Hom}(L_s, L_s[2])$ the image of $c_1(\mathcal{W}_s) \in H^2(\operatorname{Rep}_s, \Lambda)$.

- The forgetful morphism $\pi_{(s,s)} : \operatorname{Rep}_{(s,s)} \to \operatorname{Rep}_{2s}$ is the $\mathbf{P}^1$-fibration associated to the rank 2 bundle $\mathcal{W}_{2s}$. We denote by $\tau_{ss} \in \operatorname{Hom}(L_s \circ L_s, L_s \circ L_s[-2])$ the image of the composition $\pi_{(s,s)*}\Lambda \xrightarrow{\text{trace}} \Lambda[-2] \xrightarrow{\text{can}} \pi_{(s,s)*}\Lambda[-2]$.

- Let $s \neq t \in I$. Consider the morphism

$$f_{st} : \operatorname{Rep}_{(s,t)} \to \operatorname{Rep}_s \times \operatorname{Rep}_t,\ (V \subset V') \mapsto (V, V'/V).$$

Let $\mathcal{M}_{st} = f_{st*}\mathcal{O}$: this is the vector bundle $\operatorname{Ext}^1(V', V)$ over $\operatorname{Rep}_s \times \operatorname{Rep}_t = \{(V, V')\}$. We have $\mathcal{M}_{st} \simeq (\mathcal{W}_s \boxtimes \mathcal{W}_t^{-1})^{\oplus d_{ts}}$.

The vector bundle $f_{st}^*\mathcal{M}_{st}$ has a section given by assigning to $(V \subset V')$ the class of the extension $0 \to V \to V' \to V'/V \to 0$. The zero substack of that section is $Z_{st} = \operatorname{Rep}_{(s+t)_{\mathrm{ss}}}$, a closed substack of codimension d_{ts} in $\operatorname{Rep}_{(s,t)}$.

We denote by $\tau_{st} \in \operatorname{Hom}(L_s \circ L_t, L_t \circ L_s[m_{st}])$ the image of the composition

$$\rho_{(t,s)*}(\Lambda_{Z_{st}} \xrightarrow{\text{Gysin}} \Lambda_{\operatorname{Rep}_{(t,s)}}[2d_{st}]) \circ \rho_{(s,t)*}(\Lambda_{\operatorname{Rep}_{(s,t)}} \xrightarrow{\text{can}} \Lambda_{Z_{st}}).$$

5.4.4. *Polynomial actions.* Let us first study Hom-spaces in the category $\mathcal{P}$ under the action of polynomial rings.

Let $\nu \in I^n$ and $\nu' \in I^{n'}$. Given $i \in I$, let $n_i = \#\{r | \nu_r = i\}$ and $n'_i = \#\{r | \nu'_r = i\}$. By [ChrGi, §8.6], there is an isomorphism of $(\Lambda[x_{\nu_1}, \ldots, x_{\nu_n}], \Lambda[x_{\nu'_1}, \ldots, x_{\nu'_{n'}}])$-bimodules

$$\operatorname{Ext}^*(L_{\nu_1} \circ \cdots \circ L_{\nu_n}, L_{\nu'_1} \circ \cdots \circ L_{\nu'_{n'}}) \xrightarrow{\sim} H_c^{\sigma-*}(\operatorname{Rep}_\nu \times_{\operatorname{Rep}} \operatorname{Rep}_{\nu'})$$

where $\sigma = \dim \operatorname{Rep}_\nu + \dim \operatorname{Rep}_{\nu'}$. If $\operatorname{Ext}^*(L_{\nu_1} \circ \cdots \circ L_{\nu_n}, L_{\nu'_1} \circ \cdots \circ L_{\nu'_{n'}}) \neq 0$, then the stack $\operatorname{Rep}_\nu \times_{\operatorname{Rep}} \operatorname{Rep}_{\nu'}$ is non-empty, so $n_i = n'_i$ for all i. Assume this holds. It follows from §5.3.5 that $\operatorname{Ext}^*(L_{\nu_1} \circ \cdots \circ L_{\nu_n}, L_{\nu'_1} \circ \cdots \circ L_{\nu'_n})$ is a free graded $H^*(BG)$-module of graded rank

$$N = v^\sigma P(W, q^{-1}) \sum_{w \in W} q^{-(l(w) + \sum_{s \neq t} d_{st} \cdot \#\{a,b | \gamma_t(b) < \gamma_s(a) \text{ and } \gamma'_t(w_t^{-1}(b)) < \gamma'_s(w_s^{-1}(a))\})}.$$

We have

$$\sigma = 2 \sum_s n_s(n_s - 1) + \sum_{s \neq t} d_{st} \left(\#\{a, b | \gamma_t(b) < \gamma_s(a)\} + \#\{a, b | \gamma'_t(b) < \gamma'_s(a)\}\right).$$

On the other hand,

$$l(w) = \sum_s \#\{a, b | \gamma_s(b) < \gamma_s(a) \text{ and } \gamma'_s(w_s^{-1}(b)) > \gamma'_s(w_s^{-1}(a))\}.$$

It follows that

$$N = P(W, q) \sum_{w \in W} q^{\frac{1}{2} \sum_{s,t \in I} m_{st} \cdot \#\{a,b | \gamma_s(a) < \gamma_t(b) \text{ and } \gamma'_s(w_s(a)) > \gamma'_t(w_t(b))\}}$$

and we deduce from Lemma 3.10 that the graded dimensions of the free $\left(\bigotimes_i \Lambda[X_{i,1}, \ldots, X_{i,n_i}]^{\mathfrak{S}_{n_i}}\right)$-modules $\operatorname{Hom}_{\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} \mathcal{B}(\Gamma)}(F_{\nu_1} \cdots F_{\nu_n}, F_{\nu'_1} \cdots F_{\nu'_n})$ and $\operatorname{Ext}^*(L_{\nu_1} \circ \cdots \circ L_{\nu_n}, L_{\nu'_1} \circ \cdots \circ L_{\nu'_n})$ coincide.

5.4.5. *Relations* τ^2. Let $s \neq t \in I$. The self-intersection formula shows that

$$\Lambda_{Z_{st}} \xrightarrow{\text{Gysin}} \Lambda_{\mathrm{Rep}_{(t,s)}}[2d_{st}] \xrightarrow{\text{can}} \Lambda_{Z_{st}}[2d_{st}]$$

is equal to

$$c_{d_{st}}(f_{ts}^* \mathcal{M}_{ts}) = (c_1((\mathcal{W}_t)_{|Z_{st}}) - c_1((\mathcal{W}_s)_{|Z_{st}}))^{d_{st}}.$$

On the other hand, the composition

$$\Lambda_{\mathrm{Rep}_{(s,t)}} \xrightarrow{\text{can}} \Lambda_{Z_{st}} \xrightarrow{\text{Gysin}} \Lambda_{\mathrm{Rep}_{(s,t)}}[2d_{ts}]$$

is equal to

$$[Z_{st}] = c_{d_{ts}}(f_{st}^* \mathcal{M}_{st}) = (c_1((\mathcal{W}_s)_{|Z_{st}}) - c_1((\mathcal{W}_t)_{|Z_{st}}))^{d_{ts}}.$$

We have shown that

$$\tau_{ts} \circ \tau_{st} = (-1)^{d_{st}} (x_s L_t - L_s x_t)^{d_{st}+d_{ts}}.$$

It follows from §5.3.3 that $\tau_{ss}^2 = 0$.

5.4.6. *Relations* τ^3. Consider now $s, t, u \in I$.

• Assume first s, t and u are distinct. The intersection of the closed substacks $\mathrm{Rep}_{((s+t)_{\mathrm{ss}},u)}$ and $\mathrm{Rep}_{(t,(s+u)_{\mathrm{ss}})}$ of $\mathrm{Rep}_{(t,s,u)}$ is transverse, since the intersection of $0 \times \mathbf{C}^{d_{us}}$ and $\mathbf{C}^{d_{st}} \times 0$ in $\mathbf{C}^{d_{st}} \times \mathbf{C}^{d_{us}}$ is transverse.

It follows that the composition $(L_t \tau_{su}) \circ (\tau_{st} L_u)$ is equal to the image of the composition

$$\rho_{(t,u,s)*}(\mathbf{C}_Z \xrightarrow{\text{Gysin}} \mathbf{C}_{\mathrm{Rep}_{(t,u,s)}}[2(d_{su}+d_{st})]) \circ \rho_{(s,t,u)*}(\mathbf{C}_{\mathrm{Rep}_{(s,t,u)}} \xrightarrow{\text{can}} \mathbf{C}_Z)$$

where Z is the substack of $\mathrm{Rep}_{(t,u,s)}$ (resp. of $\mathrm{Rep}_{(s,t,u)}$) of triples $(L \subset L' \subset L'')$ such that $(L'')_s$ is a direct summand of L''.

Similarly, the intersection of Z and $\mathrm{Rep}_{((t+u)_{\mathrm{ss}},s)}$ in $\mathrm{Rep}_{(t,u,s)}$ is transverse and we deduce that $(\tau_{tu} L_s) \circ (L_t \tau_{su}) \circ (\tau_{st} L_u)$ is equal to the image of

$$\rho_{(u,t,s)*}(\mathbf{C}_{\mathrm{Rep}_{(s+t+u)_{\mathrm{ss}}}} \xrightarrow{\text{Gysin}} \mathbf{C}_{\mathrm{Rep}_{(u,t,s)}}[2(d_{st}+d_{su}+d_{tu})]) \circ \rho_{(s,t,u)*}(\mathbf{C}_{\mathrm{Rep}_{(s,t,u)}} \xrightarrow{\text{can}} \mathbf{C}_{\mathrm{Rep}_{(s+t+u)_{\mathrm{ss}}}}).$$

A similar calculation provides the same description of $(L_u \tau_{st}) \circ (\tau_{su} L_t) \circ (L_s \tau_{tu})$, so we have

$$(\tau_{tu} L_s) \circ (L_t \tau_{su}) \circ (\tau_{st} L_u) = (L_u \tau_{st}) \circ (\tau_{su} L_t) \circ (L_s \tau_{tu}).$$

• We consider now the case where $s = t \neq u$. As above, we obtain that the composition $(\tau_{su} L_s) \circ (L_s \tau_{su})$ is equal to the image of the composition

$$\rho_{(u,s,s)*}(\mathbf{C}_{\mathrm{Rep}_{(s,s,u)_{\mathrm{ss}}}} \xrightarrow{\text{Gysin}} \mathbf{C}_{\mathrm{Rep}_{(u,s,s)}}[4d_{su}]) \circ \rho_{(s,s,u)*}(\mathbf{C}_{\mathrm{Rep}_{(s,s,u)}} \xrightarrow{\text{can}} \mathbf{C}_{\mathrm{Rep}_{(s,s,u)_{\mathrm{ss}}}}).$$

The commutation of the action of the nil affine Hecke algebra in §5.3.3 shows that

$$(\tau_{su} L_s) \circ (L_s \tau_{su}) \circ (\tau_{ss} L_u) = (L_u \tau_{ss}) \circ (\tau_{su} L_s) \circ (L_s \tau_{su}).$$

• The case $s = t = u$ follows from the results of §5.3.3.

5.4.7. *Conclusion.* The relations (3) and (4) in §3.3.3 are clear when $s \neq t$ and follow from §5.3.3 when $s = t$. The results of §5.4.5 and 5.4.6 complete the verification of the defining relations for the category $\mathcal{B}(\Gamma)$. Thanks to §5.4.4, we obtain a monoidal $\bar{\mathbf{Q}}_l$-linear graded functor $R : (\bar{\mathbf{Q}}_l \otimes \mathcal{B}(\Gamma))^i\text{-gr} \to \mathcal{P}$. That functor is essentially surjective. It follows from Proposition 3.9 and from §5.4.4 that R is faithful.

Let $\nu \in I^n$ and $\nu' \in I^{n'}$. By Nakayama's Lemma, it follows from §5.4.4 that R induces an isomorphism

$$\mathrm{Hom}_{\mathcal{B}(\Gamma)}(F_{\nu_1} \cdots F_{\nu_n}, F_{\nu'_1} \cdots F_{\nu'_n}) \xrightarrow{\sim} \mathrm{Ext}^*(L_{\nu_1} \circ \cdots \circ L_{\nu_n}, L_{\nu'_1} \circ \cdots \circ L_{\nu'_{n'}}).$$

This completes the proof that R is an equivalence.

5.5. 2-Representations.

5.5.1. *Framed quivers and construction of representations.* Nakajima introduced new quiver varieties in order to construct irreducible representations $L(\lambda)$ of Kac-Moody algebras. We present a modification due to Hao Zheng [Zh].

Let $\hat{\Gamma}$ be the quiver obtained from Γ by adding vertices $\hat{i}$ for $i \in I$ and arrows $i \to \hat{i}$. We have $\mathrm{Rep}(\hat{\Gamma}) = \coprod_{\mu,\nu \in \mathbf{Z}^I_{\geq 0}} \mathrm{Rep}_{\hat{\nu}+\mu}(\hat{\Gamma})$.

Assume i is a source of Γ. Let U_i be the substack of Rep of representations V such that the canonical map $V_i \to \bigoplus_{a:i\to j} V_j$ is injective, where a runs over arrows of $\hat{\Gamma}$ starting at i. Let $\mathcal{N}_i$ be the thick subcategory of $D(\mathrm{Rep}(\hat{\Gamma}))$ of complexes of sheaves with 0 restriction to $\mathcal{N}_i$.

If i is not a source, consider a quiver Γ'_i corresponding to a different orientation of Γ and such that i is a source of Γ'_i. Define $\mathcal{N}'_i \subset D(\mathrm{Rep}(\hat{\Gamma}'_i)$ as above. Now, there is an equivalence $D(\mathrm{Rep}(\hat{\Gamma}'_i) \xrightarrow{\sim} D(\mathrm{Rep}(\hat{\Gamma}))$ given by Fourier transform and we define $\mathcal{N}_i$ to be the image of $\mathcal{N}'_i$.

Finally, let $\mathcal{N}$ be the thick subcategory of $D(\mathrm{Rep}(\hat{\Gamma}))$ generated by $\mathcal{N}_i$ for $i \in I$. This is independent of the choice of the quivers Γ'_i. Let $D = D(\mathrm{Rep}(\hat{\Gamma}))/\mathcal{N}$.

Consider now a root datum $(X, Y, \langle -, - \rangle, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ with Cartan matrix that afforded by Γ.

Let $\lambda \in X^+$. Let $\nu_i = \langle \lambda, \alpha_i^\vee \rangle$ and $\nu = (\nu_i)_i \in \mathbf{Z}^I_{\geq 0}$. We put $\mathrm{Rep}(\lambda) = \coprod_{\mu \in \mathbf{Z}^I_{\geq 0}} \mathrm{Rep}_{\hat{\nu}+\mu}(\hat{\Gamma})$ and we denote by $D(\lambda)$ the image of $D(\mathrm{Rep}(\lambda))$ in D.

The convolution functor $L_i \circ -$ stabilizes $\mathcal{N}$ and induces an endofunctor E_i of $D(\lambda)$.

It has a right adjoint F_i. Let $\mathcal{P}(\lambda)$ be the smallest full subcategory of $D(\lambda)$ containing $\mathbf{C}_{\mathrm{Rep}_{\hat{\nu}}}$, stable under E_i for $i \in I$, and stable under direct summands and direct sums.

Theorem 5.9 (Zheng)**.** *The functors E_i and F_i satisfy Serre relations, and abstract versions of the $\rho_{i,\lambda}$ and σ_{ij} isomorphisms. In particular, they induce an action of $U_q(\mathfrak{g})$ on $\mathbf{C} \otimes_{\mathbf{Z}} K_0(\mathcal{P}(\lambda))$ and the resulting module is isomorphic to the simple module of highest weight λ.*

5.5.2. 2*-representations.* The action by convolution of $\mathcal{B}(\Gamma)$ on $D(\mathrm{Rep}(\hat{\Gamma}))$ (cf Theorem 5.7) induces a graded action on $\mathcal{P}(\lambda)$.

Theorem 5.10. *The graded action of $\mathcal{B}(\Gamma)$ on $\mathcal{P}(\lambda)$ extends to a graded action of $\mathfrak{A}(\Gamma)$. There is an equivalence of graded 2-representations of $\mathfrak{A}(\Gamma)$*

$$\big(\mathcal{L}(\lambda) \otimes_k k^\Gamma \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}_l\big)^i\text{-gr} \xrightarrow{\sim} \mathcal{P}(\lambda).$$

Proof. We have $\mathrm{End}^\bullet(\Lambda_{\mathrm{Rep}_\nu}) \simeq H^*(BG_\nu)$. Given $M, N \in \mathcal{P}(\lambda)$, the space $\mathrm{Hom}^\bullet(M,N)$ is a finitely generated $H^*(BG_\nu)$-module [Zh, proof of Proposition 3.2.5]. By Theorem 5.9, the functors E_i and F_i induce an action of $\mathfrak{sl}_2$ on $K_0(\mathcal{P}(\lambda)) \otimes_{\mathbf{Z}[q^{\pm 1/2}]} \mathbf{C}[q^{\pm 1/2}]/(q^{1/2}-1)$. We deduce from Corollary 4.13 that we have a 2-representation of $\mathfrak{A}(\Gamma)$ (the grading can be forgotten to check that the maps $\rho_{s,\lambda}$ are isomorphisms).

Let $i \in I$ and $0 \le r \le \nu_i$. We have $\mathrm{Rep}_{\hat{\nu}+r\alpha_i} = \prod_{j\neq i} B\,\mathrm{GL}_{\nu_j} \times BP_r$, where P_r is the maximal parabolic subgroup of GL_{ν_i} with Levi $\mathrm{GL}_r \times \mathrm{GL}_{\nu_i - r}$. The graded action of $H^*(B\,\mathrm{GL}_{\nu_i})$ on $E_i^{(\nu_i)}(\Lambda_{\mathrm{Rep}_\nu})$ corresponds to the action of $\bar{\mathbf{Q}}_l[X_1,\ldots,X_{\nu_i}]^{\mathfrak{S}_{\nu_i}}$. We deduce that the canonical map $P_\lambda \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}_l \to H^*(BG_\nu)$ is an isomorphism. This proves the last part of the theorem. □

Remark 5.11. Zheng provides more generally a construction of tensor products of simple representations, and the first part of Theorem 5.10, and its proof, generalize immediately to that case: this provides graded 2-representations with Grothendieck group that tensor product of simple representations.

Putting Theorems 4.25 and 5.10 together, we obtain

Corollary 5.12. *There is an equivalence compatible with the graded action of* $\mathfrak{A}(\Gamma)$

$$(\mathcal{B}(\lambda) \otimes_k k^\Gamma \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}_l)^i\text{-gr} \xrightarrow{\sim} \mathcal{P}(\lambda).$$

As a consequence, the indecomposable projective modules for cyclotomic quiver Hecke algebras over $\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} k^\Gamma$ correspond to the canonical basis elements of $L(\lambda)$. When Γ has type A_n or $\tilde{A}_n$, this is Ariki's Theorem (formerly, the Lascoux-Leclerc-Thibon conjecture). Here, we used the geometry of quiver varieties, which carry the same singularities as flag varieties, in type A.

Remark 5.13. It would be interesting to extend Theorems 5.7 and 5.10 to the case of coefficients $\mathbf{Z}$ or $\mathbf{F}_p$.

Lauda has given an independent proof of the results of §5.5.2 for $\mathfrak{sl}_2$: in this case, the geometry is that of flag varieties of type A [Lau2]. An earlier geometrical approach has been given by Cautis, Kamnitzer and Licata for $\mathfrak{sl}_2$, based on coherent sheaves on cotangent bundles of flag varieties and compactifications of those [CauKaLi1], later generalized to arbitrary Γ [CauKaLi2].

Webster has given a presentation of our results and constructions in §5.5.2 and has used this to develop a categorification of the Reshetikhin-Turaev invariants [We1, We2]. He has also constructed a counterpart of the categories $\mathcal{B}(\lambda)$ for tensor products.

References

[Ari] S. Ariki, "Representations of Quantum Algebras and Combinatorics of Young Tableaux", Amer. Math. Soc., 2000.

[BeFrKho] J. Bernstein, I.B. Frenkel and M. Khovanov, *A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}2)$ via projective and Zuckerman functors*, Selecta Math. (N.S.) **5** (1999), 199–241.

[BrKl1] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009), 451–484.

[BrKl2] J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), 1883–1942.

[BrStr] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, Moscow Mathematical Journal **11** (2011).

[CauKaLi1] S. Cautis, J. Kamnitzer and A. Licata, *Coherent sheaves and categorical $\mathfrak{sl}_2$-actions*, Duke Math. J. **154** (2010), 135–179.

[CauKaLi2] S. Cautis, J. Kamnitzer and A. Licata, *Coherent Sheaves on Quiver Varieties and Categorification*, preprint arXiv:1104.0352.

[ChrGi] N. Chriss and V. Ginzburg, "Representation theory and complex geometry", Birkhäuser, 1997.

[ChRou] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals of Math. **167** (2008), 245–298

[GaRoi] P. Gabriel and A.V. Roiter, "Representations of finite-dimensional algebras", Encylopedia of Mathematical Sciences, Algebra VIII, Springer Verlag, 1997.

[Gr] I. Grojnowski, *Affine $\hat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric group and related Hecke algebras*, math.RT/9907129.

[Hi] H. Hiller, "Geometry of Coxeter groups", Pitman, 1982.

[Kac] V. Kac, "Infinite-dimensional Lie algebras", Cambridge University Press, 1990.

[KanKas] S.-J. Kang and M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier Algebras*, preprint arXiv:1102.4677v3.

[KhoLau1] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups, I*, Represent. Theory **13** (2009), 309–347.

[KhoLau2] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups, II*, Trans. Amer. Math. Soc. **363** (2011), 2685–2700.

[KhoLau3] M. Khovanov and A. Lauda, *A categorification of quantum* $\mathfrak{sl}(n)$, Quantum Topol. **1** (2010), 1–92.

[Kl] A. Kleshchev, "Linear and Projective Representations of Symmetric Groups", Cambridge University Press, 2005.

[Ku] S. Kumar, "Kac-Moody groups, their flag varieties and representation theory", Birkhäuser, 2002.

[LaOl1] Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks II: Adic Coefficients*, Publ. Math. Inst. Hautes Études Sci. **107** (2008), 169–210.

[LaOl2] Y. Laszlo and M. Olsson, *Perverse t-structure on Artin stacks*, Math. Z. **261** (2009), 737–748.

[Lau1] A. Lauda, *A categorification of quantum* $\mathfrak{sl}(2)$, Adv. Math. **225** (2010), 3327–3424.

[Lau2] A. Lauda, *Categorified quantum sl(2) and equivariant cohomology of iterated flag varieties*, Algebr. Represent. Theory **14** (2011), 253–282.

[LauVa] A. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Advances in Math. **228** (2011), 803–861.

[Lu1] G. Lusztig, "Introduction to quantum groups", Birkhäuser, 1993.

[Ma] A. Mathas, "Hecke algebras and Schur algebras of the symmetric group", Amer. Math. Soc., 1999.

[Rou1] R. Rouquier, *Derived equivalences and finite dimensional algebras*, Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol II, pp. 191–221, EMS Publishing House, 2006.

[Rou2] R. Rouquier, 2-*Kac-Moody algebras*, preprint arXiv:0812.5023.

[Sch] O. Schiffmann, *Lectures on Hall algebras*, preprint arXiv:math/0611617.

[VarVas] M. Varagnolo and E. Vasserot, *Canonical bases and Khovanov-Lauda algebras*, preprint arXiv:0901.3992.

[We1] B. Webster, *Knot invariants and higher representation theory I: diagrammatic and geometric categorification of tensor products*, preprint arXiv:1001.2020.

[We2] B. Webster, *Knot invariants and higher representation theory II: the categorification of quantum knot invariants*, preprint arXiv:1005.4559.

[Zh] H. Zheng, *Categorification of integrable representations of quantum groups*, preprint arXiv:0803.3668.

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