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Representation dimension of exterior algebras

Raphaël Rouquier

Institut de Mathématiques de Jussieu – CNRS UFR de Mathématiques, Université Denis Diderot 2, place Jussieu, 75005 Paris, France (e-mail: rouquier@math.jussieu.fr)

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Abstract. We determine the representation dimension of exterior algebras. This provides the first known examples of representation dimension > 3. We deduce that the Loewy length of the group algebra over $\mathbf{F}_2$ of a finite group is strictly bounded below by the 2-rank of the group (a conjecture of Benson). A key tool is the use of the concept of dimension of a triangulated category.

1. Introduction

In his 1971 Queen Mary College notes [Au], Auslander introduced an invariant of finite dimensional algebras, the representation dimension. It was meant to measure how far an algebra is to having only finitely many classes of indecomposable modules. Whereas many upper bounds have been found for the representation dimension, lower bounds were missing. In particular, it wasn't known whether the representation dimension could be greater than 3. A proof that all algebras have representation dimension at most 3 would have led for example to a solution of the finitistic dimension conjecture [IgTo].

We prove here that the representation dimension of the exterior algebra of a non-zero finite dimensional vector space is one plus the dimension of that vector space – in particular, the representation dimension can be arbitrarily large. Thus, the representation dimension is a useful invariant of finite dimensional algebras of infinite representation type, confirming the hope of Auslander. The case of algebras with infinite global dimension is particularly interesting.

As a consequence of our results, we prove the characteristic $p = 2$ case of a conjecture of Benson asserting that the p-rank of a finite group is less than the Loewy length of its group algebra over a field of characteristic p.

The main idea is to use a recently defined notion of dimension for a triangulated category [Rou]. In Sect. 3, we show how this relates to the representation dimension, when applied to the derived category or its stable quotient. The dimension of the derived category is related at the same time to the Loewy length and to the global dimension.

In Sect. 4, we compute these dimensions for the exterior algebra of a finite dimensional vector space, using a reduction to a commutative algebra problem, via Koszul duality. We show that $\dim \Lambda(k^n) - \overline{\text{mod}} = n - 1$ and that the representation dimension of $\Lambda(k^n)$ is $n + 1$ (Theorem 4.1). This enables us to settle the characteristic 2 case of a conjecture of Benson (Theorem 4.11).

Preliminary results have been obtained and exposed at the conference "Twenty years of tilting theory" in Fraueninsel in November 2002. I wish to thank the organizers for giving me the opportunity to report on these early results and the participants for many useful discussions, particularly Thorsten Holm for introducing me to Auslander's work.

2. Notations and terminology

Let $\mathcal{C}$ be an additive category. We denote by $K^b(\mathcal{C})$ the homotopy category of bounded complexes of objects of $\mathcal{C}$. Given $\mathcal{F}$ a family of objects of $\mathcal{C}$, we denote by $\text{add}(\mathcal{F})$ the smallest full subcategory of $\mathcal{C}$ containing the objects of $\mathcal{F}$ and closed under taking direct summands. We denote by $\mathcal{C}^\circ$ the opposite category to $\mathcal{C}$.

Let $\mathcal{T}$ be a triangulated category. A *thick* subcategory $\mathcal{I}$ of $\mathcal{T}$ is a full triangulated subcategory closed under taking direct summands.

Let A be an algebra over a field k. We denote by A-mod the category of finitely generated left A-modules and by A-proj the category of finitely generated projective A-modules. We denote by $\text{gldim}\, A$ the global dimension of A. We denote by A° the opposite algebra to A and we put $A^{\text{en}} = A \otimes_k A^\circ$.

Assume A has finite dimension over k. We denote by $J(A)$ the Jacobson radical of A and we denote by $\text{ll}(A)$ the Loewy length of A, *i.e.*, the minimal integer r such that $J(A)^r = 0$. We denote by $D^b(A)$ the derived category of bounded complexes of finitely generated A-modules and we denote by A-perf the thick subcategory of $D^b(A)$ of objects isomorphic to bounded complexes of finitely generated projective A-modules.

3. Dimensions

3.1. Auslander's representation dimension.

3.1.1. Let $\mathcal{A}$ be an abelian category.

Definition 3.1. *The (Auslander) representation dimension* repdim $\mathcal{A}$ *is the smallest integer $i \geq 2$ such that there is an object $M \in \mathcal{A}$ with the property that given any $L \in \mathcal{A}$,*

(a) *there is an exact sequence*

$$0 \to M^{-i+2} \to M^{-i+3} \to \cdots \to M^0 \to L \to 0$$

with $M^j \in \operatorname{add}(M)$ *such that the sequence*

$$0 \to \operatorname{Hom}_{\mathcal{A}}(M, M^{-i+2}) \to \operatorname{Hom}_{\mathcal{A}}(M, M^{-i+3}) \to \cdots \to \operatorname{Hom}_{\mathcal{A}}(M, M^0) \to \operatorname{Hom}_{\mathcal{A}}(M, L) \to 0$$

is exact

(b) *there is an exact sequence*

$$0 \to L \to M'^0 \to M'^1 \to \cdots \to M'^{i-2} \to 0$$

with $M'^j \in \operatorname{add}(M)$ *such that the sequence*

$$0 \to \operatorname{Hom}_{\mathcal{A}}(M'^{i-2}, M) \to \cdots \to \operatorname{Hom}_{\mathcal{A}}(M'^1, M) \to \operatorname{Hom}_{\mathcal{A}}(M'^0, M) \to \operatorname{Hom}_{\mathcal{A}}(L, M) \to 0$$

is exact.

An object M that realizes the minimal i is called an *Auslander generator*.

Note that either condition (a) or (b) implies that $\operatorname{gldim} \operatorname{End}_{\mathcal{A}}(M) \leq i$, and $\operatorname{gldim} \operatorname{End}_{\mathcal{A}}(M) = \operatorname{repdim} \mathcal{A}$ if M is an Auslander generator (cf. *e.g.* [ErHoIySc, Lemma 2.1]). Note also that if condition (a) (resp. (b)) holds for every L in a subcategory $\mathcal{I}$ of $\mathcal{A}$ such that every object of $\mathcal{A}$ is a direct summand of an object of $\mathcal{I}$, then, it holds for every object of $\mathcal{A}$.

Note that $\operatorname{repdim} \mathcal{A} = 2$ if and only if $\mathcal{A}$ has only finitely many isomorphism classes of indecomposable objects. Note also that $\operatorname{repdim} \mathcal{A} = \operatorname{repdim} \mathcal{A}^\circ$.

3.1.2. Take $\mathcal{A} = A$-mod, where A is a finite dimensional algebra over a field. We write $\operatorname{repdim}(A)$ for $\operatorname{repdim}(A\text{-mod})$.

Let $M \in \mathcal{A}$ and $i \geq 2$. If M satisfies (a) of Definition 3.1, then it contains a projective generator as a direct summand (take $L = A$). More generally, the following are equivalent

- M satisfies (a) of Definition 3.1 and M contains an injective cogenerator as a direct summand
- M satisfies (b) of Definition 3.1 and M contains a projective generator as a direct summand
- M satisfies (a) and (b) of Definition 3.1.

So, the definition of representation dimension given here coincides with Auslander's original definition (cf [Au, §III.3] and [ErHoIySc, Lemma 2.1]) when A is not semi-simple. When A is semi-simple, Auslander assigns the representation dimension 0 whereas we define it to be 2 here. Iyama has shown [Iy] that the representation dimension of a finite dimensional algebra is finite.

Various classes of algebras with representation dimension 3 have been found: algebras with radical square zero [Au, §III.5, Proposition p. 56], hereditary algebras [Au, §III.5, Proposition p. 58] and more generally stably hereditary algebras [Xi, Theorem 3.5], special biserial algebras [ErHoIySc], local algebras of quaternion type [Ho].

3.1.3. One can weaken the requirements in the definition of the representation dimension as follows:

Definition 3.2. *The weak representation dimension of* $\mathcal{A}$*, denoted by* $\mathrm{wrepdim}(\mathcal{A})$*, is the smallest integer* $i \geq 2$ *such that there is an object* $M \in \mathcal{A}$ *with the property that given any* $L \in \mathcal{A}$*, there is a bounded complex* $C = 0 \to C^r \to \cdots \to C^s \to 0$ *of* $\mathrm{add}(M)$ *with*

- L *isomorphic to a direct summand of* $H^0(C)$
- $H^d(C) = 0$ *for* $d \neq 0$ *and*
- $s - r \leq i - 2$.

Note that $\mathrm{wrepdim}\,\mathcal{A} = \mathrm{wrepdim}\,\mathcal{A}^\circ$ and $\mathrm{repdim}\,\mathcal{A} \geq \mathrm{wrepdim}\,\mathcal{A}$.

In order to obtain lower bounds for the representation dimension of certain algebras, we will actually construct lower bounds for the weak representation dimension.

Remark 3.3. One could also study intermediate versions, left (resp. right) weak representation dimension, by requiring $C^d = 0$ for $d > 0$ (resp. $d < 0$) in the definition.

Remark 3.4. Note that the representation dimension as well as the invariants of Definition 3.2 are not invariant by derived equivalence (consider for instance a derived equivalence between an algebra with finite representation type and an algebra with infinite representation type, cf. for example [Ha, §III.4.14]).

Remark 3.5. All the definitions given here for abelian categories make sense for exact categories.

3.2. Dimension of a triangulated category.

3.2.1. We recall the definition of the dimension of a triangulated category, following [Rou]. The notion of finite-dimensionality corresponds to Bondal-Van den Bergh's notion of strong finite generation [BoVdB], which was introduced in order to obtain representability Theorems of Brown type.

Let $\mathcal{T}$ be a triangulated category. Let $M \in \mathcal{T}$. Let us define inductively a family of full additive subcategories of $\mathcal{T}$.

We put $\langle M \rangle_1 = \mathrm{add}(\{M[i]\}_{i \in \mathbf{Z}})$. For $i \geq 2$, we define $\langle M \rangle_i$ as the full subcategory of $\mathcal{T}$ whose objects are direct summands of objects L such that there is a distinguished triangle $M_1 \to L \to M_2 \rightsquigarrow$ with $M_1 \in \langle M \rangle_1$ and $M_2 \in \langle M \rangle_{i-1}$. We write $\langle M \rangle_{\mathcal{T},i}$ when there is a possible ambiguity on the ambient triangulated category.

Definition 3.6. *The dimension* $\dim \mathcal{T}$ *of* $\mathcal{T}$ *is the smallest integer* $d \geq 0$ *such that there is* $M \in \mathcal{T}$ *with* $\mathcal{T} = \langle M \rangle_{d+1}$.

If $\mathcal{T} \to \mathcal{T}'$ is a triangulated functor such that every object of $\mathcal{T}'$ is a direct summand of the image of an object of $\mathcal{T}$, then $\dim \mathcal{T}' \leq \dim \mathcal{T}$.

3.2.2. Let k be a field and A be a finite dimensional k-algebra.

The dimensions of $D^b(A)$ and $D^b(A)/A$-perf are related both to the Loewy length and to the global dimension (none of these two are invariant under derived equivalence, while the dimensions are preserved).

Proposition 3.7. *We have the inequalities* $\dim D^b(A)/A\text{-perf} \leq \mathrm{ll}(A) - 1$ *and* $\dim D^b(A)/A\text{-perf} \leq \mathrm{wrep}(A) - 2$*. If k is perfect, then* $\dim D^b(A) \leq \inf(\mathrm{gldim}\, A, \mathrm{repdim}(A))$.

Proof. Let $n = \mathrm{wrepdim}\, A$. There is $N \in A$-mod with the property that given $L \in A$-mod, there is a bounded complex $C = 0 \to C^r \to \cdots \to C^s \to 0$ of $\mathrm{add}(N)$ with $H^i(C) = 0$ for $i \neq 0$, L is a direct summand of $H^0(C)$ and $s - r \leq n - 2$. Then, $L \in \langle N \rangle_{s-r+1}$. Every object of $D^b(A)/A$-perf is isomorphic to an object $L[r]$ for some $L \in A$-mod and $r \in \mathbf{Z}$. Consequently, $D^b(A)/A\text{-perf} = \langle N \rangle_{n-1}$. We have also $L \in \langle A/JA \rangle_{\mathrm{ll}(A)}$, hence $D^b(A)/A\text{-perf} = \langle A/JA \rangle_{\mathrm{ll}(A)}$.

Let $0 \to P^{-r} \to \cdots \to P^0 \to A \to 0$ be a minimal projective resolution of A as an A^{en}-module. Then, $r = \mathrm{gldim}\, A$, since $\mathrm{Ext}^i_{A^{\mathrm{en}}}(A, \mathrm{Hom}_k(T, S)) \xrightarrow{\sim} \mathrm{Ext}^i_A(T, S)$ for S, T two simple A-modules (note that since k is perfect, every simple A^{en}-module is isomorphic to a module of the form $\mathrm{Hom}_k(T, S)$). Given $C \in D^b(A)$, then $P^i \otimes_A C \in \langle A \rangle_1$ since $P^i \in \mathrm{add}(A^{\mathrm{en}})$, so $C \in \langle A \rangle_{r+1}$. We have shown that $D^b(A) = \langle A \rangle_{1+\mathrm{gldim}\, A}$.

Let M be an Auslander generator for A-mod. Let C be a bounded complex of objects of $\mathrm{add}(M)$. Let $L \in A$-mod and let D be a bounded complex of $\mathrm{add}(M)$ together with a map $f : D \to L$ such that $\mathrm{Hom}^\bullet_A(M, f)$ is a quasi-isomorphism between the complexes $\mathrm{Hom}^\bullet_A(M, D)$ and $\mathrm{Hom}^\bullet_A(M, L)$ (in particular, f is a quasi-isomorphism). Then, $\mathrm{Hom}^\bullet_A(C, f) : \mathrm{Hom}^\bullet_A(C, D) \to \mathrm{Hom}^\bullet_A(C, L)$ is a quasi-isomorphism, hence $\mathrm{Hom}(C, f) : \mathrm{Hom}_{K^b(A)}(C, D) \to \mathrm{Hom}_{K^b(A)}(C, L)$ is an isomorphism.

It follows by induction that every bounded complex of A-mod is quasi-isomorphic to a bounded complex of $\mathrm{add}(M)$, *i.e.*, the canonical functor $K^b(\mathrm{add}(M)) \to D^b(A)$ is essentially surjective. We have canonical equivalences $K^b(\mathrm{End}_A(M)\text{-proj}) \xrightarrow{\sim} D^b(\mathrm{End}_A(M))$ and $K^b(\mathrm{End}_A(M)\text{-proj}) \xrightarrow{\sim} K^b(\mathrm{add}(M))$. We know that $\dim D^b(\mathrm{End}_A(M)) \leq \mathrm{gldim}\, \mathrm{End}_A(M)$. So, $\dim D^b(A) \leq \mathrm{repdim}(A)$. □

Remark 3.8. The proof of Proposition 3.7 shows that the assumption that k is perfect can be weakened. If $Z(A/JA)$ is a product of fields that are separable extensions of k, then $\dim D^b(A) \leq \mathrm{gldim}\, A$.

3.3. Stable categories of self-injective algebras. Let k be a field. Let A a self-injective finite dimensional k-algebra. Given M an A-module, we denote by ΩM the kernel of a surjective map from a projective cover of M to M and by $\Omega^{-1} M$ the cokernel of an injective map from M to an injective hull of M.

We denote by $A-\overline{\mathrm{mod}}$ the stable category of A. This is the quotient of the additive category A-mod by the additive subcategory A-proj. It has a triangulated category structure where the distinguished triangles come from short exact sequences of modules and where the translation functor is Ω^{-1}. The canonical functor A-mod $\to D^b(A\text{-mod})$ induces an equivalence of triangulated categories $A-\overline{\text{mod}} \xrightarrow{\sim} D^b(A\text{-mod})/A\text{-perf}$ ([KeVo, Example 2.3], [Ri, Theorem 2.1]).

Proposition 3.9. *Let A be a non-semisimple self-injective k-algebra. Then,*

$$\mathrm{ll}(A) \geq \mathrm{repdim}\, A \geq \mathrm{wrepdim}\, A \geq 2 + \dim A - \overline{\text{mod}}.$$

Proof. The first inequality is [Au, §III.5, Proposition p. 55] (use $M = A \oplus A/J(A) \oplus A/J(A)^2 \oplus \cdots$). Note that Auslander's result only claims that $\mathrm{repdim}\, A \leq \mathrm{ll}(A) + 1$, but the proof actually shows the stronger inequality (in the proof of the Theorem p. 45, the case $m = n$ does not occur if C has no projective indecomposable summand, for A is self-injective). The second inequality is trivial (cf. Sect. 3.1.3). The last inequality is given by Proposition 3.7. □

4. Exterior algebras

4.1. Dimension. The aim of Sect. 4.1 is the proof of the following theorem, which gives the first known examples of algebras with representation dimension > 3.

Theorem 4.1. *Let $n \geq 1$ be an integer. Then,* $\dim \Lambda(k^n) - \overline{\text{mod}} = \mathrm{repdim}\, \Lambda(k^n) - 2 = n - 1$ *and* $\dim D^b(\Lambda(k^n)\text{-mod}) = n$.

In Sect. 4.1.1, we give an elementary and triangulated category-free proof of the bound $\mathrm{repdim}\, \Lambda(k^n) \geq 4$ if $n \geq 3$ and k is uncountable. This gives a direct proof that $\mathrm{repdim}\, \Lambda(k^3) = 4$, via Proposition 3.9. Note that Sect. 4.1.1 is not used for the proof of the general case, which requires the more sophisticated Koszul duality equivalence.

4.1.1. Let $A = \Lambda(V)$ with V a finite dimensional vector space of dimension ≥ 3 over an uncountable field k. Given $z \in \mathbf{P}(V)$, we put $X(z) = A/(Az)$ where we view z as a line in V.

Proposition 4.2. *Let* $M, N \in A$-mod. *Let Y be a subset of* $\mathbf{P}(V)$ *such that for every* $y \in Y$*, there is an exact sequence*

$$0 \to M \to N \to X(y) \to 0.$$

Then, Y is contained in a proper closed subvariety of $\mathbf{P}(V)$.

Proof. We have a canonical isomorphism $S(V^*) \xrightarrow{\sim} \mathrm{Ext}_A^*(k,k)$. Consider an exact sequence $0 \to M \xrightarrow{f_y} N \to X(y) \to 0$. Let $\phi_y = \mathrm{Ext}_A^*(k, f_y)$:

$\mathrm{Ext}^*_A(k, M) \to \mathrm{Ext}^*_A(k, N)$, a morphism between two graded finitely generated $S(V^*)$-modules. We have an exact sequence

$$\mathrm{Ext}^{*-1}_A(k, X(y)) \to \mathrm{Ext}^*_A(k, M) \xrightarrow{\phi_y} \mathrm{Ext}^*_A(k, N) \to \mathrm{Ext}^*_A(k, X(y)).$$

It follows that the supports of $\ker\phi_y$ and $\operatorname{coker}\phi_y$ are contained in the line y, and the support of $\ker\phi_y \oplus \operatorname{coker}\phi_y$ is thus exactly y, since the support of $\mathrm{Ext}^*_A(k, X(y))$ is y. The proposition follows now from Lemma 4.3 below. □

Lemma 4.3. *Let M and N be two finitely generated graded $S(V^*)$-modules. Let Y be a subset of $\mathbf{P}(V)$ such that for every $y \in Y$, there is $\phi_y : M \to N$ a graded morphism of $S(V^*)$-modules with the property that the support of* $\ker\phi \oplus \operatorname{coker}\phi_y$ *is y. Then, Y is contained in a proper closed subvariety of* $\mathbf{P}(V)$.

Proof. Let M_{tor} be the torsion submodule of M (elements whose support is distinct from V). Replacing M and N by M/M_{tor} and N/N_{tor}, one can assume that M and N are torsion-free and the maps ϕ_y are injective. Let now M_0 (resp. N_0) be a free graded sumodule of M (resp. N) such that M/M_0 (resp. N/N_0) is a torsion module. There is a non-zero homogeneous $Q \in S(V^*)$ such that $QN \subset N_0$. Now, replacing $\phi_y : M \to N$ by $Q\phi_y : M_0 \to N_0$ and removing to Y its intersection with the union of $Q = 0$ and the support of M/M_0, we are reduced to the case where M and N are free and non-zero. Consider a non-zero injective map $\phi : M \to N$ with torsion cokernel. Then, M and N have the same rank and the support of ϕ is the zero locus of $\det\phi$, a hypersurface. We get a contradiction, since $\dim V \geq 3$. □

Corollary 4.4. *Let $M \in A$-mod. Then, there is $z \in \mathbf{P}(V)$ such that there is no exact sequence $0 \to M_1 \to M_0 \to X(z) \to 0$ with $M_0, M_1 \in \mathrm{add}(M)$. In particular,* $\operatorname{repdim} A \geq 4$.

Proof. There are only countably many M_0 and M_1's (up to isomorphism). For each of them, the $X(z)$ that can resolved corresponding to points z in a strict closed subvariety of $\mathbf{P}(V)$ (Proposition 4.2). Since $\mathbf{P}(V)$ is not a countable union of strict closed subvarieties, the corollary follows. □

4.1.2. We need to consider derived categories of differential modules. The theory of such derived categories mirrors that of the usual derived category of complexes of modules (forget the grading). We state here the needed constructions and results.

Let A be a k-algebra. A differential A-module is an $(A \otimes_k k[\varepsilon]/(\varepsilon^2))$-module. We view a differential A-module as a pair (M, d) where M is an A-module and $d \in \mathrm{End}_A(M)$ satisfying $d^2 = 0$ is given by the action of ε. The cohomology of a differential A-module is the A-module $\ker d/\operatorname{im} d$.

The category of differential A-modules has the structure of an exact category, where the exact sequences are those exact sequences of $(A \otimes_k k[\varepsilon]/(\varepsilon^2))$-modules that split by restriction to A. This is a Frobenius category and its associated stable category is called the homotopy category of differential A-modules.

A morphism of differential A-modules is a quasi-isomorphism if the induced map on cohomology is an isomorphism. We now define the derived category of differential A-modules, denoted by $D\mathrm{diff}(A)$, as the localization of the homotopy category of differential A-modules in the class of quasi-isomorphisms. These triangulated categories have a trivial shift functor.

We have a triangulated forgetful functor $D(A) \to D\mathrm{diff}(A)$. Let X, Y be two A-modules and $i \geq 0$. Then, the canonical map $\mathrm{Ext}^i_A(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{D(A)}(X, Y[i]) \to \mathrm{Hom}_{D\mathrm{diff}(A)}(X, Y)$ is injective and we have an isomorphism $\prod_{n\geq 0} \mathrm{Ext}^n_A(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{D\mathrm{diff}(A)}(X, Y)$.

4.1.3. A more general but less explicit treatment of the next two lemmas is given in [Rou, Lemmas 7.12 and 7.13, Proposition 7.14].

Lemma 4.5. *Let V be a finite dimensional vector space over k. Let A be the localisation of the algebra $k[V]$ of polynomial functions on V at the maximal ideal of functions vanishing at* 0.

Then, the functor $(A \otimes k) \otimes_{A^{\mathrm{en}}} - : A^{\mathrm{en}}\text{-mod} \to A\text{-mod}$ *induces a surjective morphism* $\mathrm{Ext}^\bullet_{A^{\mathrm{en}}}(A, A) \to \mathrm{Ext}^\bullet_A(k, k)$. *Furthermore,* $\mathrm{Ext}^\bullet_A(k, k)$ *is generated by* $\mathrm{Ext}^1_A(k, k)$ *as an algebra.*

Proof. Let I be the kernel of the multiplication map $A \otimes A \to A$. Then, the canonical map $\mathrm{Hom}_{A^{\mathrm{en}}}(I, A) \to \mathrm{Ext}^1_{A^{\mathrm{en}}}(A, A)$ induces an isomorphism $\mathrm{Hom}_A(I/I^2, A) \xrightarrow{\sim} \mathrm{Ext}^1_{A^{\mathrm{en}}}(A, A)$ (note that the left and right actions of A on I/I^2 coincide). Furthermore, the canonical map $\Lambda^\bullet_A \mathrm{Hom}_A(I/I^2, A) \to \mathrm{Ext}^\bullet_{A^{\mathrm{en}}}(A, A)$ is an isomorphism.

Let $\mathfrak{m}$ be the maximal ideal of A. We have canonical isomorphisms $V \xrightarrow{\sim} \mathrm{Hom}_A(\mathfrak{m}, k) \to \mathrm{Ext}^1_A(k, k)$. This gives rise to an isomorphism $\Lambda^\bullet_k V \xrightarrow{\sim} \mathrm{Ext}^\bullet_A(k, k)$.

We have a canonical surjective map $\mathrm{Hom}_A(I/I^2, A) \to V$ sending f to

$$V^* \xrightarrow{\xi \mapsto \xi\otimes 1 - 1\otimes\xi} I/I^2 \xrightarrow{f} A \xrightarrow{\mathrm{can}} k.$$

The functor $(A \otimes A/\mathfrak{m}) \otimes_{A^{\mathrm{en}}} - : A^{\mathrm{en}}\text{-mod} \to A\text{-mod}$ gives a morphism $\mathrm{Ext}^\bullet_{A^{\mathrm{en}}}(A, A) \to \mathrm{Ext}^\bullet_A(k, k)$ and we have a commutative diagram

$$\begin{array}{ccc} \Lambda^\bullet_A \mathrm{Hom}_A(I/I^2, A) & \twoheadrightarrow & \Lambda^\bullet_k V \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Ext}^\bullet_{A^{\mathrm{en}}}(A, A) & \longrightarrow & \mathrm{Ext}^\bullet_A(k, k). \end{array}$$

In particular, the canonical map $\mathrm{Ext}^\bullet_{A^{\mathrm{en}}}(A, A) \to \mathrm{Ext}^\bullet_A(k, k)$ is surjective. □

Lemma 4.6. *Let V be a finite dimensional vector space over k. Let A be the localisation of the algebra $k[V]$ of polynomial functions on V at the maximal ideal of functions vanishing at* 0.

Let d be the smallest integer such that $k \in \langle A\rangle_{D\mathrm{diff}(A),d}$. Then, $d = 1 + \dim V$.

Proof. Since k has a projective resolution with $1 + \dim V$ terms, it follows that $k \in \langle A\rangle_{D\mathrm{diff}(A),1+\dim V}$.

Let $m = \dim V$. Let n be an integer with $k \in \langle A\rangle_{D\mathrm{diff}(A),n}$. We apply now Lemma 4.5. Let $\zeta_1, \ldots, \zeta_m \in \mathrm{Ext}^1_{A^{\mathrm{en}}}(A, A)$ whose image in $\mathrm{Ext}^1_A(k, k)$ is a base over k. The image of $\zeta_1 \cdots \zeta_m$ in $\mathrm{Ext}^m_A(k, k)$ is not zero. One sees by induction on r that $\zeta_1 \cdots \zeta_r$ gives the 0 endomorphism of the identity functor of $\langle A\rangle_{D\mathrm{diff}(A),r}$. It follows that $n > m$. □

Proof of Theorem 4.1. Let $A = \Lambda(k^n)$ and $B = k[x_1, \ldots, x_n]$. Consider the derived category of differential graded B-modules. Let $\mathcal{T}$ be its smallest thick subcategory containing B and let $\mathcal{I}$ be the smallest thick subcategory of $\mathcal{T}$ containing k. Koszul duality [Ke, §10.5, Lemma "The 'exterior' case"] provides an equivalence

$$R\,\mathrm{Hom}^\bullet(k, -) : D^b(\Lambda(k^n)) \xrightarrow{\sim} \mathcal{T}.$$

Since the functor sends A to k, it restricts to an equivalence A-perf $\xrightarrow{\sim} \mathcal{I}$ and passing to quotients, it gives an equivalence $A-\overline{\mathrm{mod}} \xrightarrow{\sim} \mathcal{T}/\mathcal{I}$.

Denote by $F : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ the quotient functor. Let $M \in \mathcal{T}$ such that $\mathcal{T}/\mathcal{I} = \langle F(M)\rangle_{\mathcal{T}/\mathcal{I},r+1}$. Up to isomorphism, we can assume M is finitely generated and projective as a B-module. Let $\mathcal{F}$ be the sheaf over $\mathbf{P}^{n-1}$ corresponding to the graded B-module M. The differential on M gives a map $d : \mathcal{F} \to \mathcal{F}(1)$. Let $\mathcal{G} = \ker d(1)/\operatorname{im} d$, a coherent sheaf on $\mathbf{P}^{n-1}$. Let x be a closed point of $\mathbf{P}^{n-1}$ such that $\mathcal{G}_x$ is a projective $\mathcal{O}_x$-module. Then, there is a projective $\mathcal{O}_x$-module R such that $\ker d_x = \operatorname{im} d_x \oplus R$. We have an exact sequence $0 \to R \to \mathcal{F}_x \to \mathcal{F}_x/R \to 0$ of differential $\mathcal{O}_x$-modules. Since $\mathcal{F}_x/R$ is acyclic, it follows that $R \to \mathcal{F}_x$ is an isomorphism in $D\mathrm{diff}(\mathcal{O}_x)$, the derived category of differential $\mathcal{O}_x$-modules. Let $I(x)$ be the prime ideal of B corresponding to the line x of $\mathbf{A}^n$. Note that the differential graded B-module $B/I(x)$ (the differential is 0) is in $\mathcal{T}$. So, $F(B/I(x)) \in \langle F(M)\rangle_{\mathcal{T}/\mathcal{I},r+1}$, hence $k_x \in \langle \mathcal{F}_x\rangle_{D\mathrm{diff}(\mathcal{O}_x),r+1}$, and finally $k_x \in \langle \mathcal{O}_x\rangle_{D\mathrm{diff}(\mathcal{O}_x),r+1}$. By Lemma 4.6, we get $r \geq n-1$. Hence, $\dim A-\overline{\mathrm{mod}} \geq n-1 = \mathrm{ll}(A)-2$. Now, Proposition 3.9 gives the conclusion.

The proof of the inequality $\dim D^b(\Lambda(k^n)\text{-mod}) \geq n$ is similar (and easier). Proposition 3.7 (cf also Remark 3.8) gives the inequality $\dim D^b(\Lambda(k^n)\text{-mod}) \leq n$. □

4.2. Applications.

We assume here that k is a field of characteristic $p > 0$.

Proposition 4.7. *Let G be a finite group and B a block of kG. Let D be a defect group of B. Then,* $\dim B - \overline{\mathrm{mod}} = \dim(kD) - \overline{\mathrm{mod}}$ *and* $\dim D^b(B\text{-mod}) = \dim D^b((kD)\text{-mod})$.

Proof. Recall that a defect group D of B is a (smallest) subgroup such that the identity functor of B-mod is a direct summand of $\mathrm{Ind}_D^G \mathrm{Res}_D^G$. So, every object of $B-\underline{\mathrm{mod}}$ is a direct summand of an object in the image of $B \otimes_{kD} - : kD-\underline{\mathrm{mod}} \to B-\underline{\mathrm{mod}}$. It follows that $\dim B-\underline{\mathrm{mod}} \leq \dim kD-\underline{\mathrm{mod}}$. Also, kD is a direct summand of B as a (kD, kD)-bimodule, hence $\dim kD-\underline{\mathrm{mod}} \leq \dim B-\underline{\mathrm{mod}}$. The case of derived categories is similar. □

Given P a finite p-group and Q a maximal subgroup of P, we denote by $\beta_Q \in H^2(P, \mathbf{Z}/p)$ the class of the exact sequence

$$0 \to \mathbf{Z}/p \to \mathrm{Ind}_Q^P \mathbf{Z}/p \xrightarrow{x-1} \mathrm{Ind}_Q^P \mathbf{Z}/p \to \mathbf{Z}/p \to 0$$

where $x \in P - Q$.

The following proposition gives a recursive bound for the dimension of the stable category.

Proposition 4.8. *Let G be a finite group and B a block of kG. Let D be a defect group of B. Let $D_1, \ldots, D_n$ be a family of maximal subgroups of D such that $\beta_{D_1} \cdots \beta_{D_n} = 0$ (such a family exists and one can assume $n \leq \frac{p+1}{p^2}|D : \Phi(D)|$).*

Then, $\dim B-\underline{\mathrm{mod}} < 2\sum_i(1+\dim kD_i-\underline{\mathrm{mod}})$ *and* $\dim D^b(B\text{-mod}) < 2\sum_i(1 + \dim D^b(kD_i\text{-mod}))$.

Proof. By Proposition 4.7, it is enough to consider the case where $G = D$. Then, [Ca, Lemma 3.9] asserts that there is a kG-module M which has k as a direct summand and has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{2n} = M$ with $M_{2i-1}/M_{2i-2} \simeq \mathrm{Ind}_{D_i}^G \Omega^{t_i} k$ and $M_{2i}/M_{2i-1} \simeq \mathrm{Ind}_{D_i}^G \Omega^{t'_i} k$ for some integers t_i, t'_i. Let $X_i \in D^b(kD_i\text{-mod})$ such that $D^b(kD_i\text{-mod}) = \langle X_i \rangle_{r_i}$. Consider now $L \in D^b(kG\text{-mod})$. We have $L \otimes (M_{2i-1}/M_{2i-2}) \in \langle kG \oplus \mathrm{Ind}_{D_i}^G \Omega^{t_i} X_i \rangle_{r_i}$ and $L \otimes (M_{2i}/M_{2i-1}) \in \langle kG \oplus \mathrm{Ind}_{D_i}^G \Omega^{t'_i} X_i \rangle_{r_i}$. The result about the derived category follows. The proof for the stable category is similar.

The existence of the family is Serre's Theorem on product of Bockstein's, cf *e.g.* [Ben, Theorem 7.4.3]. □

Theorem 4.9. *Let G be a finite group, B a block of kG over a field k of characteristic* 2. *Let D be a defect group of B. Then,* $\mathrm{repdim}\, B \geq 2 + \dim B-\underline{\mathrm{mod}} > r$ *and* $\dim D^b(B\text{-mod}) \geq r$, *where r is the* 2*-rank of D.*

Proof. The first inequality is given by Proposition 3.9. By Proposition 4.7, it suffices to prove the theorem for $G = D$ and $B = kD$. Let P be an elementary abelian 2-subgroup of D with rank the 2-rank of D. Then, $\dim kP-\underline{\mathrm{mod}} \leq \dim kD-\underline{\mathrm{mod}}$, since kD is a direct summand of kP as a (kD, kD)-bimodule. Now, $kP \simeq \Lambda(k^r)$ and the theorem follows from Theorem 4.1. The derived category assertion has a similar proof. □

Let us recall a conjecture of D. Benson:

Conjecture 4.10 (Benson). Let G be a finite group, B a block of kG over a field k of characteristic p with defect group D. Then, $\mathrm{ll}(B) > p\text{-rank}(D)$.

From Theorem 4.9 and Proposition 3.9, we deduce:

Theorem 4.11. *Benson's conjecture 4.10 holds for $p = 2$.*

References

[Au] Auslander, M.: Representation dimension of Artin algebras. Queen Mary College Mathematics Notes, London 1971

[Ben] Benson, D.J.: Representations and Cohomology, vol. II. Cambridge University Press 1991

[BoVdB] Bondal, A., Van den Bergh, M.: Generators and representability of functors in commutative and noncommutative geometry. Moscow Math. J. **3**, 1–36 (2003)

[Ca] Carlson, J.F.: Cohomology and induction from elementary abelian subgroups. Q. J. Math., Oxf. II. Ser. **51**, 169–181 (2000)

[ErHoIySc] Erdmann, K., Holm, T., Iyama, O., Schröer, J.: Radical embeddings and representation dimension. Adv. Math. **185**, 159–177 (2004)

[Ha] Happel, D.: Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. Cambridge University Press 1988

[Ho] Holm, T.: Representation dimension of some tame blocks of finite groups. Algebra Colloq. **10**, 275–284 (2003)

[IgTo] Igusa, K., Todorov, G.: On the finitistic global dimension conjecture for Artin algebras. In: Representations of Algebras and Related Topics, pp. 201–204. Am. Math. Soc. 2005

[Iy] Iyama, O.: Finiteness of representation dimension. Proc. Am. Math. Soc. **131**, 1011–1014 (2003)

[Ke] Keller, B.: Deriving DG categories. Ann. Sci. Éc. Norm. Supér., IV. Sér. **27**, 63–102 (1994)

[KeVo] Keller, B., Vossieck, D.: Sous les catégories dérivées. C. R. Acad. Sci., Paris, Sér. I, Math. **305**, 225–228 (1987)

[Ri] Rickard, J.: Derived categories and stable equivalence. J. Pure Appl. Algebra **61**, 303–317 (1989)

[Rou] Rouquier, R.: Dimensions of triangulated categories. Preprint 2004, http://www.math.jussieu.fr/∼rouquier/preprints

[Xi] Xi, C.: Representation dimension and quasi-hereditary algebras. Adv. Math. **168**, 193–212 (2002)