MICROLOCALIZATION OF RATIONAL CHEREDNIK ALGEBRAS

MASAKI KASHIWARA and RAPHAËL ROUQUIER

Abstract

We construct a microlocalization of the rational Cherednik algebras H of type S_n. This is achieved by a quantization of the Hilbert scheme $\mathrm{Hilb}^n \mathbb{C}^2$ *of n points in $\mathbb{C}^2$. We then prove the equivalence of the category of H-modules and that of modules over its microlocalization under certain conditions on the parameter.*

Contents

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1. Introduction

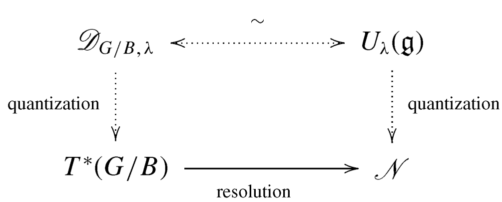
Let us recall that $\mathrm{Hilb}^n \mathbb{C}^2$, the Hilbert scheme of n points in $\mathbb{C}^2$, is a symplectic (in particular, crepant) resolution of $\mathbb{C}^{2n}/S_n = S^n\mathbb{C}^2$. On the other hand, the orbifold $[\mathbb{C}^{2n}/S_n]$ (or the corresponding algebra $\mathbb{C}[\mathbb{C}^{2n}] \rtimes S_n$) is a noncommutative crepant resolution of $\mathbb{C}^{2n}/S_n$. There is an equivalence between derived categories of coherent sheaves on $\mathrm{Hilb}^n \mathbb{C}^2$ and finitely generated modules over $\mathbb{C}[\mathbb{C}^{2n}] \rtimes S_n$ (McKay's correspondence; cf. [12]).

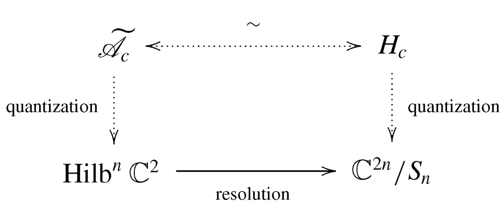
The rational Cherednik algebra H_c associated with S_n is a one-parameter quantization of $\mathbb{C}[\mathbb{C}^{2n}] \rtimes S_n$. We construct a one-parameter quantization $\widetilde{\mathscr{A}_c}$ of $\mathscr{O}_{\mathrm{Hilb}^n \mathbb{C}^2}$ and an equivalence of categories between a certain category of $\widetilde{\mathscr{A}_c}$-modules (good modules with F-action) and the category of finitely generated H_c-modules (under certain conditions on the parameter c). Note that this is an equivalence of abelian categories, while the nonquantized McKay's correspondence is only an equivalence of derived categories.

The quantization $\widetilde{\mathscr{A}_c}$ is a sheaf over $\mathrm{Hilb}^n \mathbb{C}^2$. Locally on an open subset isomorphic to T^*U, it is isomorphic to the sheaf of microdifferential operators $\mathscr{W}$ with a homogenizing parameter $\hbar$.

Note that our construction is an analog of the Beĭlinson-Bernstein localization theorem for universal enveloping algebras upon flag varieties:

nilpotent cone $\mathscr{N}$	$\mathbb{C}^{2n}/S_n$
enveloping algebra quotients $U_\lambda(\mathfrak{g})$	H_c
$T^*(G/B)$	$\mathrm{Hilb}^n \mathbb{C}^2$
$\mathscr{D}_{G/B.\lambda}$	$\widetilde{\mathscr{A}_c}$





Let us mention that our constructions give rise to the spherical subalgebra eH_ce of H_c, and under certain assumptions on c, the two algebras are Morita equivalent. It would be interesting to quantize directly the Procesi bundle to obtain H_c.

Let us now describe some earlier results related to our work. An important achievement of Etingof and Ginzburg [6] and of Gan and Ginzburg [7] is a construction of a deformation of the Harish-Chandra morphism for $\mathrm{GL}_n(\mathbb{C})$, providing a construction of the spherical subalgebra eH_ce of H_c as a quantum Hamiltonian reduction. This provides a quantization of the Calogero-Moser space, which is itself obtained by classical Hamiltonian reduction (see Kazhdan, Kostant, and Sternberg [21]).

Gordon and Stafford [8], [9] constructed a one-parameter family of graded ($\mathbb{Z}$)-algebras $\mathscr{B}_c$ that quantize (a graded ($\mathbb{Z}$)-algebra Morita equivalent to) the homogeneous coordinate ring of $\mathrm{Hilb}^n \mathbb{C}^2$.

In positive characteristic, Bezrukavnikov, Finkelberg, and Ginzburg [4] constructed a sheaf of Azumaya algebras on the Hilbert scheme whose algebra of global sections is isomorphic to H_c and obtained an equivalence of derived categories between modules over that Azumaya algebra and representations of H_c.

Let us explain the type of sheaf of algebras used to quantize $\mathrm{Hilb}^n \mathbb{C}^2$. On a complex contact manifold, Kashiwara [17] constructed the stack $\mathscr{E}$ of microdifferential operators. Locally, a model for a contact manifold is the projectivized cotangent bundle P^*X, and the stack $\mathscr{E}$ comes from the sheaf $\mathscr{E}_X$ of microdifferential operators of Sato, Kawai, and Kashiwara.

On a symplectic variety, Kontsevich [22] and Polesello and Schapira [24] defined a stack $\mathscr{W}$ of microdifferential operators with a homogenizing parameter $\hbar$ (making all objects modules over $\mathbb{C}((\hbar))$). Locally, a model is T^*X, and $\mathscr{W}$ comes from microdifferential operators on $P^*(X \times \mathbb{C})$ which do not depend on the extra variable.

For applications to representation theory, these constructions are unsatisfactory:

- the first construction forgets about the zero section; and
- the second construction gives "too-large" objects (defined over $\mathbb{C}((\hbar))$ instead of $\mathbb{C}$).

To overcome these difficulties, we consider here symplectic manifolds X with a $\mathbb{C}^\times$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight. We consider the case where the stack $\mathscr{W}$ comes from a sheaf of algebras together with a compatible action of $\mathbb{C}^\times$, and we study the corresponding structure, a W-*algebra with* F-*action*. The category of its modules is defined over $\mathbb{C}$, as the F-action induces a $\mathbb{C}^\times$-action on $\mathbb{C}((\hbar))$ whose invariant field is $\mathbb{C}$.

Let us now describe the structure of the article.

In the first part of this article, §2, we study a general setting for the quantization of symplectic manifolds X with a $\mathbb{C}^\times$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight. We first review the theory of W-algebras on symplectic manifolds in §2.2. In §2.3, we introduce the notion of W-algebra with F-action. An important point of this construction is that the category of $\mathscr{W}$-modules with F-action on a cotangent bundle (for the canonical structure) is equivalent to the category of modules over the sheaf $\mathscr{D}$ of differential operators. We adapt in §2.4 the study of equivariance and its twisted version for the action of a complex Lie group, and we explain how to construct W-algebras with F-action by symplectic reduction in §2.5. Finally, in §2.6, we provide sufficient conditions to ensure $\mathscr{W}$-affinity (a counterpart of Beĭlinson-Bernstein's result for $\mathscr{D}$-modules).

We devote §3 to the construction of $\mathscr{D}$-modules with an action of the rational Cherednik algebra H_c of type A_{n-1} or of its spherical subalgebra eH_ce. This is related to the constructions of [4] and [7]. Let $V = \mathbb{C}^n$, and let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. We construct in §3.2 a quasi-coherent $\mathscr{D}_{\mathfrak{g}\times V}$-module $\mathscr{M}_c$ together with an action of H_c, building on the explicit description of the $\mathscr{D}$-module arising in Springer's correspondence given in [14]. We construct a coherent $\mathscr{D}_{\mathfrak{g}\times V}$-submodule $\mathscr{L}_c$ of $\mathscr{M}_c$ which is stable under the action of the spherical subalgebra of H_c, and we construct a shift operator in §3.3. This is achieved by reduction to rank 2.

In §4, we construct a W-algebra with F-action on $\operatorname{Hilb}^n \mathbb{C}^2$ by symplectic reduction from the previous constructions. After recalling some properties of $\operatorname{Hilb}^n \mathbb{C}^2$ in §4.1, we construct in §4.2 a W-algebra $\widetilde{\mathscr{A}_c}$ on $\operatorname{Hilb}^n \mathbb{C}^2$ by symplectic reduction of $\mathscr{L}_c$ for the action of $\mathrm{GL}_n(\mathbb{C})$. In §4.3, we present our main results: $\widetilde{\mathscr{A}_c}$-affinity of $\operatorname{Hilb}^n \mathbb{C}^2$, an isomorphism between global sections of $\widetilde{\mathscr{A}_c}$ and the spherical algebra, and an equivalence between the category of good $\widetilde{\mathscr{A}_c}$-modules with F-action and the one of finitely generated modules over the spherical algebra. We also describe similar results for H_c. So, we have obtained a microlocalization of the rational Cherednik algebras: we have constructed a W-algebra with F-action over the Hilbert scheme whose algebra of global sections is isomorphic to H_c and whose modules are equivalent to representations of H_c. Those results are obtained under certain assumptions on c. We explain in §4.4 how to view sections of our W-algebras over open subsets of the Hilbert schemes as appropriate fractions in the Cherednik algebra. Finally, we describe explicitly the constructions for $n = 2$ in §4.5.

2. F-actions on W-algebras

2.1. Notation

By a manifold M, we mean a complex manifold, equipped with the classical topology, and $\mathscr{O}_M$ is the sheaf of holomorphic functions. We denote by $\mathscr{D}_M$ the sheaf of differential operators with holomorphic coefficients and by $\mathscr{E}_M$ the sheaf of formal microdifferential operators on the cotangent bundle T^*M.

We denote by $\mathbb{G}_{\mathrm{m}}$ the multiplicative group $\mathbb{C}^\times$.

Given a ring A, we denote by $\mathrm{Mod}_{\mathrm{coh}}(A)$ the category of coherent left A-modules.

2.2. W-algebras

We review some results on W-algebras. We refer the reader to [24] (where the convergent version is studied, while we use the simpler formal version).

2.2.1

Let $\mathbf{k} = \mathbb{C}((\hbar))$ be the field of formal Laurent series in an indeterminate $\hbar$, and let $\mathbf{k}(0) = \mathbb{C}[[\hbar]]$. Given $m \in \mathbb{Z}$, we define $\mathscr{W}_{T^*\mathbb{C}^n}(m)$ as the sheaf of formal series $\sum_{k\geqslant -m} \hbar^k a_k$ ($a_k \in \mathscr{O}_{T^*\mathbb{C}^n}$) on the cotangent bundle $T^*\mathbb{C}^n$ of $\mathbb{C}^n$, and we set $\mathscr{W}_{T^*\mathbb{C}^n} = \bigcup_m \mathscr{W}_{T^*\mathbb{C}^n}(m)$. Then $\mathscr{W}_{T^*\mathbb{C}^n}$ has a structure of $\mathbf{k}$-algebra given by

$$a \circ b = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} \hbar^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha a \cdot \partial_x^\alpha b.$$

We have a ring homomorphism $\mathscr{D}_{\mathbb{C}^n}(\mathbb{C}^n) \to \mathscr{W}_{T^*\mathbb{C}^n}(T^*\mathbb{C}^n)$ given by $x_i \mapsto x_i$, $\frac{\partial}{\partial x_i} \mapsto \hbar^{-1}\xi_i$.

2.2.2

Let X be a complex symplectic manifold with symplectic form ω_X. We denote by X^{opp} the symplectic manifold X with symplectic form $-\omega_X$.

A *W-algebra* is a $\mathbf{k}$-algebra $\mathscr{W}$ on X such that for any point $x \in X$, there are an open neighbourhood U of x, a symplectic map $f\colon U \to T^*\mathbb{C}^n$, and a $\mathbf{k}$-algebra isomorphism $g\colon \mathscr{W}|_U \xrightarrow{\sim} f^{-1}\mathscr{W}_{T^*\mathbb{C}^n}$.

A W-algebra $\mathscr{W}$ satisfies the following properties.

(i) The algebra $\mathscr{W}$ is a coherent and Noetherian algebra.

(ii) $\mathscr{W}$ contains a canonical subalgebra $\mathscr{W}(0)$ that is locally isomorphic to $\mathscr{W}_{T^*\mathbb{C}^n}(0)$ (via the maps g). We set $\mathscr{W}(m) = \hbar^{-m}\mathscr{W}(0)$.

(iii) We have a canonical $\mathbb{C}$-algebra isomorphism $\mathscr{W}(0)/\mathscr{W}(-1) \xrightarrow{\sim} \mathscr{O}_X$ (coming from the canonical isomorphism via the maps g). The corresponding morphism $\sigma_m\colon \mathscr{W}(m) \to \hbar^{-m}\mathscr{O}_X$ is called the *symbol map*.

(iv) We have

$$\sigma_0(\hbar^{-1}[a, b]) = \{\sigma_0(a), \sigma_0(b)\}$$

for any $a, b \in \mathscr{W}(0)$. Here, $\{\cdot, \cdot\}$ is the Poisson bracket.

(v) The canonical map $\mathscr{W}(0) \to \varprojlim_{m\to\infty}\mathscr{W}(0)/\mathscr{W}(-m)$ is an isomorphism.

(vi) A section a of $\mathscr{W}(0)$ is invertible in $\mathscr{W}(0)$ if and only if $\sigma_0(a)$ is invertible in $\mathscr{O}_X$.

(vii) Given ϕ a $\mathbf{k}$-algebra automorphism of $\mathscr{W}$, we can find locally an invertible section a of $\mathscr{W}(0)$ so that $\phi = \mathrm{Ad}(a)$. Moreover, a is unique up to a scalar multiple. In other words, we have canonical isomorphisms

$$\begin{array}{ccc} \mathscr{W}(0)^\times/\mathbf{k}(0)^\times & \xrightarrow[\mathrm{Ad}]{\sim} & \mathrm{Aut}\big(\mathscr{W}(0)\big) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathscr{W}^\times/\mathbf{k}^\times & \xrightarrow[\mathrm{Ad}]{\sim} & \mathrm{Aut}(\mathscr{W}) \end{array}$$

(viii) Let v be a $\mathbf{k}$-linear filtration-preserving derivation of $\mathscr{W}$. Then there exists locally a section a of $\mathscr{W}(1)$ such that $v = \mathrm{ad}(a)$. Moreover, a is unique up to a scalar. In other words, we have an isomorphism

$$\mathscr{W}(1)/\hbar^{-1}\mathbf{k}(0) \xrightarrow[\mathrm{ad}]{\sim} \mathrm{Der}_{\mathrm{filtered}}(\mathscr{W}).$$

(ix) If $\mathscr{W}$ is a W-algebra, then its opposite ring $\mathscr{W}^{\mathrm{opp}}$ is a W-algebra on X^{opp}.
Conjecturally, (iii) – (v) characterize $\mathscr{W}(0)$.

Note that two W-algebras on X are locally isomorphic.

2.2.3

Assume that there exist $a_i, b_i \in \mathscr{W}(0)$ $(i = 1, \dots, n)$ such that $[a_i, a_j] = [b_i, b_j] = 0$ and $[b_i, a_j] = \hbar\delta_{ij}$. They induce a symplectic map

$$f = \big(\sigma_0(a_1), \dots, \sigma_0(a_n); \sigma_0(b_1), \dots, \sigma_0(b_n)\big) : X \to T^*\mathbb{C}^n.$$

Then there exists a unique isomorphism

$$\mathscr{W} \xrightarrow{\sim} f^{-1}\mathscr{W}_{T^*\mathbb{C}^n}, \qquad a_i \mapsto x_i, \qquad b_i \mapsto \xi_i.$$

We call $(a_1, \dots, a_n; b_1, \dots, b_n)$ *quantized symplectic coordinates* of $\mathscr{W}$.

Let M be a complex manifold M, and let $\pi_M : T^*M \to M$ be the projection. We can associate canonically a W-algebra $\mathscr{W}_{T^*M}$ with a morphism $\pi_M^{-1}\mathscr{D}_M \to \mathscr{W}_{T^*M}$ so that

$$\begin{array}{ccc} \pi_M^{-1}F_m(\mathscr{D}_M) & \longrightarrow & \mathscr{W}_{T^*M}(m) \\ \downarrow{\scriptstyle\sigma_m} & & \downarrow{\scriptstyle\sigma_m} \\ \mathscr{O}_{T^*M} & \xrightarrow[\hbar^{-m}]{} & \hbar^{-m}\mathscr{O}_{T^*M} \end{array}$$

commutes. Here, $F(\mathscr{D}_M)$ is the order filtration of $\mathscr{D}_M$. Note that $\pi_M^{-1}\mathscr{D}_M \to \mathscr{W}_{T^*M}$ decomposes into $\pi_M^{-1}\mathscr{D}_M \to \mathscr{E}_M \to \mathscr{W}_{T^*M}$. The ring $\mathscr{W}_{T^*M}$ is flat over $\pi_M^{-1}\mathscr{D}_M$ and faithfully flat over $\mathscr{E}_M$. In particular, for a coherent $\mathscr{D}_M$-module $\mathscr{M}$, the characteristic variety $\mathrm{Ch}(\mathscr{M})$ coincides with $\mathrm{Supp}(\mathscr{W}_{T^*M} \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M})$.

Let X and Y be two symplectic manifolds. The product $X \times Y$ is also a symplectic manifold. For a W-algebra $\mathscr{W}_X$ on X and a W-algebra $\mathscr{W}_Y$ on Y, there is a W-algebra $\mathscr{W}_X \boxtimes \mathscr{W}_Y$ on $X \times Y$. Letting both $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the projections, $\mathscr{W}_X \boxtimes \mathscr{W}_Y$ contains $p_1^{-1}\mathscr{W}_X \otimes_{\mathbf{k}} p_2^{-1}\mathscr{W}_Y$ as a $\mathbf{k}$-subalgebra, and it is faithfully flat over it.

For a $\mathscr{W}$-module $\mathscr{M}$, a $\mathscr{W}(0)$-*lattice* is a coherent $\mathscr{W}(0)$-submodule $\mathscr{N}$ of $\mathscr{M}$ such that the canonical map $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{N} \to \mathscr{M}$ is an isomorphism.

We say that a $\mathscr{W}$-module $\mathscr{M}$ is *good* if, for any relatively compact open subset U of X, there exists a coherent $\mathscr{W}(0)|_U$-lattice of $\mathscr{M}|_U$. The full subcategory of good $\mathscr{W}$-modules is an abelian subcategory of the category of $\mathscr{W}$-modules.

The following fact is used in this article (see [20, Theorem 1.2.2], where the convergent version is proved).

LEMMA 2.1
Let r be an integer, and let $\mathscr{M}$ be a coherent $\mathscr{W}$-module so that $\mathscr{E}xt^j_{\mathscr{W}}(\mathscr{M}, \mathscr{W}) = 0$ for any $j > r$. Then $\mathscr{H}^j_S(\mathscr{M}) = 0$ for any closed analytic subset S and any $j <$ codim $S - r$.

Let $\bar{\mathbf{k}} := \bigcup_{n>0} \mathbb{C}((\hbar^{1/n}))$ be an algebraic closure of $\mathbf{k}$. We sometimes need to replace $\mathscr{W}$ with $\mathbf{k}' \otimes_{\mathbf{k}} \mathscr{W}$ for some field $\mathbf{k}'$ with $\mathbf{k} \subset \mathbf{k}' \subset \bar{\mathbf{k}}$.

2.3. F-actions

2.3.1

Let X be a symplectic manifold. Consider an action of $\mathbb{G}_\mathrm{m}$ on X, viewed as a manifold: $\mathbb{C}^\times \ni t \mapsto T_t \in \mathrm{Aut}(X)$. We assume that $\mathbb{G}_\mathrm{m}$ stabilizes the line $\mathbb{C}\omega_X \subset H^0(X, \Omega_X^2)$ with a positive weight m (i.e., $T_t^*\omega_X = t^m\omega_X$ for all $t \in \mathbb{C}^\times$).

We denote by v the vector field given by the $\mathbb{G}_\mathrm{m}$-action: $v(a)(x) = \frac{d}{dt}a(T_t(x))\big|_{t=1}$. The Poisson bracket $\{\bullet, \bullet\}$ is homogeneous of degree $-m$:

$$T_t^*\{a, b\} = t^{-m}\{T_t^*a, T_t^*b\}$$

and

$$v\{a, b\} = \big\{v(a), b\big\} + \big\{a, v(b)\big\} - m\{a, b\} \quad \text{for } a, b \in \mathscr{O}_X.$$

Let $\mathscr{W}$ be a W-algebra.

Definition 2.2

An *F-action with exponent m* on $\mathscr{W}$ is an action of $\mathbb{G}_\mathrm{m}$ on the $\mathbb{C}$-algebra $\mathscr{W}$, $\mathscr{F}_t\colon T_t^{-1}\mathscr{W} \xrightarrow{\sim} \mathscr{W}$ for $t \in \mathbb{C}^\times$, so that $\mathscr{F}_t(\hbar) = t^m\hbar$ and $\mathscr{F}_t(a)$ depends holomorphically on t for any $a \in \mathscr{W}$.

Let us fix an F-action with exponent m on $\mathscr{W}$. The $\mathbb{G}_m$-action induces an order-preserving derivation v_F of $\mathscr{W}$ given by $v_F(a) = \frac{d}{dt}\mathscr{F}_t(a)\big|_{t=1}$. It satisfies the following properties:

$$\begin{aligned} &v_F(\hbar) = m\hbar, \\ &\sigma_0\big(v_F(a)\big) = v\big(\sigma_0(a)\big) \quad \text{for } a \in \mathscr{W}(0). \end{aligned} \tag{2.1}$$

Remark 2.3

Here, F stands for *Frobenius.* Note that v_F determines the F-action on $\mathscr{W}$. However, for a given v_F satisfying (2.1), we cannot always find an F-action on $\mathscr{W}$.

The action of $\mathbb{G}_m$ on $\mathscr{W}$ extends to an action on $\mathscr{W}[\hbar^{1/m}] = \mathbf{k}(\hbar^{1/m}) \otimes_{\mathbf{k}} \mathscr{W}$ given by $\mathscr{F}_t(\hbar^{1/m}) = t\,\hbar^{1/m}$.

Definition 2.4

A $\mathscr{W}[\hbar^{1/m}]$-module with an F-action (or simply a $(\mathscr{W}[\hbar^{1/m}], \mathscr{F})$-module) is a $\mathbb{G}_m$-equivariant $\mathscr{W}[\hbar^{1/m}]$-module: we have isomorphisms $\mathscr{F}_t \colon T_t^{-1}\mathscr{M} \xrightarrow{\sim} \mathscr{M}$ for $t \in \mathbb{C}^\times$, and we assume that

(a) $\mathscr{F}_t(u)$ depends holomorphically on t for any $u \in \mathscr{M}$ (i.e., there exist locally finitely many u_i such that $\mathscr{F}_t(u) = \sum_i a_i(t)u_i$, where $a_i(t) \in \mathscr{W}[\hbar^{1/m}]$ depends holomorphically on t);

(b) $\mathscr{F}_t(au) = \mathscr{F}_t(a)\mathscr{F}_t(u)$ for $a \in \mathscr{W}[\hbar^{1/m}]$, $u \in \mathscr{M}$; and

(c) $\mathscr{F}_t \circ \mathscr{F}_{t'} = \mathscr{F}_{tt'}$ for $t, t' \in \mathbb{C}^\times$.

We denote by $\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}])$ the category of $(\mathscr{W}[\hbar^{1/m}], \mathscr{F})$-modules: morphisms are morphisms of $\mathscr{W}[\hbar^{1/m}]$-modules compatible with the $\mathbb{G}_m$-action. We denote by $\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}[\hbar^{1/m}])$ its full subcategory of good $(\mathscr{W}[\hbar^{1/m}], \mathscr{F})$-modules. These are $\mathbb{C}$-linear abelian categories. Note that if there is a relatively compact open subset U of X such that $\mathbb{C}^\times \cdot U = X$, then a good $(\mathscr{W}[\hbar^{1/m}], \mathscr{F})$-module admits a coherent $(\mathscr{W}(0)[\hbar^{1/m}], \mathscr{F})$-lattice.

Assume that $X = \{\mathrm{pt}\}$ so that $\mathscr{W} = \mathbf{k}$. We have an equivalence $\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}]) \xrightarrow{\sim} \mathrm{Mod}(\mathbb{C})$, $\mathscr{M} \mapsto \mathscr{M}^{\mathbb{G}_m}$, with quasi-inverse given by $V \mapsto \mathbb{C}((\hbar^{1/m})) \otimes_{\mathbb{C}} V$.

Remark 2.5

Kaledin [15] as well as Kontsevich have also studied quantization for a symplectic variety with a $\mathbb{G}_m$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight.

2.3.2

Let $\mathscr{W}$ be a W-algebra with an F-action with exponent m. Let n be a positive integer, and consider the restriction of the F-action via $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^n$: we have a new action given by $T'_t = T_{t^n}$ and $\mathscr{F}'_t = \mathscr{F}_{t^n}$. This defines an F-action on $\mathscr{W}$ with exponent mn. Then we have quasi-inverse equivalences of categories

$$\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}]) \xleftrightarrow{\sim} \mathrm{Mod}_F(\mathscr{W}[\hbar^{1/nm}]),$$
$$\mathscr{M} \mapsto \mathscr{W}[\hbar^{1/nm}] \otimes_{\mathscr{W}[\hbar^{1/m}]} \mathscr{M},$$
$$\big\{s \in \mathscr{N}\ ;\ \mathscr{F}'_\zeta(s) = s \text{ for any } \zeta \in \mathbb{C} \text{ with } \zeta^n = 1\big\} \leftarrow\!\shortmid \mathscr{N}.$$

Remark 2.6

The equivalence above shows that the category depends only on the one-parameter subgroup of $\mathrm{Aut}(X, \mathscr{W})$ given by the $\mathbb{G}_\mathrm{m}$-action.

Let $\hat{\mathbb{G}}_\mathrm{m} = \varprojlim_n \mathbb{G}_\mathrm{m}$, where the limit is taken over maps $f_{n,n'} : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m},\ t \mapsto t^{n/n'}$ for positive integers n, n' with $n'|n$. This is a pro-algebraic group (some sort of universal covering group of $\mathbb{G}_\mathrm{m}$). In terms of functions, we have $\hat{\mathbb{G}}_\mathrm{m} = \mathrm{Spec}\big(\bigoplus_{a\in\mathbb{Q}} \mathbb{C}t^a\big)$ with multiplication coming from the coproduct $t^a \mapsto t^a \otimes t^a$. Instead of considering $\mathbb{G}_\mathrm{m}$-actions as above, we could consider $\hat{\mathbb{G}}_\mathrm{m}$-actions on X so that $T_t^*\omega_X = t\omega_X$. Although theoretically more satisfactory, this more complicated formulation is not used in the present article.

2.3.3

Let us now give two examples.

Let M be a manifold, let $X = T^*M$, and let $\mathscr{W} = \mathscr{W}_{T^*M}$. We consider the canonical $\mathbb{G}_\mathrm{m}$-action given by $T_t(x, \xi) = (x, t\xi)$. There is a unique F-action with exponent 1 on $\mathscr{W}$ with $\mathscr{F}|_{\mathscr{D}_M} = \mathrm{id}$. Then, for any $\mathbb{G}_\mathrm{m}$-invariant open subset U of X, we have an equivalence

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}|_U) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{good}}(\mathscr{E}_M|_U), \qquad \mathscr{M} \mapsto \mathscr{M}^{\mathbb{G}_\mathrm{m}}.$$

In particular, we have an equivalence

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_M).$$

Let $X = T^*\mathbb{C}^n$, and let $\mathscr{W} = \mathscr{W}_{T^*\mathbb{C}^n}$. Fix $m > 1$, and fix $l_1, \ldots, l_n \in \{1, \ldots, m-1\}$. We define a $\mathbb{G}_\mathrm{m}$-action by $T_t((x_i), (\xi_i)) = ((t^{l_i}x_i), (t^{m-l_i}\xi_i))$. Then $T_t^*(\omega_X) = t^m\omega_X$. We define an F-action on $\mathscr{W}$ with exponent m by $\mathscr{F}_t(x_i) = t^{l_i}x_i$, $\mathscr{F}_t(\partial_i) = t^{-l_i}\partial_i$, and $\mathscr{F}_t(\hbar) = t^m\hbar$. (Note that the relation $[\partial_i, x_i] = 1$ is preserved by $\mathscr{F}_t$.) Then

$$\mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}])}(\mathscr{W}[\hbar^{1/m}])^{\mathrm{opp}} = \mathbb{C}[\hbar^{-l_i/m}x_i, \hbar^{l_i/m}\partial_i; i = 1, \ldots, n] \subset \mathscr{W}[\hbar^{1/m}],$$

which is isomorphic to $\mathscr{D}(\mathbb{C}^n)$. Moreover, $\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}[\hbar^{1/m}])$ is equivalent to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}(\mathbb{C}^n))$ (see Theorem 2.10).

2.4. Equivariance

We discuss G-equivariance of $\mathscr{W}$ by adapting [19] and [16], where the $\mathscr{D}$-module version is studied.

2.4.1

Let G be a complex Lie group acting on a symplectic manifold X. Given $g \in G$, let T_g be the corresponding symplectic automorphism of X. Let $\mathfrak{g}$ be the Lie algebra of G, and assume that a moment map $\mu_X \colon X \to \mathfrak{g}^*$ is given.

A W-algebra with G-action is a W-algebra with an action of G: we have $\mathbf{k}$-algebra isomorphisms $\rho_g \colon \mathscr{W} \xrightarrow{\sim} T_g^{-1}\mathscr{W}$ for $g \in G$ so that for any $a \in \mathscr{W}$, $\rho_g(a)$ depends holomorphically on $g \in G$. Moreover, we assume that there is a *quantized moment map* $\mu_{\mathscr{W}} \colon \mathfrak{g} \to \mathscr{W}(1)$, so that

$$[\mu_{\mathscr{W}}(A), a] = \frac{d}{dt}\rho_{\exp(tA)}(a)\Big|_{t=0},$$

$$\sigma_0\big(\hbar\mu_{\mathscr{W}}(A)\big) = A \circ \mu_X,$$

$$\mu_{\mathscr{W}}\big(\mathrm{Ad}(g)A\big) = \rho_g\big(\mu_{\mathscr{W}}(A)\big),$$

for any $A \in \mathfrak{g}$ and $a \in \mathscr{W}$. Note that $\mu_{\mathscr{W}}$ is a Lie algebra homomorphism.

2.4.2

A quasi-G-equivariant $\mathscr{W}$-module is a $\mathscr{W}$-module $\mathscr{M}$ with an action of G:

$$\rho_g \colon \mathscr{M} \xrightarrow{\sim} T_g^{-1}\mathscr{M}$$

depending holomorphically on $g \in G$ and such that $\rho_g(au) = \rho_g(a)\rho_g(u)$ for $a \in \mathscr{W}$ and $u \in \mathscr{M}$. Then we have a Lie algebra homomorphism $\alpha \colon \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(\mathscr{M})$ given by $\alpha(A)(u) = \frac{d}{dt}\rho_{\exp(tA)}u\big|_{t=0}$ for $A \in \mathfrak{g}$ and $u \in \mathscr{M}$. It satisfies

$$\alpha(A)(au) = [\mu_{\mathscr{W}}(A), a]u + a \cdot \alpha(A)(u).$$

It follows that we have a Lie algebra homomorphism

$$\gamma_{\mathscr{M}} \colon \mathfrak{g} \to \mathrm{End}_{\mathscr{W}}(\mathscr{M}), \qquad A \mapsto \alpha(A) - \mu_{\mathscr{W}}(A). \tag{2.2}$$

The $\mathscr{W}$-module $\mathscr{W}$ is regarded as a quasi-G-equivariant $\mathscr{W}$-module. We have $\alpha(A) = \mathrm{ad}(\mu_{\mathscr{W}}(A))$ and $\gamma_{\mathscr{W}}(A)(a) = -a\mu_{\mathscr{W}}(A)$ ($a \in \mathscr{W}$, $A \in \mathfrak{g}$). Given a G-module V and a quasi-G-equivariant $\mathscr{W}$-module $\mathscr{M}$, the tensor product $\mathscr{M} \otimes V$ has a natural structure of a quasi-G-equivariant $\mathscr{W}$-module. The corresponding γ is given by

$$\gamma_{\mathscr{M}\otimes V}(A)(u\otimes v)=\gamma_{\mathscr{M}}(A)u\otimes v+u\otimes Av \quad \text{for } u\in\mathscr{M},\, v\in V, \text{ and } A\in\mathfrak{g}.$$

Let $\lambda\in(\mathfrak{g}^*)^G$. If $\gamma_{\mathscr{M}}$ coincides with the composition $\mathfrak{g}\xrightarrow{\lambda}\mathbb{C}\xrightarrow{z\mapsto z\cdot \mathrm{Id}_{\mathscr{M}}}\mathrm{End}_{\mathscr{W}}(\mathscr{M})$, we say that $\mathscr{M}$ is a twisted G-equivariant $\mathscr{W}$-module with twist λ. For such a coherent module $\mathscr{M}$, we have $\mathrm{Supp}(\mathscr{M})\subset\mu_X^{-1}(0)$.

We denote by $\mathrm{Mod}(\mathscr{W},G)$ the category of quasi-G-equivariant $\mathscr{W}$-modules, and we denote by $\mathrm{Mod}^G_\lambda(\mathscr{W})$ its full subcategory of twisted G-equivariant $\mathscr{W}$-modules with twist λ. We denote by $\mathrm{Mod}^{G,\,\mathrm{good}}_\lambda(\mathscr{W})$ the category of good twisted G-equivariant $\mathscr{W}$-modules with twist λ.

The embedding $\mathrm{Mod}^G_\lambda(\mathscr{W})\to\mathrm{Mod}(\mathscr{W},G)$ has a left adjoint

$$\begin{aligned}\Phi_\lambda\colon\ &\mathrm{Mod}(\mathscr{W},G)\to\mathrm{Mod}^G_\lambda(\mathscr{W}),\\ \Phi_\lambda(\mathscr{M})&=\mathscr{M}/\Big(\sum_{A\in\mathfrak{g}}(\gamma_{\mathscr{M}}(A)-\lambda(A))\mathscr{M}\Big).\end{aligned}\tag{2.3}$$

Let V be a one-dimensional G-module, and let $\chi\in(\mathfrak{g}^*)^G$ be its infinitesimal character. Then we have an equivalence

$$\mathrm{Mod}^G_\lambda(\mathscr{W})\xrightarrow{\sim}\mathrm{Mod}^G_{\lambda+\chi}(\mathscr{W}),\qquad \mathscr{M}\mapsto\mathscr{M}\otimes V.\tag{2.4}$$

Let $\mathscr{W}$ be a W-algebra with an F-action with exponent m. A G-action on $(\mathscr{W},\mathscr{F})$ is a G-action on $\mathscr{W}$ such that T_t and T_g commute, $\mathscr{F}_t$ and $\rho(g)$ commute, and $\mu_{\mathscr{W}}(A)$ is $\mathscr{F}_t$-invariant, for $t\in\mathbb{C}^\times$, $g\in G$, and $A\in\mathfrak{g}$.

We define similarly the notion of twisted G-equivariant $(\mathscr{W}[\hbar^{1/m}],\mathscr{F})$-modules. We denote by $\mathrm{Mod}^{G,\,\mathrm{good}}_{F,\,\lambda}(\mathscr{W}[\hbar^{1/m}])$ the category of good twisted G-equivariant $(\mathscr{W}[\hbar^{1/m}],\mathscr{F})$-modules with twist $\lambda\in(\mathfrak{g}^*)^G$.

2.5. Symplectic reduction

Let X be a symplectic manifold with a symplectic action of G and a moment map $\mu_X\colon X\to\mathfrak{g}^*$. *Assume that G acts properly and freely on X* (i.e., the map $G\times X\to X\times X$ defined by $(g,x)\mapsto(gx,x)$ is a closed embedding). Then $\mu_X^{-1}(0)$ is an involutive submanifold. Let $Z=\mu_X^{-1}(0)/G$, and let $p\colon\mu_X^{-1}(0)\to Z$ be the projection. Then Z carries a natural symplectic structure such that p preserves the symplectic form (i.e., denoting by ω_Z the symplectic form of Z, we have $p^*\omega_Z=\omega_X|_{\mu_X^{-1}(0)}$). The local form of X is given by the following lemma.

LEMMA 2.7 (see [10, §41])

Locally on Z, the manifold X is isomorphic to $T^*G\times Z$. More precisely, for any point $x\in\mu_X^{-1}(0)$, there exist a G-invariant open neighbourhood U of x in X and a

G-equivariant open symplectic embedding $U \to T^*G \times T^*\mathbb{C}^n$ compatible with the moment maps.

Let $\mathscr{W}$ be a W-algebra on X with a G-action. Let $\lambda \in (\mathfrak{g}^*)^G$. Set

$$\mathscr{L}_\lambda := \Phi_\lambda(\mathscr{W}) = \mathscr{W} / \sum_{A \in \mathfrak{g}} \mathscr{W}\big(\mu_{\mathscr{W}}(A) + \lambda(A)\big).$$

Then $\mathscr{L}_\lambda$ is a coherent twisted G-equivariant $\mathscr{W}$-module with twist λ. The support of $\mathscr{L}_\lambda$ coincides with $\mu_X^{-1}(0)$. Let $\mathscr{L}_\lambda(0)$ be the $\mathscr{W}(0)$-lattice $\mathscr{W}(0)/\sum_{A\in\mathfrak{g}}\mathscr{W}(-1)(\mu_{\mathscr{W}}(A) + \lambda(A))$ of $\mathscr{L}_\lambda$.

Let $\mathscr{W}_Z = \big((p_* \mathscr{E}nd_{\mathscr{W}}(\mathscr{L}_\lambda))^G\big)^{\mathrm{opp}}$, a sheaf of $\mathbf{k}$-algebras on Z.

PROPOSITION 2.8

(i) *$\mathscr{W}_Z$ is a W-algebra on Z, and $\mathscr{W}_Z(0) \simeq \big((p_* \mathscr{E}nd_{\mathscr{W}(0)}(\mathscr{L}_\lambda(0)))^G\big)^{\mathrm{opp}}$.*

(ii) *We have quasi-inverse equivalences of categories*

$$\mathrm{Mod}^{\mathrm{good}}(\mathscr{W}_Z) \xleftrightarrow{\sim} \mathrm{Mod}_\lambda^{G,\,\mathrm{good}}(\mathscr{W}),$$
$$\mathscr{N} \mapsto \mathscr{L}_\lambda \otimes_{p^{-1}\mathscr{W}_Z} p^{-1}\mathscr{N},$$
$$\big(p_*\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M})\big)^G \leftarrow\!\shortmid \mathscr{M}.$$

(iii) *Let V be a one-dimensional representation with infinitesimal character χ. Then $\mathscr{N}_{\lambda,\chi}(0) := \big(p_*\mathscr{H}om_{\mathscr{W}(0)}(\mathscr{L}_\lambda(0), \mathscr{L}_{\lambda-\chi}(0) \otimes V)\big)^G$ is a $\mathscr{W}_Z(0)$-lattice of a coherent $\mathscr{W}_Z$-module $\mathscr{N}_{\lambda,\chi} := (p_*\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{L}_{\lambda-\chi} \otimes V))^G$ and $\mathscr{N}_{\lambda,\chi}(0)/\hbar\mathscr{N}_{\lambda,\chi}(0)$ is isomorphic to $(p_*(\mathscr{O}_{\mu_X^{-1}(0)} \otimes V))^G$, the line bundle on Z associated with V.*

(iv) *Assume that $\mathscr{W}$ has an F-action with exponent m compatible with the G-action. Then $\mathscr{W}_Z$ has a natural F-action with exponent m, and we have an equivalence of categories:*

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}_Z[\hbar^{1/m}]) \simeq \mathrm{Mod}_{F,\lambda}^{G,\mathrm{good}}(\mathscr{W}[\hbar^{1/m}]).$$

Note that $\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}) \simeq p^{-1}\big((p_*\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}))^G\big)$. Hence, if G is connected, we have $p_*\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}) \simeq (p_*\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}))^G$.

2.6. $\mathscr{W}$-affinity

2.6.1

Let X be a symplectic manifold. Let S be a variety, let $f\colon X \to S$ be a projective morphism, and let L be a relatively ample line bundle on X. Let $\mathscr{W}$ be a W-algebra on X. The following theorem is an analog of the result of Beĭlinson and Bernstein [1] on $\mathscr{D}$-modules on flag manifolds. We follow the formulation of [16].

THEOREM 2.9

For $n > 0$, let $\mathscr{L}_n(0)$ be a locally free $\mathscr{W}(0)$-module of rank 1 so that $\mathscr{L}_n(0)/\hbar\mathscr{L}_n(0) = L^{\otimes(-n)}$. Set $\mathscr{L}_n = \mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{L}_n(0)$.

Consider the following conditions:

$$\text{for } n \gg 0, \text{ there exist a vector space } V_n \text{ and a split epimorphism } \mathscr{L}_n \otimes V_n \twoheadrightarrow \mathscr{W}; \text{ that is, } \mathscr{W} \text{ is a direct summand of the direct sum of finitely many copies of } \mathscr{L}_n; \tag{2.5}$$

$$\text{for } n \gg 0, \text{ there exist a vector space } V_n \text{ and an epimorphism } \mathscr{W} \otimes V_n \twoheadrightarrow \mathscr{L}_n. \tag{2.6}$$

(i) *Assume (2.5). Then, for every good $\mathscr{W}$-module $\mathscr{M}$, we have $R^i f_*(\mathscr{M}) = 0$ for $i \neq 0$.*

(ii) *Assume (2.6). Then every good $\mathscr{W}$-module is generated by its global sections (locally on S).*

The proof is given in §§2.6.2 and 2.6.3.

Assume that $\mathscr{W}$ has an F-action with exponent m, and assume that S has a $\mathbb{G}_m$-action so that f is $\mathbb{G}_m$-equivariant. Assume, moreover, that there exists $\mathrm{o} \in S$ such that every point of S shrinks to o (i.e., $\lim_{t\to 0} tx = \mathrm{o}$ for any $x \in S$).

Let $\widetilde{\mathscr{W}} = \mathscr{W}[\hbar^{1/m}]$, and let $A = \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}})^{\mathrm{opp}}$.

THEOREM 2.10

Assume that conditions (2.5) and (2.6) hold. Then A is a left Noetherian ring, and we have quasi-inverse equivalences of categories between $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ and $\mathrm{Mod}_{\mathrm{coh}}(A)$,

$$\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}}) \xleftrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(A),$$
$$\mathscr{M} \mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}),$$
$$\widetilde{\mathscr{W}} \otimes_A M \hookleftarrow M.$$

The proof is given in §2.6.4.

2.6.2. Vanishing theorem

Let $\mathscr{W}$ be a W-algebra on a symplectic manifold X. Let $\mathscr{M}$ be a coherent $\mathscr{W}$-module. Recall that $\mathscr{M}(0)$ is a $\mathscr{W}(0)$-*lattice* of $\mathscr{M}$ if $\mathscr{M}(0)$ is a coherent $\mathscr{W}(0)$-submodule of $\mathscr{M}$ such that $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{M}(0) \xrightarrow{\sim} \mathscr{M}$.

We start with the following lemma.

LEMMA 2.11
For any coherent $\mathscr{W}(0)$-module $\mathscr{N}$, the canonical map is an isomorphism

$$\mathscr{N} \xrightarrow{\sim} \varprojlim_{m} \mathscr{N}/\hbar^m \mathscr{N}. \tag{2.7}$$

Proof
Let us first show that $\mathscr{N} \to \varprojlim_m \mathscr{N}/\hbar^m \mathscr{N}$ is a monomorphism. For any $x \in X$, we have morphisms of $\mathscr{W}(0)_x$-modules:

$$\mathscr{N}_x \to (\varprojlim_{m} \mathscr{N}/\hbar^m \mathscr{N})_x \to \varprojlim_{m} (\mathscr{N}_x/\hbar^m \mathscr{N}_x).$$

Since the composition is injective (by the Artin-Rees argument; see, e.g., [25]), the map $\mathscr{N}_x \to (\varprojlim_m \mathscr{N}/\hbar^m \mathscr{N})_x$ is injective.

Let us show now that $\mathscr{N} \to \varprojlim_m \mathscr{N}/\hbar^m \mathscr{N}$ is surjective. The question being local, we can take an exact sequence of coherent $\mathscr{W}(0)$-modules

$$0 \to \mathscr{M} \to \mathscr{L} \to \mathscr{N} \to 0,$$

where $\mathscr{L}$ is a free $\mathscr{W}(0)$-module of finite rank. For any Stein open subset U and $m > 0$, we have

$$H^1\big(U, \mathscr{M}/(\hbar^m \mathscr{L} \cap \mathscr{M})\big) = 0,$$

and

$$\Gamma\big(U, \mathscr{M}/(\hbar^m \mathscr{L} \cap \mathscr{M})\big) \to \Gamma\big(U, \mathscr{M}/(\hbar^{m-1} \mathscr{L} \cap \mathscr{M})\big) \text{ is surjective.}$$

Indeed, in the exact sequence

$$\begin{aligned}\Gamma\big(U; \mathscr{M}/(\hbar^m \mathscr{L} \cap \mathscr{M})\big) &\to \Gamma\big(U; \mathscr{M}/(\hbar^{m-1} \mathscr{L} \cap \mathscr{M})\big)\\ &\to H^1\big(U; (\hbar^{m-1} \mathscr{L} \cap \mathscr{M})/(\hbar^m \mathscr{L} \cap \mathscr{M})\big)\\ &\to H^1\big(U; \mathscr{M}/(\hbar^m \mathscr{L} \cap \mathscr{M})\big)\\ &\to H^1\big(U; \mathscr{M}/(\hbar^{m-1} \mathscr{L} \cap \mathscr{M})\big),\end{aligned}$$

$H^1(U; (\hbar^{m-1} \mathscr{L} \cap \mathscr{M})/(\hbar^m \mathscr{L} \cap \mathscr{M}))$ vanishes because $(\hbar^{m-1} \mathscr{L} \cap \mathscr{M})/(\hbar^m \mathscr{L} \cap \mathscr{M})$ is a coherent $\mathscr{O}_X$-module.

It follows that the next sequence is exact:

$$0 \to \Gamma\big(U, \mathscr{M}/(\hbar^m \mathscr{L} \cap \mathscr{M})\big) \to \Gamma(U, \mathscr{L}/\hbar^m \mathscr{L}) \to \Gamma(U, \mathscr{N}/\hbar^m \mathscr{N}) \to 0.$$

Since $\{\Gamma(U, \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M}))\}_m$ satisfies the Mittag-Leffler (ML) condition, the bottom row of the following commutative diagram is exact:

$$\begin{array}{ccccccc} & & \Gamma(U,\mathscr{L}) & \longrightarrow & \Gamma(U,\mathscr{N}) & & \\ & & \downarrow \wr & & \downarrow & & \\ 0 \longrightarrow \varprojlim_m \Gamma\big(U, \mathscr{M}/(\hbar^m\mathscr{L}\cap\mathscr{M})\big) & \longrightarrow & \varprojlim_m \Gamma(U,\mathscr{L}/\hbar^m\mathscr{L}) & \longrightarrow & \varprojlim_m \Gamma(U,\mathscr{N}/\hbar^m\mathscr{N}) & \longrightarrow & 0 \end{array}$$

It follows that $\Gamma(U, \mathscr{N}) \to \varprojlim_m \Gamma(U, \mathscr{N}/\hbar^m\mathscr{N}) \simeq \Gamma(U, \varprojlim_m \mathscr{N}/\hbar^m\mathscr{N})$ is surjective. □

LEMMA 2.12
Let $\mathscr{M}$ be a coherent $\mathscr{W}$-module, and let $\mathscr{M}(0)$ be a $\mathscr{W}(0)$-lattice of $\mathscr{M}$. Set $\mathscr{M}(m) = \hbar^{-m}\mathscr{M}(0)$, and set $\overline{\mathscr{M}} = \mathscr{M}(0)/\mathscr{M}(-1)$. Assume that

$$H^i(X, \overline{\mathscr{M}}) = 0 \quad \text{for } i \neq 0.$$

Then

(i) *the canonical morphism*

$$\Gamma\big(X, \mathscr{M}(0)\big)/\,\Gamma\big(X, \mathscr{M}(-m)\big) \longrightarrow \Gamma\big(X, \mathscr{M}(0)/\mathscr{M}(-m)\big)$$

is an isomorphism for any $m \geqslant 0$; and

(ii) $H^i(X, \mathscr{M}(0)) = 0$ *for any $i \neq 0$.*

Proof
Given $m \geqslant 0$, the exact sequence

$$0 \to \overline{\mathscr{M}} \xrightarrow{\hbar^m} \mathscr{M}(0)/\mathscr{M}(-m-1) \to \mathscr{M}(0)/\mathscr{M}(-m) \to 0$$

induces exact sequences

$$\Gamma\big(X, \mathscr{M}(0)/\mathscr{M}(-m-1)\big) \to \Gamma\big(X, \mathscr{M}(0)/\mathscr{M}(-m)\big) \to H^1(X, \overline{\mathscr{M}})$$

and

$$H^i(X, \overline{\mathscr{M}}) \to H^i\big(X, \mathscr{M}(0)/\mathscr{M}(-m-1)\big) \to H^i\big(X, \mathscr{M}(0)/\mathscr{M}(-m)\big).$$

It follows that $\Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m-1)) \to \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m))$ is surjective for any $m \geqslant 0$ and $H^i(X, \mathscr{M}(0)/\mathscr{M}(-m)) = 0$ for any $i > 0$. Since $\Gamma(X, \mathscr{M}(0)) = \varprojlim_m \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m))$ by Lemma 2.11, we obtain (i).

For $i > 0$, we have

$$H^i\big(X, \mathscr{M}(0)\big) = \varprojlim_m H^i\big(X, \mathscr{M}(0)/\mathscr{M}(-m)\big) = 0$$

because $\{H^{i-1}(X, \mathscr{M}(0)/\mathscr{M}(-m))\}_m$ satisfies the ML condition. □

2.6.3. Proof of Theorem 2.9

Let us prove (i). The question being local on S, we may assume that there exists a $\mathscr{W}(0)$-lattice $\mathscr{M}(0)$ of $\mathscr{M}$. Set $\overline{\mathscr{M}} = \mathscr{M}(0)/\hbar\mathscr{M}(0)$. Then, for $m \gg 0$, we have $R^i f_*(L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}) = 0$ for $i \neq 0$. It follows that

$$H^i(f^{-1}U, L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}) = 0$$

for any $i \neq 0$ and any Stein open subset U of S. From now on, we assume that m is large enough so that the vanishing above holds.

Let $\mathscr{A}_m = \mathscr{E}nd_{\mathscr{W}}(\mathscr{L}_m)^{\mathrm{opp}}$, a W-algebra on X. We have $\mathscr{A}_m(0) = \mathscr{E}nd_{\mathscr{W}(0)}(\mathscr{L}_m(0))^{\mathrm{opp}}$. Let $\mathscr{L}_m(0)^* = \mathscr{H}om_{\mathscr{W}(0)}(\mathscr{L}_m(0), \mathscr{W}(0))$, an $(\mathscr{A}_m(0), \mathscr{W}(0))$-bimodule, and let $\mathscr{L}_m^* = \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_m, \mathscr{W})$, an $(\mathscr{A}_m, \mathscr{W})$-bimodule. We have

$$\mathscr{L}_m^* \simeq \mathscr{A}_m \otimes_{\mathscr{A}_m(0)} \mathscr{L}_m(0)^* \simeq \mathscr{L}_m(0)^* \otimes_{\mathscr{W}(0)} \mathscr{W}.$$

Note that the bimodules $\mathscr{L}_m$ and $\mathscr{L}_m^*$ give inverse Morita equivalences between $\mathscr{A}_m$ and $\mathscr{W}$.

Let $\mathscr{M}_m(0) = \mathscr{L}_m^*(0) \otimes_{\mathscr{W}(0)} \mathscr{M}(0)$, an $\mathscr{A}_m(0)$-lattice in the $\mathscr{A}_m$-module $\mathscr{M}_m = \mathscr{L}_m^* \otimes_{\mathscr{W}} \mathscr{M}$. We have $\mathscr{M}_m(0)/\hbar\mathscr{M}_m(0) \simeq L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}$; hence, $H^i(f^{-1}U, \mathscr{M}_m(0)/\hbar\mathscr{M}_m(0)) = 0$ for $i \neq 0$. Lemma 2.12(ii) implies that $H^i(f^{-1}U, \mathscr{M}_m(0)) = 0$ for $i \neq 0$. Taking the inductive limit with respect to Stein open neighbourhoods U of $s \in S$, we obtain $H^i(f^{-1}(s), \mathscr{M}_m(0)) = 0$, and hence,

$$H^i\big(f^{-1}(s), \mathscr{M}_m\big) \simeq \mathbf{k} \otimes_{\mathbf{k}(0)} H^i\big(f^{-1}(s), \mathscr{M}_m(0)\big) = 0. \tag{2.8}$$

By condition (2.5), $\mathscr{W}$ is a direct summand of a direct sum of finitely many copies of the left $\mathscr{W}$-module $\mathscr{L}_m$. So, $\mathscr{W}$ is a direct summand of a direct sum of finitely many copies of the right $\mathscr{W}$-module $\mathscr{L}_m^*$, and $\mathscr{M}$ is a direct summand of a direct sum of finitely many copies of $\mathscr{M}_m$ (as a sheaf). Then (2.8) implies that $H^i(f^{-1}(s), \mathscr{M}) = 0$. This completes the proof of (i).

We now prove (ii). We keep the same notation as in the proof of (i). Since L is relatively ample, given $s \in S$ there exists a surjective map $(\mathscr{O}_X|_{f^{-1}(s)})^{\oplus N} \twoheadrightarrow$

$(L^{\otimes m} \otimes \overline{\mathscr{M}})|_{f^{-1}(s)}$ for some N. On the other hand, Lemma 2.12(i) implies that $\Gamma(f^{-1}(s), \mathscr{M}_m(0)) \to \Gamma(f^{-1}(s), \mathscr{M}_m(0)/\hbar\mathscr{M}_m(0))$ is surjective. Hence, we have a morphism $\phi_m\colon \mathscr{A}_m(0)^{\oplus N}|_{f^{-1}(s)} \to \mathscr{M}_m(0)|_{f^{-1}(s)}$ such that the composition $\mathscr{A}_m(0)^{\oplus N}|_{f^{-1}(s)} \to (\mathscr{M}_m(0)/\hbar\mathscr{M}_m(0))|_{f^{-1}(s)}$ is an epimorphism. It follows that ϕ_m is an epimorphism. Thus, there exists an epimorphism $\mathscr{A}_m^{\oplus N}|_{f^{-1}(s)} \twoheadrightarrow \mathscr{M}_m|_{f^{-1}(s)}$. By applying the exact functor $\mathscr{L}_m \otimes_{\mathscr{A}_m} \bullet\colon \mathrm{Mod}(\mathscr{A}_m) \to \mathrm{Mod}(\mathscr{W})$, we obtain an epimorphism $\mathscr{L}_m^{\oplus N}|_{f^{-1}(s)} \twoheadrightarrow \mathscr{M}|_{f^{-1}(s)}$. The assertion follows now from condition (2.6). □

2.6.4. Proof of Theorem 2.10

By Theorem 2.9, $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}}) \ni \mathscr{M} \mapsto f_*(\mathscr{M}) \in \mathrm{Mod}(f_*(\widetilde{\mathscr{W}}))$ is an exact functor.

By the assumption, o has a neighbourhood system consisting of relatively compact Stein open neighbourhoods U such that U is stable by T_t $(0 < |t| \leqslant 1)$. For such a U, we have $S = \bigcup_{t\in\mathbb{C}^*} T_t U$. For any $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$, we have

$$\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}) = \left\{s \in \mathscr{M}(f^{-1}U);\, s \text{ is F-invariant}\right\}.$$

Here, $s \in \mathscr{M}(f^{-1}U)$ is F-invariant if $\mathscr{F}_t(s) = s$ for any $t \in \mathbb{C}^\times$ with $|t| = 1$.

For $s \in \mathscr{M}(f^{-1}U)$, let

$$p_n(s) = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=1} t^{-n}\mathscr{F}_t(s)\,\frac{dt}{t}.$$

We have $s = \sum_n p_n(s)$, and $\hbar^{-n/m} p_n(s) = p_0(\hbar^{-n/m}s)$ is F-invariant.

LEMMA 2.13

$\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \bullet)$ *is an exact functor.*

Proof

Let $\varphi\colon \mathscr{M} \to \mathscr{M}' \to 0$ be an epimorphism in $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$, and let $s' \in \mathscr{M}'(f^{-1}U)$ so that $\mathscr{F}_t(s') = s'$ for any t with $|t| = 1$. By Theorem 2.9, there exists $s \in \mathscr{M}(f^{-1}U)$ such that $\varphi(s) = s'$. We have $\varphi(p_0(s)) = s'$, and $p_0(s)$ is F-invariant. □

LEMMA 2.14

Any $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ *is generated by F-invariant global sections.*

Proof

By Theorem 2.9, $\mathscr{M}$ is generated by global sections $s_i \in \mathscr{M}(f^{-1}U)$. Then $\mathscr{M}$ is generated by the $\hbar^{-n/m}p_n(s_i)$'s. Indeed, let $\mathscr{N}$ be the submodule of $\mathscr{M}$ generated by the $p_n(s_i)$'s. This is a coherent submodule of $\mathscr{M}$. Let $\psi\colon \mathscr{M} \to \mathscr{M}/\mathscr{N}$ be the quotient morphism. Then $p_n\psi(s_i) = \psi(p_n(s_i)) = 0$ for any n, and hence, $\psi(s_i) = 0$. It follows that $\mathscr{N} = \mathscr{M}$. □

We deduce that $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M})$ is an A-module of finite presentation for any $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$.

LEMMA 2.15
A is left Noetherian.

Proof
Let I be a left ideal of A. Let $\mathscr{I} \subset \widetilde{\mathscr{W}}$ be the image of $\widetilde{\mathscr{W}} \otimes_A I \to \widetilde{\mathscr{W}}$. Note that $\mathscr{I}$ belongs to $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$. Since $\widetilde{\mathscr{W}}$ is coherent, there exist finitely many $a_i \in I$ such that $\mathscr{I} = \sum \widetilde{\mathscr{W}} a_i$. We have $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{I}) = \sum_i A a_i \subset I$ by Lemma 2.13. Since we have injective maps $I \to \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{I}) \hookrightarrow \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \widetilde{\mathscr{W}}) = A$, we obtain $I = \sum_i A a_i$. □

Since the good $(\widetilde{\mathscr{W}}, F)$-modules are generated by their F-invariant sections, $\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \bullet)$ sends $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ to $\mathrm{Mod}_{\mathrm{coh}}(A)$.

Given $M \in \mathrm{Mod}_{\mathrm{coh}}(A)$, the canonical morphism

$$M \to \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \widetilde{\mathscr{W}} \otimes_A M)$$

is an isomorphism because both sides are right exact functors of M and the morphism is an isomorphism for $M = A$.

Given $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$, the canonical map $\widetilde{\mathscr{W}} \otimes_A \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}) \to \mathscr{M}$ is an isomorphism because both sides are right exact functors of $\mathscr{M}$ and $\mathscr{M}$ has a resolution $\widetilde{\mathscr{W}}^{\oplus m_1} \to \widetilde{\mathscr{W}}^{\oplus m_0} \to \mathscr{M} \to 0$ in $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ by Lemma 2.14.

This completes the proof of Theorem 2.10. □

3. Rational Cherednik algebras and $\mathscr{D}$-modules

3.1. Definitions, notation, and recollections

3.1.1
Let $V = \mathbb{C}^n$, let $G = \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$, and let $\mathfrak{g} = \mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$. We denote by $e_{rs} \in \mathfrak{g}$ the elementary matrix with zero coefficients everywhere except in row r and column s, where the coefficient is 1. We denote by $A_{rs} \in \mathbb{C}[\mathfrak{g}]$ the corresponding coordinate function.

We denote by $\mathfrak{t} = \mathbb{C}^n$ the Cartan subalgebra of diagonal matrices of $\mathfrak{g}$, and we denote by $W = S_n$ the Weyl group. We denote by s_{ij} the transposition (ij) for $1 \leqslant i \neq j \leqslant n$. We have $\mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\mathfrak{t}^*] = \mathbb{C}[y_1, \ldots, y_n]$.

We put $\mathfrak{d}(x) = \prod_{i<j}(x_i - x_j) \in \mathbb{C}[\mathfrak{t}]$. We denote by $\mathfrak{g}_{\mathrm{reg}}$ the open subset of regular semisimple elements of $\mathfrak{g}$, and we put $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}} = \{x \in \mathfrak{t}\,;\, \mathfrak{d}(x) \neq 0\}$.

We identify $\mathbb{C}[\mathfrak{t}]^W$ and $\mathbb{C}[\mathfrak{g}]^G$ via the restriction map.

Given M a graded vector space, we denote by M_k its component of degree k.

3.1.2

Let X be a manifold, let $i\colon Y \hookrightarrow X$ be a submanifold, and let $f\colon \mathscr{M} \to \mathscr{N}$ be a morphism of coherent $\mathscr{D}_X$-modules. Assume that Y is noncharacteristic for $\mathscr{M}$ and $\mathscr{N}$ (i.e., for $Z = \mathrm{Ch}(\mathscr{M})$ or $Z = \mathrm{Ch}(\mathscr{N})$, we have $Z \cap T^*_Y X \subset T^*_X X$). If $i^*(f)\colon i^*\mathscr{M} \to i^*\mathscr{N}$ is an isomorphism (resp., monomorphism, epimorphism), then so is f on a neighbourhood of Y (see, e.g., [18, Theorem 4.7]).

3.1.3

Let $f \in H^0(X; \mathscr{O}_X)$ be nonzero. We denote by $\delta(f)$ the element f^{-1} of the $\mathscr{D}_X$-module $\mathscr{O}_X[f^{-1}]/\mathscr{O}_X$. So, $\mathscr{D}_X\delta(f) \subset \mathscr{O}_X[f^{-1}]/\mathscr{O}_X$. More generally, let S be a closed subvariety of complete intersection of codimension r given by $f_1 = \cdots = f_r = 0$ for $f_1, \ldots, f_r \in H^0(X; \mathscr{O}_X)$. Then

$$\mathscr{H}^j_S(\mathscr{O}_X) = 0 \quad \text{for } j \neq r$$

and

$$\mathscr{H}^r_S(\mathscr{O}_X) \simeq \mathscr{O}[(f_1 \cdots f_r)^{-1}] / \sum_{1 \leqslant i \leqslant r} \mathscr{O}[(f_1 \cdots \hat{f}_i \cdots f_r)^{-1}].$$

We denote the last $\mathscr{D}_X$-module by $\mathscr{B}_{S|X}$. We denote by $\delta(f_1)\cdots\delta(f_r)$ the section $1/(f_1 \cdots f_r)$ of $\mathscr{B}_{S|X}$.

3.2. Construction of some $\mathscr{D}$-modules

3.2.1

Given $c \in \mathbb{C}$, we denote by H_c the rational Cherednik algebra of $(\mathfrak{t}, W)$ with parameter c: this is the $\mathbb{C}$-algebra quotient of $T(\mathfrak{t}^* \oplus \mathfrak{t}) \rtimes W$ by the relations

$$\begin{aligned} &[x_i, x_j] = 0, \qquad [y_i, y_j] = 0, \\ &[y_i, x_j] = c s_{ij} \quad \text{for } i \neq j, \\ &[y_i, x_i] = 1 - c \sum_{k \neq i} s_{ik}. \end{aligned}$$

We have a vector space decomposition ("PBW property") (see [6, Theorem 1.3])

$$H_c = \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}^*] \otimes \mathbb{C}[W].$$

There is an injective algebra morphism (given by Dunkl operators) (see [6, Proposition 4.5])

$$\theta_c\colon H_c \hookrightarrow \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}])$$

given by the canonical map on $\mathbb{C}[\mathfrak{t}] \rtimes W$ and by

$$\theta_c(y_i) = \partial_{x_i} - c \sum_{k \neq i} \frac{1}{x_i - x_k}(1 - s_{ik}). \tag{3.1}$$

It induces an isomorphism of algebras after localization

$$\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{t}]} H_c \xrightarrow{\sim} \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W.$$

We introduce the idempotents $e := (1/n!) \sum_{w \in W} w \in \mathbb{C}[W] \subset H_c$ and $e_{\det} := (1/n!) \sum_{w \in W} \det(w) w \in \mathbb{C}[W] \subset H_c$ corresponding to the trivial representation and the sign representation of W.

We have an injective morphism $\mathbb{C}[\mathfrak{t}]^W \to eH_ce$, $a \mapsto ae$, and we identify $\mathbb{C}[\mathfrak{t}]^W$ with its image. We put $\mathbf{y}^2 = \sum_{i=1}^n y_i^2 \in H_c$. Recall that eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W e$ and $\mathbb{C}[\mathfrak{t}^*]^W e$ (cf., e.g., [4, proof of Proposition 5.4.4]). On the other hand, we have an isomorphism of $\mathbb{C}[W]$-modules (cf., e.g., [2, Corollary 4.9])

$$\big(\mathrm{ad}(\mathbf{y}^2)\big)^k : \mathbb{C}[\mathfrak{t}]_k \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}^*]_k. \tag{3.2}$$

It sends $a(x_1, \dots, x_n)$ to $2^k k! a(y_1, \dots, y_n)$. Hence, eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W e$ and $\mathbf{y}^2 e$.

We denote by $h \mapsto h^*$ the anti-involution of H_c given by $x_i \mapsto x_i$, $y_i \mapsto -y_i$, $w \mapsto w^{-1}$ ($w \in W$).

3.2.2

We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the G-invariant bilinear symmetric form $\mathfrak{g} \times \mathfrak{g} \ni (A, A') \mapsto \mathrm{tr}(AA')$.

A pair (A, z) denotes a point of $\mathfrak{g} \times V$. We identify $T^*(\mathfrak{g} \times V)$ with $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$; accordingly, we denote a point in $T^*(\mathfrak{g} \times V)$ by (A, B, z, ζ). Let $\mu : T^*(\mathfrak{g} \times V) \to \mathfrak{g}^*$ be the moment map. It is given by $\mu(A, B, z, \zeta) = -[A, B] - z \circ \zeta$.

Let us denote by

$$\mu_D : \mathfrak{g} \to \mathscr{D}_{\mathfrak{g} \times V}(\mathfrak{g} \times V)$$

the Lie algebra homomorphism associated with the diagonal action of G on $\mathfrak{g} \times V$. Let us consider the $\mathscr{D}_{\mathfrak{g} \times V}$-module $\mathscr{L}_c = \mathscr{D}_{\mathfrak{g} \times V} u_c$ given by the defining equation

$$\big(\mu_D(C) + c\, \mathrm{tr}(C)\big) u_c = 0 \quad (C \in \mathfrak{g}).$$

More formally, we have $\mathscr{L}_c = \mathscr{D}_{\mathfrak{g} \times V} / (\mathscr{D}_{\mathfrak{g} \times V}(\mu_D + c\, \mathrm{tr})(\mathfrak{g}))$, and u_c is the image of 1 in $\mathscr{L}_c$.

We consider $\mathscr{L}_c$ as a twisted G-equivariant $\mathscr{D}_{\mathfrak{g} \times V}$-module with twist $c\, \mathrm{tr}$, where u_c is a G-invariant section of $\mathscr{L}_c$. Since any $a \in \mathbb{C}[\mathfrak{g}]^G$ commutes with $\mu_D(C)$ ($C \in \mathfrak{g}$), the map $u_c \mapsto au_c$ extends to a $\mathscr{D}_{\mathfrak{g}\times V}$-linear endomorphism of $\mathscr{L}_c$. Hence, $\mathscr{L}_c$ has a $(\mathbb{C}[\mathfrak{t}]^W \otimes \mathscr{D}_{\mathfrak{g}\times V})$-module structure.

The characteristic variety $\mathrm{Ch}(\mathscr{L}_c)$ of $\mathscr{L}_c$ is the almost-commuting variety:

$$\mathrm{Ch}(\mathscr{L}_c) = \mu^{-1}(0) = \big\{(A, B, z, \zeta); [A, B] + z \circ \zeta = 0\big\}.$$

This is a complete intersection in $T^*(\mathfrak{g} \times V)$ (see [7, Theorem 1.1]).

LEMMA 3.1
Let $\mathfrak{g}_1$ be the open subset of $\mathfrak{g}$ of elements that have at least $(n-1)$ distinct eigenvalues. We have

$$\mathscr{H}^0_{(\mathfrak{g}\setminus\mathfrak{g}_{\mathrm{reg}})\times V}(\mathscr{L}_c) = 0 \qquad \textit{and} \qquad \mathscr{H}^1_{(\mathfrak{g}\setminus\mathfrak{g}_1)\times V}(\mathscr{L}_c) = 0.$$

Proof
Since $\mathrm{Ch}(\mathscr{L}_c)$ is a complete intersection, we have (see [18, (2.23)])

$$\mathscr{E}xt^j_{\mathscr{D}_{\mathfrak{g}\times V}}(\mathscr{L}_c, \mathscr{D}_{\mathfrak{g}\times V}) = 0 \quad \text{for } j \neq \operatorname{codim}_{T^*(\mathfrak{g}\times V)} \mu^{-1}(0) = n^2. \tag{3.3}$$

Let $\gamma\colon \mathfrak{g} \to \mathfrak{t}/W$ be the canonical map associating to $A \in \mathfrak{g}$ the eigenvalues of A. Let $\tilde{\gamma}\colon \mu^{-1}(0) \to \mathfrak{t}/W$ be given by $(A, B, i, j) \mapsto \gamma(A)$. Then $\tilde{\gamma}$ is a flat morphism (see [7, Corollary 2.7]).

Let S be a closed subset of $\mathfrak{t}/W$. Since $\tilde{\gamma}$ is flat, we have

$$\operatorname{codim}_{T^*(\mathfrak{g}\times V)}\big(\gamma^{-1}(S) \times_{\mathfrak{g}} \mathrm{Ch}(\mathscr{L}_c)\big) - \operatorname{codim}_{T^*(\mathfrak{g}\times V)} \mathrm{Ch}(\mathscr{L}_c) = \operatorname{codim}_{\mathfrak{t}/W} S.$$

Lemma 2.1 applied to $\gamma^{-1}(S) \times_{\mathfrak{g}} \mathrm{Ch}(\mathscr{L}_c)$ implies that

$$\mathscr{H}^j_{\gamma^{-1}(S)\times V}(\mathscr{L}_c) = 0 \quad \text{for } j < \operatorname{codim}_{\mathfrak{t}/W} S,$$

and the lemma follows. □

3.2.3
Let us recall some constructions and results of [14]. Let $\mu_0\colon \mathfrak{g} \to \mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}(\mathfrak{t} \times \mathfrak{g})$ be the morphism given by the action of G on $\mathfrak{t} \times \mathfrak{g}$, $g \cdot (x, A) = (x, \mathrm{Ad}(g)A)$. We consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}$-module generated by $\delta_0(x, A)$ with the following defining equations:

$$\mu_0(C)\delta_0(x, A) = 0 \quad \text{for any } C \in \mathfrak{g}$$

and

$$\big(P(A) - P(x)\big)\delta_0(x, A) = 0,$$
$$\big(P(\partial_A) - P(-\partial_x)\big)\delta_0(x, A) = 0,$$

for any $P \in \mathbb{C}[\mathfrak{g}]^G$.

Then $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A)$ is a simple holonomic $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}$-module with support $\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{g}$. Its characteristic variety is the set of (x, y, A, B) such that $[A, B] = 0$, and there exists $g \in G$ such that $\mathrm{Ad}(g)A$ and $\mathrm{Ad}(g)B$ are upper triangular and x and y are the diagonal components of $\mathrm{Ad}(g)A$ and $\mathrm{Ad}(g)B$. Note that $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A) \subset \mathscr{B}_{\mathfrak{t}\times_{\mathfrak{t}/W}\mathfrak{g}|\mathfrak{t}\times\mathfrak{g}}$ by $\delta_0(x, A) \mapsto \prod_{i=1}^{n} \delta(P_i(x) - P_i(A))$ (see §3.1.3), where $P_i \in \mathbb{C}[\mathfrak{g}]^G$ $(i = 1, \ldots, n)$ are the fundamental invariants given by $\det(1 + tA) = \sum_{i=0}^{n} P_i(A)t^i$.

We need to consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A) \boxtimes \mathscr{O}_V$, generated by $\delta(x, A) := \delta_0(x, A) \boxtimes 1$ which satisfies the same equations as $\delta_0(x, A)$ and $\partial_{z_i}\delta(x, A) = 0$. In particular, $\mu_D(C)\delta(x, A) = 0$ for any $C \in \mathfrak{g}$.

3.2.4
Let us set

$$q(A, z) = \det(A^{n-1}z, A^{n-2}z, \ldots, Az, z).$$

We have $q(\mathrm{Ad}(g)A, gz) = \det(g)q(A, z)$ for $g \in G$ and $[\mu_D(C), q(A, z)] = -\mathrm{tr}(C)q(A, z)$ for $C \in \mathfrak{g}$.

Consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}q(A, z)^c\delta(x, A)$. A precise definition is as follows. Let us consider the left ideal $\mathscr{I}$ of $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V} \otimes \mathbb{C}[s]$ (s being an indeterminate) consisting of those $P(s)$ such that $P(m)q(A, z)^m\delta(x, A) = 0$ for any $m \in \mathbb{Z}_{\geqslant 0}$. We now define $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}q(A, z)^c\delta(x, A)$ as $(\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}\otimes\mathbb{C}[s])/(\mathscr{I}+\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}\otimes\mathbb{C}[s](s-c))$. It is a holonomic $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module.

The element $q(A, z)^c\delta(x, A)$ satisfies

$$\big(\mu_D(C) + c\,\mathrm{tr}(C)\big)q(A, z)^c\delta(x, A) = 0 \quad \text{for any } C \in \mathfrak{g},$$
$$\big(P(A) - P(x)\big)q(A, z)^c\delta(x, A) = 0 \quad \text{for any } P \in \mathbb{C}[\mathfrak{g}]^G.$$

We put $v_c = q(A, z)^c\delta(x, A)$. Let $p_0\colon \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{g} \times V \to \mathfrak{g} \times V$ be the projection. Let us consider the $\mathscr{D}_{\mathfrak{g}\times V}$-module

$$\mathscr{M}_c = (p_0)_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}v_c) = (p_0)_*\big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}q(A, z)^c\delta(x, A)\big).$$

By the definition, we have an isomorphism $\mathscr{M}_c \xrightarrow{\sim} j_*j^{-1}\mathscr{M}_c$, where $j\colon \mathfrak{g}_{\mathrm{reg}} \times V \hookrightarrow \mathfrak{g} \times V$ is the open embedding. This is a quasi-coherent $\mathscr{D}_{\mathfrak{g}\times V}$-module whose characteristic variety is contained in the almost-commuting variety $\mu^{-1}(0)$.

The action of W on $\mathfrak{t}_{\mathrm{reg}}$ induces a W-action on $\mathscr{M}_c$. Here, W acts trivially on v_c. Hence, the $\mathscr{D}_{\mathfrak{g}\times V}$-module $\mathscr{M}_c$ has a module structure over $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$. Therefore, H_c acts on $\mathscr{M}_c$ via the canonical embedding $\theta_c\colon H_c \hookrightarrow \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$.

3.3. Spherical constructions and shift

3.3.1

There is a $\mathscr{D}_{\mathfrak{g}\times V}$-linear homomorphism

$$\iota\colon \mathscr{L}_c \to \mathscr{M}_c, \qquad u_c \mapsto v_c. \tag{3.4}$$

We regard $\mathscr{M}_c$ as a twisted G-equivariant $\mathscr{D}_{\mathfrak{g}\times V}$-module with twist $c\,\mathrm{tr}$, where sections in $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})v_c$ are G-invariant. Then the morphism above is G-equivariant. Moreover, it is $\mathbb{C}[\mathfrak{t}]^W$-linear. Hence, ι induces an epimorphism of $((\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W) \otimes \mathscr{D}_{\mathfrak{g}\times V})$-modules:

$$\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathscr{L}_c \twoheadrightarrow \mathscr{M}_c.$$

LEMMA 3.2

The morphism of ($\mathbb{C}[W] \otimes \mathscr{D}_{\mathfrak{g}\times V}$)*-modules*

$$1 \otimes \iota\colon \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathscr{L}_c \to \mathscr{M}_c$$

is an isomorphism on $\mathfrak{g}_{\mathrm{reg}} \times V$.

In particular, the induced morphisms $\mathscr{L}_c \xrightarrow{u_c \mapsto v_c} e\mathscr{M}_c$ *and* $\mathscr{L}_c \xrightarrow{u_c \mapsto \mathfrak{d}(x)v_c} e_{\det}\mathscr{M}_c$ *are isomorphisms on* $\mathfrak{g}_{\mathrm{reg}} \times V$.

Proof

Let $i\colon \mathfrak{t}_{\mathrm{reg}} \times V \hookrightarrow \mathfrak{g} \times V$ be the embedding. Note that i is noncharacteristic for $\mathscr{L}_c$ and $\mathscr{M}_c$. Since $G \cdot \mathfrak{t}_{\mathrm{reg}} = \mathfrak{g}_{\mathrm{reg}}$, it is enough to prove that the canonical map $\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W} i^*\mathscr{L}_c \to i^*\mathscr{M}_c$ is an isomorphism (cf. §3.1.2).

We have $i^*\mu_D(e_{rs}) = (A_{rr} - A_{ss})\partial_{A_{rs}} - z_s\partial_{z_r}$. It follows that we have an isomorphism

$$\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}\big/\Big(\sum_i \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}(z_i\partial_{z_i} - c)\Big) \xrightarrow{\sim} i^*\mathscr{L}_c, \qquad 1 \mapsto i^*u_c.$$

Let $i''\colon \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}} \hookrightarrow \mathfrak{t} \times \mathfrak{g}$ be the embedding. Since the Jacobian

$$\partial\big(P_1(x), \ldots, P_n(x)\big)/\partial(x_1, \ldots, x_n)$$

is equal to $\mathfrak{d}(x)$ (e.g., see [5, Chapter V, §5.4, Proposition 5]), we have an isomorphism

$$i''^*\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A) \xrightarrow[\sim]{\delta_0(x,A)\mapsto \sum_w \mathfrak{d}(a)^{-1}\delta(w^{-1}x-a)} \bigoplus_{w\in W} \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{t}_{\mathrm{reg}}}\delta(w^{-1}x - a),$$

where $\delta(w^{-1}x - a) = \delta(x_{w(1)} - a_1)\cdots\delta(x_{w(n)} - a_n)$.

Let us denote by $i'\colon \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}} \times V \hookrightarrow \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{g} \times V$ the embedding. We have an isomorphism

$$i'^{*}\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}v_c \xrightarrow[\sim]{v_c\mapsto\sum_w v'_w} \bigoplus_{w\in W}\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{t}_{\mathrm{reg}}\times V}v'_w, \tag{3.5}$$

where $v'_w = \mathfrak{d}(a)^{c-1}(z_1\cdots z_n)^c\delta(w^{-1}x - a)$ has the defining equations

$$\begin{aligned}&\Big(\partial_{x_{w(i)}} + \partial_{a_i} - (c-1)\sum_{j\neq i}\frac{1}{a_i - a_j}\Big)v'_w = 0,\\ &(x_{w(i)} - a_i)v'_w = 0,\\ &(z_i\partial_{z_i} - c)v'_w = 0,\end{aligned}$$

for any $i = 1, \ldots, n$. In particular, we have

$$f(x)v'_w = (w^{-1}f)(a)v'_w \quad \text{for any } f \in \mathbb{C}[\mathfrak{t}]. \tag{3.6}$$

We obtain finally an isomorphism

$$i^*\mathscr{M}_c \xrightarrow[\sim]{v_c\mapsto\sum_w v'_w} \bigoplus_{w\in W}\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}v'_w.$$

This is compatible with the action of W, where $w'(v'_w) = v'_{w'w}$. Moreover, each $\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}v'_w$ is isomorphic to $i^*\mathscr{L}_c$ by $v'_w \mapsto u_c$. Hence, we obtain an isomorphism of $(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V} \otimes \mathbb{C}[W])$-modules

$$i^*\mathscr{M}_c \xrightarrow{\sim} \mathbb{C}[W] \otimes i^*\mathscr{L}_c.$$

The composition $i^*(\mathbb{C}[\mathfrak{t}]\otimes_{\mathbb{C}[\mathfrak{t}]^W}\mathscr{L}_c) \to i^*\mathscr{M}_c \xrightarrow{\sim} \mathbb{C}[W]\otimes i^*\mathscr{L}_c$ is given by $a\otimes u_c \mapsto \sum_{w\in W} w \otimes (w^{-1}a)u_c$ in virtue of (3.6). Then the lemma follows from the fact that $\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}] \to \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]$ given by $a \otimes b \mapsto \sum_{w\in W} w \otimes (w^{-1}a)b$ is an isomorphism. □

LEMMA 3.3
The morphism $\iota\colon \mathscr{L}_c \to \mathscr{M}_c$ is injective, and its image is stable by eH_ce. Furthermore, eH_ce acts faithfully on $\mathscr{L}_c$.

Proof
The injectivity of ι follows from Lemma 3.2 because $\mathscr{L}_c$ does not have a nonzero submodule supported in $(\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}) \times V$ by Lemma 3.1.

Since eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W$ and $\mathbf{y}^2e$ (cf. §3.2.1), the stability result follows from the following result (cf. [4, Proposition 5.4.1], [6, Proposition 6.2]):

$$\mathbf{y}^2 v_c = \Delta_{\mathfrak{g}} v_c. \tag{3.7}$$

Here, $\Delta_{\mathfrak{g}} = \sum_{i,j=1,\dots,n} \frac{\partial^2}{\partial A_{ij}\partial A_{ji}}$ is the Laplacian on $\mathfrak{g}$.

Finally, the faithfulness of the action of eH_ce follows from the faithfulness of the action of H_c on $H_cv_c \subset \mathscr{M}_c$. With the notation of the proof of Lemma 3.2, we have an isomorphism $i^*\mathscr{M}_c \simeq \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V} \rtimes W$ compatible with the action of $\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}} \rtimes W$, and the faithfulness follows from that of θ_c. □

Remark 3.4

(i) In other words, the subalgebra of $\mathrm{End}_{\mathscr{D}_{\mathfrak{g}\times V}}(\mathscr{L}_c)$ generated by $\mathbb{C}[\mathfrak{t}]^W$ and by the endomorphism $u_c \mapsto \Delta_{\mathfrak{g}} u_c$ is isomorphic to eH_ce.

(ii) The action of the algebra eH_ce on $\mathscr{L}_c$ can be described as follows. Let $\kappa_0 \colon \mathbb{C}[\mathfrak{t}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathscr{D}(\mathfrak{g})$ and $\kappa_1 \colon \mathbb{C}[\mathfrak{t}^*]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathscr{D}(\mathfrak{g})$ be the canonical morphisms. We have

$$\begin{aligned} (ae)u_c &= \kappa_0(a)u_c \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W, \\ (be)u_c &= \kappa_1(b^*)u_c \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W. \end{aligned} \tag{3.8}$$

The first equality is clear. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{t}]^W_k & \xrightarrow{\kappa_0} & \mathbb{C}[\mathfrak{g}]^G_k \\ {\scriptstyle (\mathrm{ad}(\mathbf{y}^2))^k}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{ad}(\Delta_{\mathfrak{g}}))^k} \\ \mathbb{C}[\mathfrak{t}^*]^W_k & \xrightarrow[\kappa_1]{} & \mathbb{C}[\mathfrak{g}^*]^G_k \end{array} \tag{3.9}$$

From (3.7) and the first equality, we deduce that

$$\big(\mathrm{ad}(\Delta_{\mathfrak{g}})\big)^k\big(\kappa_0(a)\big)v_c = (-1)^k\big(\mathrm{ad}(\mathbf{y}^2)\big)^k(a)v_c$$

for $a \in \mathbb{C}[\mathfrak{t}]^W_k$. This gives the second equality.

3.3.2

The morphism ι gives rise to an $(H_c \otimes \mathscr{D}_{\mathfrak{g}\times V})$-linear morphism

$$H_ce \otimes_{eH_ce} \mathscr{L}_c \to \mathscr{M}_c. \tag{3.10}$$

Consider the following conditions:

$$H_c e H_c = H_c, \tag{3.11}$$

$$eH_c e_{\det} H_c e = eH_c e \quad \text{and} \quad e_{\det} H_c e H_c e_{\det} = e_{\det} H_c e_{\det}. \tag{3.12}$$

LEMMA 3.5
If (3.11) is satisfied, then the morphism (3.10) is injective.

Proof
Since $H_c e$ is a projective $eH_c e$-module, any coherent submodule of $H_c e \otimes_{eH_c e} \mathscr{L}_c$ vanishes as soon as it is zero on $\mathfrak{g}_{\mathrm{reg}} \times V$ by Lemma 3.1. Hence, it is enough to show that the morphism (3.10) is injective on $\mathfrak{g}_{\mathrm{reg}} \times V$. Then the result follows from Lemma 3.2 and the fact that the multiplication map gives an isomorphism of right $(eH_c e \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W)$-modules

$$\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} eH_c e \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W \xrightarrow{\sim} H_c e \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W. \qquad \square$$

PROPOSITION 3.6
Condition (3.11) holds if and only if eH_c gives a Morita equivalence between H_c and $eH_c e$. Similarly, condition (3.12) holds if and only if $eH_c e_{\det}$ gives a Morita equivalence between $e_{\det} H_c e_{\det}$ and $eH_c e$.

This follows from the next lemma.

LEMMA 3.7
Let A be a ring, and let e_1 and e_2 be idempotents in A. Assume that

$$e_1 A e_2 A e_1 = e_1 A e_1 \quad \textit{and} \quad e_2 A e_1 A e_2 = e_2 A e_2.$$

(i) *For any A-module M, we have*

$$e_2 A e_1 \otimes_{e_1 A e_1} e_1 M \xrightarrow{\sim} e_2 M.$$

(ii) *Two bimodules $e_1 A e_2$ and $e_2 A e_1$ give a Morita equivalence between* $\mathrm{Mod}(e_1 A e_1)$ *and* $\mathrm{Mod}(e_2 A e_2)$.

Proof
(i) The surjectivity follows from $e_2 M = e_2 A e_2 M = e_2 A e_1 A e_2 M \subset (e_2 A e_1)(e_1 M)$.

Let us show its injectivity. By the assumption, there exist finitely many elements $a_i \in e_2Ae_1$ and $b_i \in e_1Ae_2$ such that $e_2 = \sum_i a_ib_i$. Consider now $u = \sum_j x_j \otimes v_j \in e_2Ae_1 \otimes_{e_1Ae_1} e_1M$ (where $x_j \in e_2Ae_1$, $v_j \in e_1M$). Assume that $\sum_j x_jv_j = 0$. Then

$$u = \sum_{j,i} a_ib_ix_j \otimes v_j = \sum_{j,i} a_i \otimes b_ix_jv_j = 0.$$

(ii) It is enough to show that the multiplication maps $e_2Ae_1 \otimes_{e_1Ae_1} e_1Ae_2 \to e_2Ae_2$ and $e_1Ae_2 \otimes_{e_2Ae_2} e_2Ae_1 \to e_1Ae_1$ are isomorphisms. For the first one, we apply (i) to $M = Ae_2$. The second one can be handled similarly. □

The previous result can be expressed in terms of bimodules.

PROPOSITION 3.8

Let A and B be rings, let P be an (A, B)-bimodule, let Q be a (B, A)-bimodule, let $\varphi\colon P \otimes_B Q \to A$ be a morphism of (A, A)-bimodules, and let $\psi\colon Q \otimes_A P \to B$ be a morphism of (B, B)-bimodules. Assume that φ and ψ are surjective, and assume that the following diagrams commute:

$$\begin{array}{ccc} P \otimes_B Q \otimes_A P & \xrightarrow{\varphi\otimes P} & A \otimes_A P \\ \downarrow{\scriptstyle P\otimes\psi} & & \downarrow{\scriptstyle \mathrm{can}} \\ P \otimes_B B & \xrightarrow[\mathrm{can}]{} & P \end{array} \quad \text{and} \quad \begin{array}{ccc} Q \otimes_A P \otimes_B Q & \xrightarrow{\psi\otimes Q} & B \otimes_B Q \\ \downarrow{\scriptstyle Q\otimes\varphi} & & \downarrow{\scriptstyle \mathrm{can}} \\ Q \otimes_A A & \xrightarrow[\mathrm{can}]{} & Q \end{array}$$

(i) *Then φ and ψ are isomorphisms, and P and Q give a Morita equivalence between* Mod(A) *and* Mod(B).

(ii) *Let M be an A-module, let N be a B-module, and let $f\colon Q \otimes_A M \to N$ and $g\colon P \otimes_B N \to M$ be morphisms so that the diagrams*

$$\begin{array}{ccc} P \otimes_B Q \otimes_A M & \xrightarrow{\varphi\otimes M} & A \otimes_A M \\ \downarrow{\scriptstyle P\otimes f} & & \downarrow{\scriptstyle \mathrm{can}} \\ P \otimes_B N & \xrightarrow{g} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} Q \otimes_A P \otimes_B N & \xrightarrow{\psi\otimes N} & B \otimes_B N \\ \downarrow{\scriptstyle Q\otimes g} & & \downarrow{\scriptstyle \mathrm{can}} \\ Q \otimes_A M & \xrightarrow{f} & N \end{array}$$

are commutative. Then f and g are isomorphisms.

Proof
Apply Lemma 3.7 to the ring $\begin{pmatrix} A & P \\ Q & B \end{pmatrix}$, its module $\begin{pmatrix} M \\ N \end{pmatrix}$, and $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. □

Remark 3.9
(i) It would be interesting to describe the image of the morphism (3.10).
(ii) Let

$$\mathscr{Y} = \left\{ \frac{m}{d} \,\middle|\, m, d \in \mathbb{Z}, 2 \leqslant d \leqslant n, (m, d) = 1, m < 0 \right\}.$$

It is known that condition (3.11) holds for $c \notin \mathscr{Y}$, while condition (3.12) holds when $c - 1 \notin \mathscr{Y}$ (cf. [8, Theorem 3.3], [2, Theorem 8.1], [3]).

3.3.3
Let us consider the $(\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \otimes \mathscr{D}_{\mathfrak{g}\times V})$-linear morphism

$$\sigma : \mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V),$$
$$v_c = q(A, z)^c \delta(x, A) \mapsto q(A, z) \cdot q(A, z)^{c-1} \delta(x, A) \otimes l = q(A, z) v_{c-1} \otimes l.$$

Here, $l \in \det(V) := \bigwedge^n V$ is the element such that $q(A, z)l = A^{n-1}z \wedge A^{n-2}z \wedge \cdots \wedge Az \wedge z$. In particular, $q(A, z) \otimes l$ is a G-invariant section of $\mathscr{O}_{\mathfrak{g}\times V} \otimes \det(V)$.

So, the morphism σ is G-equivariant. We endow $\mathscr{M}_{c-1}$ with an H_c-module structure via the embedding $\theta_c : H_c \hookrightarrow \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$. Then σ is H_c-linear.

Remark 3.10
Note that $\mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ is an isomorphism on $\{q(A, z) \neq 0\}$. However, with our definition of $\mathscr{M}_c$, the morphism $\mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ is not a monomorphism for certain c (e.g., $c = 0$). Let us show this after restriction to $\mathfrak{t}_{\mathrm{reg}} \times V$. We have $q({}^tA, \partial_z) q(A, z) v_{c-1} = 0$ for $c = 0$ by (3.5), while the support of $q({}^tA, \partial_z) v_c$ is the subvariety $\{q(A, z) = 0\}$.

Let $\mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x) v_{c-1})$ be the $\mathscr{D}_{\mathfrak{g}\times V}$-submodule of $\mathscr{M}_{c-1}$ generated by $\mathfrak{d}(x) v_{c-1}$.

LEMMA 3.11
(i) *$\mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x) v_{c-1})$ is invariant by $e_{\det} H_c e_{\det}$.*
(ii) *The morphism $\mathscr{L}_{c-1} \to \mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x) v_{c-1})$ given by $u_{c-1} \mapsto \mathfrak{d}(x) v_{c-1}$ is an isomorphism.*

Proof
Note that $e_{\det}\mathfrak{d}(x)v_{c-1} = \mathfrak{d}(x)v_{c-1}$. The key point is the following (cf., e.g., [13, Theorem 3.1]):

$$\mathbf{y}^2\big(\mathfrak{d}(x)v_{c-1}\big) = \Delta_{\mathfrak{g}}\big(\mathfrak{d}(x)v_{c-1}\big). \tag{3.13}$$

The proof is then similar to that of Lemma 3.3. □

By [2, Proposition 4.1], there is a (unique) isomorphism

$$f : e_{\det}H_c e_{\det} \xrightarrow{\sim} eH_{c-1}e$$

such that $\theta_{c-1}(f(a)) = \mathfrak{d}(x)^{-1}\theta_c(a)\mathfrak{d}(x)$ for $a \in e_{\det}H_c e_{\det}$.

The isomorphism $\mathscr{L}_{c-1} \xrightarrow{\sim} \mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x)v_{c-1})$ of Lemma 3.11 is compatible with f, and we sometimes view $\mathscr{L}_{c-1}$ as an $(e_{\det}H_c e_{\det} \otimes \mathscr{D}_{\mathfrak{g}\times V})$-module.

By Lemma 3.2, the image of the morphism

$$e_{\det}H_c e \otimes_{eH_c e} \mathscr{L}_c|_{\mathfrak{g}_{\text{reg}}\times V} \to \mathscr{M}_c|_{\mathfrak{g}_{\text{reg}}\times V}, \qquad a \otimes u_c \mapsto av_c$$

is contained in $\mathscr{D}_{\mathfrak{g}_{\text{reg}}\times V}(\mathfrak{d}(x)v_c)$. It follows from Lemma 3.11 that over $\mathfrak{g}_{\text{reg}} \times V$, the composite morphism $e_{\det}H_c e \otimes_{eH_c e} \mathscr{L}_c \to \mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ factors through a morphism

$$\varphi\colon e_{\det}H_c e \otimes_{eH_c e} \mathscr{L}_c|_{\mathfrak{g}_{\text{reg}}\times V} \longrightarrow \mathscr{L}_{c-1} \otimes \det(V)|_{\mathfrak{g}_{\text{reg}}\times V}. \tag{3.14}$$

Similarly, we have the morphism

$$\begin{aligned}\psi : eH_c e_{\det} \otimes_{e_{\det}H_c e_{\det}} \mathscr{L}_{c-1} \otimes \det(V)|_{\{q(A,z)\neq 0\}} &\to \mathscr{L}_c|_{\{q(A,z)\neq 0\}},\\ a \otimes u_{c-1} \otimes l &\mapsto \big(a\mathfrak{d}(x)\big)q(A,z)^{-1}u_c.\end{aligned} \tag{3.15}$$

The morphism φ is linear over $e_{\det}H_c e_{\det} \simeq eH_{c-1}e$, and the morphism ψ is linear over $eH_c e$. We have

$$\varphi\big(\mathfrak{d}(x)e \otimes u_c\big) = q(A,z)u_{c-1} \otimes l$$

and

$$q(A,z)\psi\big(\mathfrak{d}(x)e_{\det} \otimes u_{c-1} \otimes l\big) = \mathfrak{d}^2(A)u_c,$$

where $\mathfrak{d}^2(A)$ is the discriminant of the characteristic polynomial of A.

Note that the following diagrams commute on $\mathfrak{g}_{\mathrm{reg}} \times V \cap \{q(A, z) \neq 0\}$:

$$\begin{array}{ccc} eH_c e_{\det} \underset{e_{\det} H_c e_{\det}}{\otimes} e_{\det} H_c e \underset{eH_c e}{\otimes} \mathscr{L}_c & \longrightarrow & eH_c e \underset{eH_c e}{\otimes} \mathscr{L}_c \\ \downarrow \varphi & & \downarrow \mathrm{can} \\ eH_c e_{\det} \underset{e_{\det} H_c e_{\det}}{\otimes} \big(\mathscr{L}_{c-1} \otimes \det(V)\big) & \xrightarrow{\ \psi\ } & \mathscr{L}_c \end{array} \tag{3.16}$$

and

$$\begin{array}{ccc} e_{\det} H_c e \underset{eH_c e}{\otimes} eH_c e_{\det} \underset{e_{\det} H_c e_{\det}}{\otimes} \big(\mathscr{L}_{c-1} \otimes \det(V)\big) & \longrightarrow & e_{\det} H_c e_{\det} \underset{e_{\det} H_c e_{\det}}{\otimes} \big(\mathscr{L}_{c-1} \otimes \det(V)\big) \\ \downarrow \psi & & \downarrow \mathrm{can} \\ e_{\det} H_c e \underset{eH_c e}{\otimes} \mathscr{L}_c & \xrightarrow{\ \varphi\ } & \mathscr{L}_{c-1} \otimes \det(V) \end{array} \tag{3.17}$$

PROPOSITION 3.12
The morphism φ extends uniquely to a morphism of $\mathscr{D}_{\mathfrak{g}\times V}$-modules:

$$\varphi\colon e_{\det} H_c e \otimes_{eH_c e} \mathscr{L}_c \longrightarrow \mathscr{L}_{c-1} \otimes \det(V). \tag{3.18}$$

The proof proceeds by reduction to rank 2. Recall that $\mathfrak{g}_1$ denotes the open subset of $\mathfrak{g}$ of matrices with at least $(n-1)$ distinct eigenvalues. Then $\mathfrak{g} \setminus \mathfrak{g}_1$ is a closed subset of $\mathfrak{g}$ of codimension 2.

We prove first the following lemma.

LEMMA 3.13
After restriction to $\mathfrak{g}_1 \times V$, we have an inclusion of submodules of $\mathscr{M}_{c-1}$,

$$H_c \mathscr{D}_{\mathfrak{g}\times V} \overline{v}_c \subset \mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1},$$

where $\overline{v}_c = q(A, z)v_{c-1}$.

Proof
Since $H_c = \mathbb{C}[\mathfrak{t}]\mathbb{C}[\mathfrak{t}^*]\mathbb{C}[W]$, it is enough to show that

$$\mathbb{C}[\mathfrak{t}^*]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c \in \mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1} \quad \text{on } \mathfrak{g}_1 \times V. \tag{3.19}$$

Here, the action of $\mathbb{C}[\mathfrak{t}^*]$ is through $\mathbb{C}[\mathfrak{t}^*] \hookrightarrow H_c \xrightarrow{\theta_c} \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$.

Let us assume first that $n = 2$. We have

$$q(A, z) = -A_{21}z_1^2 + (A_{11} - A_{22})z_1z_2 + A_{12}z_2^2.$$

We put

$$q(\partial_A, z) = -z_1^2\partial_{A_{12}} + z_1z_2(\partial_{A_{11}} - \partial_{A_{22}}) + z_2^2\partial_{A_{21}}.$$

We show that

$$(\partial_{x_1} - \partial_{x_2})q(A, z)v_{c-1} = -q(\partial_A, z)(x_1 - x_2)v_{c-1}. \tag{3.20}$$

This is an equality in the $\mathscr{D}_{\mathfrak{g}\times V}$-submodule $\iota(\mathscr{L}_{c-1})$ of $\mathscr{M}_{c-1}$. Note that $(y_1 - y_2)v_{c-1} = (\partial_{x_1} - \partial_{x_2})v_{c-1}$.

By §3.2.3, we have

$$v_{c-1} = q(A, z)^{c-1}\delta\big(x_1 + x_2 - \mathrm{tr}(A)\big)\delta\big(x_1x_2 - \det(A)\big).$$

Since $q(\partial_A, z)q(A, z) = q(\partial_A, z)\,\mathrm{tr}(A) = 0$ and $q(\partial_A, z)\det(A) = -q(A, z)$, we obtain

$$q(\partial_A, z)v_{c-1} = q(A, z)^c\delta\big(x_1 + x_2 - \mathrm{tr}(A)\big)\delta'\big(x_1x_2 - \det(A)\big).$$

On the other hand, we have

$$(\partial_{x_1} - \partial_{x_2})q(A, z)v_{c-1} = (x_2 - x_1)q(A, z)^c\delta\big(x_1 + x_2 - \mathrm{tr}(A)\big)\delta'\big(x_1x_2 - \det(A)\big).$$

Equality (3.20) then follows.

We assume now that $n \geqslant 2$. Let S be the locally closed subset of $\mathfrak{g}$ of matrices

$$\begin{pmatrix} A' & 0 & 0 & \cdots & \\ 0 & a_3 & 0 & \cdots & \\ 0 & 0 & a_4 & & \\ \vdots & \vdots & & \ddots & \\ & & & & a_n \end{pmatrix},$$

where A' is a (2×2)-matrix, $a_i \neq a_j$ $(3 \leqslant i < j \leqslant n)$, and a_i is not an eigenvalue of A' for $3 \leqslant i \leqslant n$. Let $\mathfrak{t}_1 = \mathfrak{t} \cap S = \{x \in \mathfrak{t}\,; x_i \neq x_j \text{ for } i < j \text{ and } 3 \leqslant j\}$. Let $x' = (x_1, x_2)$, let $x'' = (x_3, \ldots, x_n)$, and let $a'' = (a_3, \ldots, a_n)$.

We have $G \cdot S = \mathfrak{g}_1$. Let $i\colon S \times V \hookrightarrow \mathfrak{g} \times V$ be the inclusion map. Then i is noncharacteristic for $\mathscr{L}_c$ and $\mathscr{M}_{c-1}$ because we have $T_xS + T_x(G \cdot x) = T_x\mathfrak{g}$ for any $x \in S$.

Denote by $\mathfrak{g}'$ the subalgebra of $\mathfrak{g}$ of matrices (A_{ij}) with $A_{ij} = 0$ whenever $i > 2$ or $j > 2$. We identify $\mathfrak{g}'$ with $\mathfrak{gl}_2(\mathbb{C})$. Given an object $\mathscr{X}$ defined earlier for $\mathfrak{g}$, we denote by $\mathscr{X}'$ the corresponding objects for $\mathfrak{g}'$ (i.e., the case where $n = 2$). For example, W' is the subgroup of W generated by s_{12}.

Let $i'' : \mathfrak{t} \times S \to \mathfrak{t} \times \mathfrak{g}$ be the embedding. We have an isomorphism of $\mathscr{D}_{\mathfrak{t}\times S}$-modules compatible with the action of W (cf. proof of Lemma 3.2):

$$i''^{*}\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta(x, A) \xrightarrow[\sim]{\delta(x,A)\mapsto\sum_w T_w^*\mathfrak{d}_1(A',a'')^{-1}\delta(x',A')\delta(x''-a'')} \bigoplus_{w\in W'\backslash W} T_w^*\mathscr{D}_{\mathfrak{t}\times S}\delta(x', A')\delta(x''-a'').$$

Here, T_w is the automorphism of $\mathfrak{t}$ given by w, and $\mathfrak{d}_1(A', a'') = \mathfrak{d}(a'')\prod_{i=3}^{n}\det(a_i I_2 - A')$, $\delta(x', A') = \delta(x_1 + x_2 - \operatorname{tr}(A'))\delta(x_1x_2 - \det(A'))$.

Let $A \in S$. We have

$$q(A, z) = q'(A', z') \cdot q_1(A, z),$$

where

$$q_1(A, z) = (z_3 \cdots z_n)\mathfrak{d}_1(A', a'').$$

Note that $\mathfrak{d}_1(A', a'')$ is invertible on S.

Let $p : \mathfrak{t}_{\mathrm{reg}} \times S \times V \to S \times V$ be the projection. We have a $(\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \otimes \mathscr{D}_{S\times V})$-linear isomorphism compatible with the action of W:

$$i^*\mathscr{M}_c \xrightarrow[\sim]{v_c\mapsto e\otimes\widetilde{v}_c} \mathbb{C}[W] \otimes_{\mathbb{C}[W']} p_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c), \tag{3.21}$$

where $\widetilde{v}_c = v'_c\, q_1(A, z)^c\, \mathfrak{d}_1(A', a'')^{-1}\, \delta(x'' - a'')$ with $v'_c = q'(A', z')^c\, \delta(x', A')$. Note that s_{12} acts trivially on $\widetilde{v}_c$. The action of $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$ on $\mathbb{C}[W] \otimes_{\mathbb{C}[W']} p_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c)$ is given by

$$(a \otimes w)(w' \otimes s) = (ww') \otimes \big(((ww')^{-1}a)s\big) \quad \text{for } w, w' \in W, a \in \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}), s \in p_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c).$$

Note that $\mathscr{D}_{S\times V}\widetilde{v}_c$ is stable by $\mathbb{C}[\mathfrak{t}_1]^{W'}$ as a submodule of $p_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c)$. Since $\mathbb{C}[\mathfrak{t}_1] = \mathbb{C}[\mathfrak{t}]\mathbb{C}[\mathfrak{t}_1]^{W'}$, $\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c$ is stable by $\mathbb{C}[\mathfrak{t}_1]$.

Let us still denote by $\widetilde{v}_c = q(A, z)\widetilde{v}_{c-1}$ the image of $\widetilde{v}_c$.

Let us set $\widetilde{y}_1 = \partial_{x_1} - c(x_1 - x_2)^{-1}(1 - s_{12})$ and $\widetilde{y}_2 = \partial_{x_2} - c(x_2 - x_1)^{-1}(1 - s_{12})$ as partial Dunkl operators, and let R be the algebra generated by $\widetilde{y}_1$, $\widetilde{y}_2$, and ∂_{x_i} ($i = 3, \ldots, n$). Then s_{12} acts on R by the permutation of $\widetilde{y}_1$ and $\widetilde{y}_2$. We have $R = R^{W'} \oplus (\widetilde{y}_1 - \widetilde{y}_2)R^{W'}$.

Let

$$\begin{aligned}\tilde{\mathscr{N}} &= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + (\widetilde{y}_1 - \widetilde{y}_2)\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c \\ &= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(\widetilde{y}_1 - \widetilde{y}_2)\widetilde{v}_c \\ &= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(\partial_{x_1} - \partial_{x_2})\widetilde{v}_c\end{aligned}$$

be a submodule of $p_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_{c-1})$. Since $(\widetilde{y}_1 + \widetilde{y}_2)\widetilde{v}_c$, $\widetilde{y}_1\widetilde{y}_2\widetilde{v}_c$, and $\partial_{x_i}\widetilde{v}_c$ $(i = 3, \ldots, n)$ belong to $\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c$ (cf. Lemma 3.3), $\tilde{\mathscr{N}}$ is invariant by R.

Set $\mathscr{N} = \mathbb{C}[W] \otimes_{\mathbb{C}[W']} \tilde{\mathscr{N}}$. Let us show that $\mathscr{N}$ is invariant by the action of $\mathbb{C}[\mathfrak{t}^*] \subset H_c \subset \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$. For any i, we have

$$y_i(w\otimes t) = w \otimes \partial_{x_{w^{-1}(i)}} t - c\sum_{k\neq i} w(1 + s_{w^{-1}(i),w^{-1}(k)}) \otimes (x_{w^{-1}(i)} - x_{w^{-1}(k)})^{-1} t$$

for any $w \in W$ and $t \in \tilde{\mathscr{N}}$. Since $(x_a - x_b)^{-1} \in \mathbb{C}[\mathfrak{t}_1]$ when a or b is in $\{3, \ldots, n\}$, we have $y_i(w \otimes t) \in \mathscr{N}$ when $w^{-1}(i) \neq 1, 2$. If $w^{-1}(i) = 1$, then

$$\begin{aligned} y_i(w\otimes t) &\equiv w\otimes \partial_{x_1} t - cw(1 + s_{12}) \otimes (x_1 - x_2)^{-1} t \quad (\mathrm{mod}\,\mathscr{N}) \\ &= w \otimes \widetilde{y}_1 t \in \mathscr{N}.\end{aligned}$$

The case of $w^{-1}(i) = 2$ is similar. Hence, we have shown that $\mathscr{N}$ is invariant by $\mathbb{C}[\mathfrak{t}^*]$. Thus, we obtain

$$\mathbb{C}[\mathfrak{t}^*](e \otimes \widetilde{v}_c) \subset \mathscr{N}.$$

The study of rank 2 above (i.e., (3.20)) shows that

$$(\widetilde{y}_1 - \widetilde{y}_2)\widetilde{v}_c \subset \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(x_1 - x_2)\widetilde{v}_{c-1}.$$

Hence, we obtain

$$\tilde{\mathscr{N}} \subset \tilde{\mathscr{N}}' := \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\mathfrak{d}(x)\widetilde{v}_{c-1},$$

which implies that

$$\mathbb{C}[\mathfrak{t}^*](e \otimes \widetilde{v}_c) \subset \mathscr{N}' := \mathbb{C}[W] \otimes \tilde{\mathscr{N}}'. \tag{3.22}$$

We have a commutative diagram, where the horizontal map is an isomorphism,

$$\begin{array}{ccc} W \times_{W'} \mathfrak{t}_1 & \xrightarrow[\sim]{(w,x)\mapsto(w(x),x)} & \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}_1/W' \\ & {\scriptstyle (w,x)\mapsto w(x)} \searrow \quad \swarrow {\scriptstyle (x,x')\mapsto x} & \\ & \mathfrak{t} & \end{array}$$

The diagram above is W-equivariant for the action of $g \in W$ given by

$$g \cdot (w, x) = (gw, x) \quad \text{for } (w, x) \in W \times_{W'} \mathfrak{t}_1,$$
$$g \cdot (x, x') = \big(g(x), x'\big) \quad \text{for } (x, x') \in \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}_1/W'.$$

It follows that we have an isomorphism of $\mathbb{C}[\mathfrak{t}]$-modules

$$\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_1]^{W'} \xrightarrow{\sim} \mathbb{C}[W] \otimes_{\mathbb{C}[W']} \mathbb{C}[\mathfrak{t}_1],$$
$$a \otimes a' \mapsto \sum_{w \in W/W'} w \otimes w^{-1}(a)a'.$$

In particular, we have $\mathbb{C}[W] \otimes_{\mathbb{C}[W']} \mathbb{C}[\mathfrak{t}_1] = \mathbb{C}[\mathfrak{t}] \cdot (e \otimes \mathbb{C}[\mathfrak{t}_1]^{W'})$. Since $\mathbb{C}[\mathfrak{t}_1]^{W'} \widetilde{v}_c \subset \mathscr{D}_{S \times V} \widetilde{v}_c$ and $\mathbb{C}[\mathfrak{t}_1]^{W'} \mathfrak{d}(x) \widetilde{v}_{c-1} \subset \mathscr{D}_{S \times V} \mathfrak{d}(x) \widetilde{v}_{c-1}$, we deduce that

$$\mathscr{N}' = \mathbb{C}[\mathfrak{t}]\big(e \otimes \mathscr{D}_{S \times V} \widetilde{v}_c + e \otimes \mathscr{D}_{S \times V} \mathfrak{d}(x) \widetilde{v}_{c-1}\big).$$

Together with (3.22), we obtain

$$\mathbb{C}[\mathfrak{t}^*] \mathscr{D}_{S \times V}(e \otimes \widetilde{v}_c) \subset \mathbb{C}[\mathfrak{t}]\big(\mathscr{D}_{S \times V}(e \otimes \widetilde{v}_c) + \mathscr{D}_{S \times V}(e \otimes \mathfrak{d}(x) \widetilde{v}_{c-1})\big).$$

Via the isomorphism (3.21), this shows that

$$i^*\big(\mathbb{C}[\mathfrak{t}^*] \mathscr{D}_{\mathfrak{g} \times V} \overline{v}_c\big) \subset i^*\big(\mathbb{C}[\mathfrak{t}] \mathscr{D}_{\mathfrak{g} \times V} \overline{v}_c + \mathbb{C}[\mathfrak{t}] \mathscr{D}_{\mathfrak{g} \times V} \mathfrak{d}(x) v_{c-1}\big).$$

Since $\mu^{-1}(0) \cap T^*_{S \times V}(\mathfrak{g} \times V) \subset T^*_{\mathfrak{g} \times V}(\mathfrak{g} \times V)$, the noncharacteristic condition implies the desired result (3.19) (cf. §3.1.2). □

Proof of Proposition 3.12
By Lemma 3.13, we have, on $\mathfrak{g}_1 \times V$,

$$\begin{aligned} e_{\det} H_c \mathscr{D}_{\mathfrak{g} \times V} \overline{v}_c &\subset e_{\det} \mathbb{C}[\mathfrak{t}] \mathscr{D}_{\mathfrak{g} \times V} \overline{v}_c + e_{\det} \mathbb{C}[\mathfrak{t}] \mathscr{D}_{\mathfrak{g} \times V} \mathfrak{d}(x) v_{c-1} \\ &\subset \mathbb{C}[\mathfrak{t}]^W \mathfrak{d}(x) \mathscr{D}_{\mathfrak{g} \times V} \overline{v}_c + \mathbb{C}[\mathfrak{t}]^W \mathscr{D}_{\mathfrak{g} \times V} \mathfrak{d}(x) v_{c-1} = \mathscr{D}_{\mathfrak{g} \times V} \mathfrak{d}(x) v_{c-1} \end{aligned}$$

since $e_{\det} \mathbb{C}[\mathfrak{t}] e = \mathbb{C}[\mathfrak{t}]^W \mathfrak{d}(x) e$ and $e_{\det} \mathbb{C}[\mathfrak{t}] e_{\det} = \mathbb{C}[\mathfrak{t}]^W e_{\det}$. Hence, φ extends to a morphism defined on $\mathfrak{g}_1 \times V$. Then the desired result follows from $\mathscr{H}^1_{(\mathfrak{g} \setminus \mathfrak{g}_1) \times V}(\mathscr{L}_{c-1}) = 0$ (see Lemma 3.1). □

4. Cherednik algebras and Hilbert schemes

4.1. Geometry of the Hilbert scheme
4.1.1
We refer to [23] and [11] for basic results on Hilbert schemes of points on $\mathbb{C}^2$.

Let us recall that

$$\mathfrak{X} = \big\{(A, B, z, \zeta) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^*; \mathbb{C}\langle A, B\rangle z = V\big\}$$

is the set of stable points for the action of G on $T^*(\mathfrak{g} \times V)$, relative to the character det of G. The group G acts freely on $\mathfrak{X}$. Let $\mu_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{g}$ be the moment map

$$\mu_{\mathfrak{X}}(A, B, z, \zeta) = -[A, B] - z \circ \zeta.$$

It is a smooth morphism. Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme classifying zero-dimensional closed subschemes of $\mathbb{C}^2$ with length n. Then we have an isomorphism $\mathrm{Hilb}^n(\mathbb{C}^2) \xrightarrow{\sim} \mu_{\mathfrak{X}}^{-1}(0)/G$. Note that we have $\zeta = 0$ on $\mu_{\mathfrak{X}}^{-1}(0)$ (cf. [7, Lemma 2.3]).

We write Hilb instead of $\mathrm{Hilb}^n(\mathbb{C}^2)$ for short. Let us denote by $p\colon \mu_{\mathfrak{X}}^{-1}(0) \to \mathrm{Hilb}$ the quotient map.

Let us recall the construction of p. For $(A, B, z, \zeta) \in \mu_{\mathfrak{X}}^{-1}(0)$, we regard V as a $\mathbb{C}[X, Y]$-module by $X \mapsto A$ and $Y \mapsto B$. Then z gives an epimorphism $\mathbb{C}[X, Y] \twoheadrightarrow V$ of $\mathbb{C}[X, Y]$-modules. Hence, V gives a closed subscheme of $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[X, Y])$ of length n, which is the corresponding point of Hilb.

Let $\pi\colon \mathrm{Hilb} \to (\mathfrak{t} \times \mathfrak{t}^*)/W$ be the Hilbert-Chow morphism. Then Hilb is a resolution of singularities of $(\mathfrak{t} \times \mathfrak{t}^*)/W \simeq (\mathbb{C}^2)^n/S_n$, the scheme of n unordered points in $\mathbb{C}^2$. We have canonical isomorphisms

$$\Gamma\big(\mu_{\mathfrak{X}}^{-1}(0), \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}\big)^G \xrightarrow{\sim} \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}}) \xrightarrow{\sim} \Gamma\big((\mathfrak{t}\times\mathfrak{t}^*)/W, \mathscr{O}_{(\mathfrak{t}\times\mathfrak{t}^*)/W}\big) \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}\times\mathfrak{t}^*]^W.$$

Let $(\mathfrak{t} \times \mathfrak{t}^*)_{\mathrm{reg}}$ be the open subset of $\mathfrak{t} \times \mathfrak{t}^*$ where the action of W is free. The Hilbert-Chow morphism π is an isomorphism over $(\mathfrak{t} \times \mathfrak{t}^*)_{\mathrm{reg}}/W$. Let $E := \pi^{-1}\big(((\mathfrak{t} \times \mathfrak{t}^*) \setminus (\mathfrak{t} \times \mathfrak{t}^*)_{\mathrm{reg}})/W\big)$ be the exceptional divisor. It is a closed irreducible hypersurface of Hilb. The line bundle L on Hilb associated with the G-equivariant line bundle $\mathscr{O}_{\mathfrak{X}} \otimes \det(V)$ on $\mathfrak{X}$ is a very ample line bundle on Hilb.

Let us set

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} = \big\{\phi(p) \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]; \phi(gp) = \det(g)\phi(p) \text{ for any } g \in G\big\}.$$

This algebra is isomorphic to $\Gamma(\mathrm{Hilb}, L) \simeq (\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)] \otimes \det(V))^G$. Let $i\colon \mathfrak{t} \times \mathfrak{t}^* \times V \hookrightarrow \mathfrak{g} \times \mathfrak{g} \times V \times V^*$ be the embedding with the last component $\zeta = 0$. Then $i^{-1}(\mu_{\mathfrak{X}}^{-1}(0))$ contains $(\mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}^* \cup \mathfrak{t} \times \mathfrak{t}^*_{\mathrm{reg}}) \times (\mathbb{C}^*)^n$. For any $\phi \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det}$, we have $(i^*\phi)(x, y, gz) = \det(g)(i^*\phi)(x, y, z)$ for any invertible diagonal matrix g. Hence, we have

$$(i^*\phi)(x, y, z) = a(x, y)(z_1 \cdots z_n)$$

for some rational function $a(x, y)$ that is regular on $(\mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}^*) \cup (\mathfrak{t} \times \mathfrak{t}^*_{\mathrm{reg}})$, an open subset of $\mathfrak{t} \times \mathfrak{t}^*$ with complement of codimension 2. Hence, we have

$$a(x, y) \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det} = \big\{a \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]; wa = \det(w)a \text{ for any } w \in W\big\}.$$

Thus, we obtain a map that is known to be an isomorphism (cf., e.g., [7, Proposition 8.2.1]), and we denote its inverse by i_d:

$$\begin{aligned} \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} \otimes \det(V) &\xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}, \\ \phi \otimes l &\mapsto \langle l, z_1 \wedge \cdots \wedge z_n \rangle a. \end{aligned} \tag{4.1}$$

Similarly, we have an isomorphism (cf., e.g., [7, Lemma 2.7.3]) whose inverse we denote by i_s:

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G} \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W}. \tag{4.2}$$

Summarizing, we have the isomorphisms

$$\begin{aligned} i_d\colon \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det} &\xrightarrow{\sim} \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} \otimes \det(V) \simeq \Gamma(\mathrm{Hilb}, L), \\ i_s\colon \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W} &\xrightarrow{\sim} \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G} \simeq \mathscr{O}_{\mathrm{Hilb}}(\mathrm{Hilb}). \end{aligned} \tag{4.3}$$

4.1.2

For a subset Y of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ with cardinality n, set $p_Y = \det(x_k^i y_k^j)_{(i,j)\in Y,\, k=1,\dots,n} \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}$ and $s_Y(A, B, z, \zeta) = \det(A^i B^j z)_{(i,j)\in Y} \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} = L(\mathrm{Hilb})$. Then $\{p_Y\}_Y$ is a basis of $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}$ as a vector space, and $i_d(p_Y) = s_Y$. The $\mathscr{O}_{\mathrm{Hilb}}$-module L is generated by $\{s_Y\}_Y$, where Y ranges over the set of Young diagrams of size n. Here, we regard a Young diagram Y as a subset of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ so that $(i, j) \in Y$ as soon as $(i, j+1)$ or $(i+1, j)$ belongs to Y.

There is a canonical global section $\tau \in \Gamma(\mathrm{Hilb}; L^{\otimes -2})$ satisfying the following property:

$$i_d(a_1)i_d(a_2)\tau = i_s(a_1a_2) \quad \text{for any } a_1, a_2 \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}. \tag{4.4}$$

Note that τ is identified with a function on $\mu_{\mathfrak{X}}^{-1}(0)$ such that $\tau(gp) = \det(g)^{-2}\tau(p)$ ($p \in \mu_{\mathfrak{X}}^{-1}(0)$ and $g \in G$).

The exceptional divisor E coincides with the set of zeros of τ, and we obtain an isomorphism

$$L^{\otimes 2} \xrightarrow{\sim} \mathscr{O}_{\mathrm{Hilb}}(-E).$$

Let us denote by $\mathfrak{d}^2(A)$ the discriminant of the characteristic polynomial of A, and similarly for $\mathfrak{d}^2(B)$. Then we have

$$\begin{aligned} i_d\big(\mathfrak{d}(x)\big) &= q(A, z), & i_d\big(\mathfrak{d}(y)\big) &= q(B, z), \\ i_s\big(\mathfrak{d}(x)^2\big) &= \mathfrak{d}^2(A), & i_s\big(\mathfrak{d}(y)^2\big) &= \mathfrak{d}^2(B). \end{aligned}$$

Hence, we have

$$\mathfrak{d}^2(A) = q(A, z)^2\tau \qquad \text{and} \qquad \mathfrak{d}^2(B) = q(B, z)^2\tau.$$

LEMMA 4.1

(i) *The hypersurface of* $\mu_{\mathfrak{X}}^{-1}(0)$ *defined by* $q(A, z) = 0$ *is irreducible, and* $p^{-1}E \cap \{q(A, z) = 0\}$ *is of codimension* 2 *in* $\mu_{\mathfrak{X}}^{-1}(0)$.

(ii) *The hypersurface of* $\mu_{\mathfrak{X}}^{-1}(0)$ *defined by* $\mathfrak{d}^2(A) = 0$ *is* $p^{-1}E \cup \{q(A, z) = 0\}$.

(iii) *The intersection* $\mu_{\mathfrak{X}}^{-1}(0) \cap \{q(A, z) = q(B, z) = 0\}$ *is of codimension* 2 *in* $\mu_{\mathfrak{X}}^{-1}(0)$.

Note that (i) follows from the fact that $q(A, z)$ does not vanish on the irreducible hypersurface $p^{-1}E$ of $\mu_{\mathfrak{X}}^{-1}(0)$, and $q(A, z)$ is irreducible on $\mu_{\mathfrak{X}}^{-1}(0) \setminus p^{-1}E$. Statement (iii) follows from [11, Lemma 3.6.2].

4.2. W*-algebras on the Hilbert scheme*

4.2.1

In §4.1, we have regarded $\mathfrak{X}$, Hilb, and so on, as schemes. Hereafter, we regard them as complex manifolds. Note that the previous constructions and results would remain valid in the analytic category. Let $\mathscr{W}_{\mathfrak{X}}$ be the $\mathscr{W}$-algebra on $\mathfrak{X}$ associated with $\mathscr{D}_{\mathfrak{g}\times V}$. Denoting by $\pi: \mathfrak{X} \to \mathfrak{g} \times V$ the projection, we have a ring homomorphism $\pi^{-1}\mathscr{D}_{\mathfrak{g}\times V} \to \mathscr{W}_{\mathfrak{X}}$ respecting the order filtration. The ring $\mathscr{W}_{\mathfrak{X}}$ is flat over $\pi^{-1}\mathscr{D}_{\mathfrak{g}\times V}$. The action of G on $\mathfrak{g} \times V$ induces an action of G on $\mathscr{W}_{\mathfrak{X}}$, and there is a quantized moment map $\mu_{\mathscr{W}}: \mathfrak{g} \to \mathscr{W}_{\mathfrak{X}}$.

We have morphisms

$$\kappa_0: \mathbb{C}[\mathfrak{t}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \to \mathscr{W}_{\mathfrak{X}}(\mathfrak{X})$$

and

$$\kappa_1: \mathbb{C}[\mathfrak{t}^*]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathscr{D}_{\mathfrak{g}}(\mathfrak{g}) \to \mathscr{W}_{\mathfrak{X}}(\mathfrak{X}).$$

Note that $\kappa_1(\mathbf{y}^2) = \Delta_{\mathfrak{g}}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, let $\mathbb{C}[\mathfrak{t}^*]^W_k$ be the homogeneous part of $\mathbb{C}[\mathfrak{t}^*]^W$ of degree k. Then κ_0 sends $\mathbb{C}[\mathfrak{t}]^W$ to $\mathscr{W}_{\mathfrak{X}}(0)$ and κ_1 sends $\mathbb{C}[\mathfrak{t}^*]^W_k$ to $\mathscr{W}_{\mathfrak{X}}(k)$, and we have the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{t}]^W & \xrightarrow{\kappa_0} & \mathscr{W}_{\mathfrak{X}}(0) \\ & \searrow & \downarrow \sigma_0 \\ & & \mathscr{O}_{\mathfrak{X}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{C}[\mathfrak{t}^*]^W_k & \xrightarrow{\kappa_1} & \mathscr{W}_{\mathfrak{X}}(k) \\ & \searrow^{\hbar^{-k}} & \downarrow \sigma_k \\ & & \hbar^{-k}\mathscr{O}_{\mathfrak{X}} \end{array} \qquad (4.5)$$

Let us consider $\mathscr{W}_{\mathfrak{X}} \otimes_{\mathscr{D}_{\mathfrak{g}\times V}} \mathscr{L}_c$, which we denote by the same letter $\mathscr{L}_c$. With the notation of §2.4.2, we have $\mathscr{L}_c = \Phi_{c\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$. Hence, $\mathscr{L}_c$ is a twisted G-equivariant $\mathscr{W}_{\mathfrak{X}}$-module with twist $c\,\mathrm{tr}$. Let u_c be the canonical section of $\mathscr{L}_c$, and set $\mathscr{L}_c(m) = \mathscr{W}_{\mathfrak{X}}(m)u_c$. Then we have an isomorphism

$$\mathscr{L}_c(0)/\mathscr{L}_c(-1) \xrightarrow{\sim} \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}.$$

The support of $\mathscr{L}_c$ is $\mu_{\mathfrak{X}}^{-1}(0)$. The $\mathscr{W}_{\mathfrak{X}}$-module $\mathscr{L}_c$ has a left action of eH_ce by Lemma 3.3. Via the anti-involution $h \mapsto h^*$ of H_c, we regard $\mathscr{L}_c$ as a $(\mathscr{W}_{\mathfrak{X}}, eH_ce)$-bimodule. Similarly, $\mathscr{L}_{c-1}$ has a structure of $(\mathscr{W}_{\mathfrak{X}}, e_{\det}H_ce_{\det})$-bimodule (see Lemma 3.11). These actions are explicitly given by

$$\begin{aligned} u_cea &= \kappa_0(a)u_c \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W \subset H_c, \\ u_ceb &= \kappa_1(b)u_c \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W \subset H_c; \end{aligned} \tag{4.6}$$

$$\begin{aligned} u_{c-1}e_{\det}a &= \kappa_0(a)u_{c-1} \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W \subset H_c, \\ u_{c-1}e_{\det}b &= \kappa_1(b)u_{c-1} \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W \subset H_c. \end{aligned} \tag{4.7}$$

Since $\mu_{\mathfrak{X}}^{-1}(0)$ is smooth, we have

$$\mathscr{E}xt^j_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c, \mathscr{W}_{\mathfrak{X}}) = 0 \quad \text{for } j \neq \operatorname{codim}_{\mathfrak{X}}\big(\mu_{\mathfrak{X}}^{-1}(0)\big).$$

Hence, for any closed subset $S \subset \mu_{\mathfrak{X}}^{-1}(0)$, we have, by Lemma 2.1,

$$\mathscr{H}^j_S(\mathscr{L}_c) = 0 \quad \text{for } j < \operatorname{codim}_{\mu_{\mathfrak{X}}^{-1}(0)} S. \tag{4.8}$$

In (3.18) and (3.15), we defined the following morphisms:

$$\varphi\colon \mathscr{L}_c \underset{eH_ce}{\otimes} eH_ce_{\det} \longrightarrow \mathscr{L}_{c-1} \otimes \det(V) \tag{4.9}$$

and

$$\psi\colon \big(\mathscr{L}_{c-1} \otimes \det(V)\big) \underset{e_{\det}H_ce_{\det}}{\otimes} e_{\det}H_ce|_{\{q(A,z)\neq 0\}} \longrightarrow \mathscr{L}_c|_{\{q(A,z)\neq 0\}}.$$

PROPOSITION 4.2
The morphism ψ extends uniquely to a morphism defined on $\mathfrak{X}$.

Proof
We have

$$q(A,z)\psi(u_{c-1} \otimes a) = u_c \cdot \big(\mathfrak{d}(x)a\big)$$

for any $a \in e_{\det}H_ce$.

Now, let us show that

$$\bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^k q(A, z)\bigr)\psi(u_{c-1} \otimes a) = u_c \cdot \bigl((\mathrm{ad}(\mathbf{y}^2)^k \mathfrak{d}(x))a\bigr) \tag{4.10}$$

holds on $\{q(A, z) \neq 0\}$ by the induction on k.

We have

$$\begin{aligned}&\bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^k q(A, z)\bigr)\psi(u_{c-1} \otimes a)\\ &\quad= \Delta_{\mathfrak{g}}\bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\psi(u_{c-1} \otimes a) - \bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\Delta_{\mathfrak{g}}\psi(u_{c-1} \otimes a).\end{aligned}$$

The first term is calculated as

$$\begin{aligned}\Delta_{\mathfrak{g}}\bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\psi(u_{c-1} \otimes a) &= \Delta_{\mathfrak{g}} u_c \cdot \bigl((\mathrm{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a\bigr)\\ &= u_c \mathbf{y}^2 \cdot \bigl((\mathrm{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a\bigr)\\ &= u_c \cdot \bigl(\mathbf{y}^2(\mathrm{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a\bigr).\end{aligned}$$

The second term is calculated as

$$\begin{aligned}\bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\Delta_{\mathfrak{g}}\psi(u_{c-1} \otimes a) &= \bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\psi(\Delta_{\mathfrak{g}} u_{c-1} \otimes a)\\ &= \bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\psi(u_{c-1}\mathbf{y}^2 \otimes a)\\ &= \bigl(\mathrm{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A, z)\bigr)\psi(u_{c-1} \otimes \mathbf{y}^2 a)\\ &= u_c \cdot \bigl((\mathrm{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))\mathbf{y}^2 a\bigr).\end{aligned}$$

Hence, we obtain (4.10). In particular, letting k be $n(n-1)/2$, the degree of $\mathfrak{d}(x)$, and using the fact that $\mathrm{ad}(\Delta_{\mathfrak{g}})^{n(n-1)/2} q(A, z)$ is equal to $q(\partial_A, z)$ up to a constant multiple (see, e.g., (3.2) and the sentence below), we obtain

$$q(\partial_A, z)\psi(u_{c-1} \otimes a) = u_c \cdot \bigl(\mathfrak{d}(y)a\bigr). \tag{4.11}$$

Hence, $\psi(u_{c-1} \otimes a)$ extends to a section of $\mathscr{L}_c$ outside $q(B, z) = 0$.

Thus, we have shown that $\psi(u_{c-1} \otimes a)$ is a section defined outside $\{q(A, z) = 0\} \cap \{q(B, z) = 0\}$. Since $\{q(A, z) = 0\} \cap \{q(B, z) = 0\} \cap \mu_{\mathfrak{X}}^{-1}(0)$ is of codimension 2 in $\mu_{\mathfrak{X}}^{-1}(0)$ (see Lemma 4.1), it follows that $\psi(u_{c-1} \otimes a)$ extends to a global section of $\mathscr{L}_c$ by (4.8). □

Remark 4.3

(i) So, we have obtained a structure of the $((e + e_{\det})H_c(e + e_{\det}))$-module on $\mathscr{L}_c \oplus \mathscr{L}_{c-1} \otimes \det(V)$.

(ii) We have

$$\varphi\big(u_c \otimes e\mathfrak{d}(x)\big) = q(A, z)u_{c-1},$$
$$\varphi\big(u_c \otimes e\mathfrak{d}(y)\big) = q(\partial_A, z)u_{c-1}$$

and (4.12)

$$q(A, z)\psi(u_{c-1} \otimes a) = u_c \cdot \big(\mathfrak{d}(x)a\big),$$
$$q(\partial_A, z)\psi(u_{c-1} \otimes a) = u_c \cdot \big(\mathfrak{d}(y)a\big),$$

for $a \in e_{\det} H_c e$.

(iii) Diagrams (3.16) and (3.17) commute on $\mathfrak{X}$.

By Propositions 3.12 and 4.2 and Remark 4.3(iii), we obtain the following proposition (see Proposition 3.8).

PROPOSITION 4.4
Assume that condition (3.12) holds. Then we have isomorphisms of twisted G-equivariant $\mathscr{W}_{\mathfrak{X}}$-modules with twist $c\,\mathrm{tr}$:

$$\varphi\colon \mathscr{L}_c \otimes_{eH_ce} eH_ce_{\det} \xrightarrow{\sim} \mathscr{L}_{c-1} \otimes \det(V)$$

and

$$\psi\colon \big(\mathscr{L}_{c-1} \otimes \det(V)\big) \underset{e_{\det}H_ce_{\det}}{\otimes} e_{\det}H_ce \xrightarrow{\sim} \mathscr{L}_c.$$

4.2.2
Let us consider

$$\mathscr{A}_c = \big(p_*(\mathscr{E}nd_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c))^G\big)^{\mathrm{opp}}.$$

It is a W-algebra on Hilb by Proposition 2.8. Let $\mathscr{A}_c(0)$ be the subring of sections of order at most zero. For $m \in \mathbb{Z}$, $\mathscr{L}_{c+m} \otimes \det(V)^{\otimes -m}$ belongs to $\mathrm{Mod}^G_{c\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$ (cf. (2.4)). Set

$$\mathscr{A}_{c,c+m} = \big(p_*\mathscr{H}om_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c, \mathscr{L}_{c+m} \otimes \det(V)^{\otimes -m})\big)^G.$$

Then $\mathscr{A}_{c,c+m}$ is an $(\mathscr{A}_c, \mathscr{A}_{c+m})$-bimodule. Let $\mathscr{A}_{c,c+m}(0) = \big(p_*\mathscr{H}om_{\mathscr{W}_{\mathfrak{X}}(0)}(\mathscr{L}_c(0), \mathscr{L}_{c+m}(0) \otimes \det(V)^{\otimes -m})\big)^G$. Then $\mathscr{A}_{c,c+m}(0)$ is an $\mathscr{A}_c(0)$-lattice of $\mathscr{A}_{c,c+m}$ and $\mathscr{A}_{c,c+m}(0)/\mathscr{A}_{c,c+m}(-1) \simeq L^{\otimes -m}$, the associated line bundle on Hilb to $\mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)} \otimes \det(V)^{\otimes -m}$ (cf. Proposition 2.8(iii)).

4.3. Affinity of $\mathscr{A}_c$

4.3.1

As an application of Theorem 2.9, we obtain the following vanishing theorem.

THEOREM 4.5

Assume that condition (3.12) holds for $c + m$ (for all $m \in \mathbb{Z}_{>0}$).

(i) *For any good $\mathscr{A}_c$-module $\mathscr{M}$, $\varprojlim_K H^i(K, \mathscr{M}) = 0$ for $i > 0$. Here, K ranges over compact subsets of* Hilb.

(ii) *Any good $\mathscr{A}_c$-module $\mathscr{M}$ is generated by global sections on any compact subset of* Hilb.

Proof

By Proposition 4.4, for any $m > 0$, $\mathscr{L}_{c+m}$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m-1} \otimes \det(V)$, and $\mathscr{L}_{c+m-1} \otimes \det(V)$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m}$ in the category $\mathrm{Mod}^G_{(c+m)\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$. Hence, $\mathscr{L}_{c+m} \otimes \det(V)^{\otimes -m}$ is a direct summand of a direct sum of copies of $\mathscr{L}_c$, and $\mathscr{L}_c$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m} \otimes \det(V)^{\otimes -m}$ in the category $\mathrm{Mod}^G_{c\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$ for any $m > 0$. It follows that $\mathscr{A}_{c,c+m}$ is a direct summand of a direct sum of copies of $\mathscr{A}_c$, and $\mathscr{A}_c$ is a direct summand of a direct sum of copies of $\mathscr{A}_{c,c+m}$ for any $m > 0$. Moreover, $\mathscr{A}_{c,c+m}$ is a good $\mathscr{A}_c$-module whose symbol is $L^{\otimes -m}$.

Theorem 2.9 now gives the conclusion. □

4.3.2

Let us give an F-action on $\mathscr{W}_{\mathfrak{X}}$ by $\mathscr{F}_t(A_{ij}) = tA_{ij}$, $\mathscr{F}_t(\partial_{A_{ij}}) = t^{-1}\partial_{A_{ij}}$, $\mathscr{F}_t(z_i) = tz_i$, $\mathscr{F}_t(\partial_{z_i}) = t^{-1}\partial_{z_i}$, and $\mathscr{F}_t(\hbar) = t^2\hbar$ for $t \in \mathbb{G}_{\mathrm{m}} = \mathbb{C}^\times$. Since $B_{ij} = \sigma_0(\hbar\partial_{A_{ji}})$ and $\zeta_i = \sigma_0(\hbar\partial_{z_i})$, the corresponding action of $\mathbb{G}_{\mathrm{m}}$ on $\mathfrak{X}$ is $T_t((A, B, z, \zeta)) = (tA, tB, tz, t\zeta)$. Its induced $\mathbb{G}_{\mathrm{m}}$-action on Hilb coincides with the action induced by the scalar $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{C}^2$. We define the F-action on $\mathscr{L}_c$ by $\mathscr{F}_t(u_c) = u_c$.

Note that

$$\begin{aligned}\mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}])}(\mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}]) &\simeq \mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}_{T^*(\mathfrak{g}\times V)}[\hbar^{1/2}])}(\mathscr{W}_{T^*(\mathfrak{g}\times V)}[\hbar^{1/2}]) \\ &\simeq \mathbb{C}[\hbar^{-1/2}A_{ij}, \hbar^{1/2}\partial_{A_{ij}}, \hbar^{-1/2}z_i, \hbar^{1/2}\partial_{z_i}] \simeq \mathscr{D}(\mathfrak{g} \times V).\end{aligned}$$

The F-action on $\mathscr{W}_{\mathfrak{X}}$ is compatible with the G-action on $\mathscr{W}$, and hence, $\mathscr{A}_c$ is also a W-algebra on Hilb with F-action (cf. Proposition 2.8(iv)). We define the F-action on $\mathscr{L}_{c-1} \otimes \det(V)$ by $\mathscr{F}_t(u_{c-1} \otimes l) = t^{-n}u_{c-1} \otimes l$. Hence, $\mathscr{A}_{c,c-1}$ has a structure of $\mathscr{A}_c$-module with F-action.

4.3.3

The $((e+e_{\det})H_c(e+e_{\det}))^{\mathrm{opp}}$-module structure on $\mathscr{L}_c \oplus (\mathscr{L}_{c-1} \otimes \det(V))$ gives a ring homomorphism

$$(e+e_{\det})H_c(e+e_{\det}) \xrightarrow{\alpha} \mathrm{End}_{\mathscr{A}_c}(\mathscr{A}_c \oplus \mathscr{A}_{c,c-1})^{\mathrm{opp}}.$$

Since it is not compatible with the F-action, we modify α.

Set

$$\widetilde{\mathscr{A}_c} = \mathscr{A}_c[\hbar^{1/2}] \qquad \text{and} \qquad \widetilde{\mathscr{A}_{c,c-1}} = \mathscr{A}_{c,c-1}[\hbar^{1/2}].$$

Let $H_c \xrightarrow{\beta} \mathbf{k}[\hbar^{1/2}] \otimes_{\mathbb{C}} H_c$ be the ring homomorphism given by $x_i \mapsto \hbar^{-1/2} \otimes x_i$, $y_i \mapsto \hbar^{1/2} \otimes y_i$, $w \mapsto 1 \otimes w$ ($w \in W$).

LEMMA 4.6

The composition

$$\Phi\colon (e+e_{\det})H_c(e+e_{\det}) \xrightarrow{\beta} \mathbf{k}[\hbar^{1/2}] \otimes_{\mathbb{C}} (e+e_{\det})H_c(e+e_{\det}) \xrightarrow{\alpha} \mathrm{End}_{\widetilde{\mathscr{A}_c}}(\widetilde{\mathscr{A}_c} \oplus \widetilde{\mathscr{A}_{c,c-1}})^{\mathrm{opp}}$$

sends $(e+e_{\det})H_c(e+e_{\det})$ *to* $\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c} \oplus \widetilde{\mathscr{A}_{c,c-1}})^{\mathrm{opp}}$.

Proof

First, let us show that Φ sends eH_ce to $\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$. For a homogeneous element $a \in \mathbb{C}[\mathfrak{t}]^W$ of degree k, $\Phi(ae)(u_c) = \hbar^{-k/2}\tilde{a}(A)u_c$, where $\tilde{a}(A)$ is the element of $\mathbb{C}[\mathfrak{g}]^G$ such that $\tilde{a}|_{\mathfrak{t}} = a$. Since $\tilde{a}(A)$ is also homogeneous of degree k, $\hbar^{-k/2}\tilde{a}(A)$ is $\mathscr{F}$-invariant, and $\Phi(ae)$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. On the other hand, we have $\Phi(\mathbf{y}^2e)(u_c) = \hbar\Delta_{\mathfrak{g}}u_c$, and $\hbar\Delta_{\mathfrak{g}}$ is $\mathscr{F}$-invariant. Hence, $\Phi(\mathbf{y}^2e)$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Since eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^We$ and $\mathbf{y}^2e$, we have $\Phi(eH_ce) \subset \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})$.

Similarly, we have $\Phi(e_{\det}H_ce_{\det}) \subset \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_{c,c-1}})$.

Let us show that $\Phi(e\mathfrak{d}(x)) \in \mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}})$. This follows from $\Phi(e\mathfrak{d}(x))(u_c) = \hbar^{-n(n-1)/4}q(A,z)u_{c-1} \otimes l$, $\mathscr{F}_t(q(A,z)) = t^{n+n(n-1)/2}q(A,z)$, and $\mathscr{F}_t(u_{c-1} \otimes l) = t^{-n}u_{c-1} \otimes l$.

For $a \in e_{\det}H_ce$, let us show that $\Phi(a)\colon \widetilde{\mathscr{A}_{c,c-1}} \to \widetilde{\mathscr{A}_c}$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Since $\Phi(ae\mathfrak{d}(x))$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$ and $\Phi(e\mathfrak{d}(x))|_{\{q(A,z)\neq 0\}}$ is an isomorphism in the category $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c}|_{\{q(A,z)\neq 0\}})$, it follows that $\Phi(a)|_{\{q(A,z)\neq 0\}}$ is in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c}|_{\{q(A,z)\neq 0\}})$. Hence, we conclude that $\Phi(a)$ is in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Similarly, one shows that $\Phi(eH_ce_{\det})$ is contained in $\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}})$. □

In particular, we obtain a morphism of algebras

$$eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}.$$

We denote by $\tilde{\varphi}$ and $\tilde{\psi}$ the modified morphisms in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$ given in Lemma 4.6:

$$\tilde{\varphi} : \widetilde{\mathscr{L}_c} \otimes_{eH_ce} eH_ce_{\det} \longrightarrow \widetilde{\mathscr{L}_{c-1}} \otimes \det(V),$$
$$\tilde{\psi} : \big(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)\big) \otimes_{e_{\det}H_ce_{\det}} e_{\det}H_ce \longrightarrow \widetilde{\mathscr{L}_c}.$$

We define the order filtration $F(eH_ce)$ on eH_ce by assigning order $1/2$ to x_i and y_i. Then the morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is compatible with the order filtrations, and the symbol map $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W \simeq \mathrm{Gr}^F(eH_ce) \to \mathrm{Gr}^F\,\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}) \subset \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})[\hbar^{\pm 1/2}]$ coincides with $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W_k \xrightarrow{\hbar^{-k} i_s} \hbar^{-k}\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})$ by (4.5). Here, $k \in \mathbb{Z}/2$.

LEMMA 4.7
The morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is an isomorphism.

Proof
Note that the subspace $\mathrm{Gr}^F\,\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}) \subset \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})[\hbar^{\pm 1/2}]$ is contained in $\bigoplus_{k\in\mathbb{Z}/2} \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k \hbar^{-k}$, where $\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k$ is the homogeneous part of weight $2k$ with respect to the $\mathbb{G}_{\mathrm{m}}$-action. Hence, we have a chain of morphisms

$$\begin{aligned}&\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W \xrightarrow{\sim} \mathrm{Gr}^F(eH_ce)\\ &\quad \to \mathrm{Gr}^F\big(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}\big) \hookrightarrow \bigoplus_{k\in\mathbb{Z}/2} \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k \hbar^{-k} \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W.\end{aligned}$$

Since the composition is the identity, the map $\mathrm{Gr}^F(eH_ce) \to \mathrm{Gr}^F(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}})$ is bijective. Hence, the morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is an isomorphism. Note that $\bigcap_k F_k(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})) = 0$. □

Remark 4.8
A similar argument shows that there is an isomorphism

$$eH_ce_{\det} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}})$$

(see §4.4).

Let $\mathrm{o} \in (\mathfrak{t} \times \mathfrak{t}^*)/W$ be the image of the origin of $\mathfrak{t} \times \mathfrak{t}^*$. Then the Hilbert-Chow morphism $\pi : \mathrm{Hilb} \to (\mathfrak{t} \times \mathfrak{t}^*)/W$ is $\mathbb{C}^\times$-equivariant, and every point of $(\mathfrak{t} \times \mathfrak{t}^*)/W$ shrinks to o.

Now, the following theorem is a consequence of Theorem 2.10.

THEOREM 4.9
Assume that condition (3.12) holds for $c + m$ for all $m \in \mathbb{Z}_{>0}$. (This is the case if $c \notin (1/n!)\mathbb{Z}_{<0}$.) We have quasi-inverse equivalences of categories between $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})$

and $\mathrm{Mod}_{\mathrm{coh}}(eH_ce)$,

$$\begin{aligned} \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c}) &\xleftrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(eH_ce), \\ \mathscr{M} &\mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}), \\ \widetilde{\mathscr{A}_c} \otimes_{eH_ce} M &\leftarrow\!\shortmid M. \end{aligned}$$

Under this equivalence, $\widetilde{\mathscr{A}_c}$ and $\widetilde{\mathscr{A}_{c,c-1}}$ correspond to eH_ce and eH_ce_{det}, respectively.

THEOREM 4.10
Assume that condition (3.12) holds for $c+m$ (for all $m \in \mathbb{Z}_{>0}$). Assume also that condition (3.11) holds. (These assumptions are satisfied if $c \notin (1/n!)\mathbb{Z}_{<0}$.) Let $\mathscr{B}_c = \mathscr{E}nd_{\widetilde{\mathscr{A}_c}}(\widetilde{\mathscr{A}_c} \otimes_{eH_ce} eH_c)^{\mathrm{opp}}$. We have quasi-inverse equivalences of categories between $\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{B}_c)$ and $\mathrm{Mod}_{\mathrm{coh}}(H_c)$,

$$\begin{aligned} \mathrm{Mod}_F^{\mathrm{good}}(\mathscr{B}_c) &\xleftrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(H_c), \\ \mathscr{M} &\mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{B}_c)}(\mathscr{B}_c, \mathscr{M}), \\ \mathscr{B}_c \otimes_{H_c} M &\leftarrow\!\shortmid M. \end{aligned}$$

Remark 4.11
It would be very interesting to have a more direct construction of $\widetilde{\mathscr{A}_c} \otimes_{eH_ce} eH_c$.

4.4. W*-algebras as fractions of eH_ce*

We explain how sections of $\widetilde{\mathscr{A}_c}$ over open subsets of Hilb can be obtained by inverting elements in the Cherednik algebra.

Let $\{F_j(H_c)\}_{j\in\mathbb{Z}/2}$ be the filtration of H_c consisting of elements of order $\leqslant j$, where we give order $1/2$ to x_i, y_i and order zero to $w \in W$. Then we have a canonical isomorphism $\sigma : \mathrm{Gr}^F(H_c) \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*] \rtimes W$. We have induced filtrations on eH_ce and eH_ce_{det}, and σ induces the isomorphisms

$$\begin{aligned} \mathrm{Gr}^F(eH_ce) &\xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W, \\ \mathrm{Gr}^F(eH_ce_{\mathrm{det}}) &\xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\mathrm{det}}. \end{aligned}$$

Composing with the morphism $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*] \to \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*][\hbar^{-1/2}]$ given by $a(x,y) \mapsto a(\hbar^{-1/2}x, \hbar^{-1/2}y)$, we obtain the homomorphisms

$$\begin{aligned} \mathrm{Gr}^F(eH_ce) &\longrightarrow \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W[\hbar^{-1/2}], \\ \mathrm{Gr}^F(eH_ce_{\mathrm{det}}) &\longrightarrow \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\mathrm{det}}[\hbar^{-1/2}]. \end{aligned}$$

We set $\widetilde{\mathscr{W}_{\mathfrak{X}}} = \mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}]$ and $\widetilde{\mathscr{W}_{\mathfrak{X}}}(0) = \mathscr{W}_{\mathfrak{X}}(0) + \hbar^{1/2}\mathscr{W}_{\mathfrak{X}}(0)$. We set $\widetilde{\mathscr{L}_c} = \widetilde{\mathscr{W}_{\mathfrak{X}}} \otimes_{\mathscr{W}_{\mathfrak{X}}} \mathscr{L}_c$. Then $\widetilde{\mathscr{L}_c} \oplus \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$ has a structure of the $(\widetilde{\mathscr{W}_{\mathfrak{X}}}, (e + e_{\det})H_c(e+e_{\det}))$-bimodule. The action of $eH_ce_{\det}$ is given by $\tilde{\varphi} \colon \widetilde{\mathscr{L}_c} \otimes_{eH_ce} eH_ce_{\det} \to \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$. On the other hand, we have canonical isomorphisms $\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \simeq \mathrm{Gr}^F(\widetilde{\mathscr{L}_{c-1}}) \xrightarrow{\sim} \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}]$. Here, $F(\widetilde{\mathscr{L}_c})$ (resp., $F(\widetilde{\mathscr{L}_{c-1}})$) is the order filtration given by $F_k(\widetilde{\mathscr{L}_c}) = \hbar^{-k}\widetilde{\mathscr{W}_{\mathfrak{X}}}(0)u_c$ (resp., $F_k(\widetilde{\mathscr{L}_{c-1}}) = \hbar^{-k}\widetilde{\mathscr{W}_{\mathfrak{X}}}(0)u_{c-1}$) for $k \in \mathbb{Z}/2$.

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \otimes \mathrm{Gr}^F(eH_ce) & \longrightarrow & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W[\hbar^{-1/2}] \\
\downarrow & & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c} \otimes eH_ce) & & \downarrow i_s \\
\downarrow & & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) & \xrightarrow{\sim} & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}]
\end{array} \tag{4.13}$$

The morphism $\tilde{\varphi}$ is order-preserving, and we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \otimes \mathrm{Gr}^F(eH_ce_{\det}) & \longrightarrow & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}[\hbar^{-1/2}] \\
\downarrow & & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c} \otimes eH_ce_{\det}) & & \downarrow i_d \\
\downarrow \tilde{\varphi} & & \\
\mathrm{Gr}^F\big(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)\big) & \xrightarrow{\sim} & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \det(V)
\end{array} \tag{4.14}$$

Hence, for any $a \in eH_ce_{\det}$, the morphism $a \colon \widetilde{\mathscr{L}_c} \to \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$ is an isomorphism on $\{i_d(\sigma(a)) \neq 0\}$. Then, for $b \in eH_ce_{\det}$, we can define

$$ba^{-1} \in \mathrm{End}_{\mathrm{Mod}^G_{F,c\,\mathrm{tr}}(\widetilde{\mathscr{W}_X}|_{\{i_d(\sigma(a))\neq 0\}})}(\widetilde{\mathscr{L}_c}\,|_{\{i_d(\sigma(a))\neq 0\}})^{\mathrm{opp}}$$

as the composition

$$\begin{array}{ccc}
\widetilde{\mathscr{L}_c} & \overset{b}{\searrow} & \\
{\scriptstyle ba^{-1}}\,\vdots\downarrow & & \widetilde{\mathscr{L}_{c-1}} \otimes \det(V) \\
\widetilde{\mathscr{L}_c} & \underset{a}{\nearrow} &
\end{array}$$

Thus, we obtain ba^{-1} as an F-invariant section of $\widetilde{\mathscr{A}_c}$ defined on $\{i_d(\sigma(a)) \neq 0\}$. Note that $ba^{-1} = bc(ac)^{-1}$ for a nonzero element $c \in e_{\det}H_ce$. Note also that the image of $ac \in eH_ce$ in $\Gamma(\mathrm{Hilb}; \widetilde{\mathscr{A}_c})$ is invertible only on $\{i_d(\sigma(a)) \neq 0\} \cap \{i_d(\sigma(c)) \neq 0\} \cap (\mathrm{Hilb} \setminus E)$.

Remark 4.12
The morphism $\tilde{\psi} : (\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)) \otimes_{e_{\det} H_c e_{\det}} e_{\det} H_c e \to \widetilde{\mathscr{L}_c}$ is also order-preserving, and it induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}^F\big(\widetilde{\mathscr{L}}_{c-1} \otimes \det(V)\big) \otimes \mathrm{Gr}^F(e_{\det} H_c e) & \longrightarrow & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \det(V) \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W, \det}[\hbar^{-1/2}] \\
\downarrow & & \\
\mathrm{Gr}^F\big(\widetilde{\mathscr{L}}_{c-1} \otimes \det(V) \otimes e_{\det} H_c e\big) & & \downarrow {\scriptstyle \tau \cdot i_d} \\
\downarrow {\scriptstyle \tilde{\psi}} & & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}}_c) & \xrightarrow{\sim} & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}]
\end{array}$$

Hence, for any $b \in e_{\det} H_c e$, the morphism $b \colon \widetilde{\mathscr{L}_{c-1}} \otimes \det(V) \to \widetilde{\mathscr{L}_c}$ is never an isomorphism on the exceptional divisor E.

4.5. Rank 2 case
Let us consider the case where $n = 2$. Let $x_0 = x_1 + x_2$, $x = x_1 - x_2$, $y_0 = (y_1 + y_2)/2$, and $y = (y_1 - y_2)/2 \in H_c$. Then $[y_0, x_0] = 1$, $[y, x] = 1 - 2cs$, where $s = s_{12}$. Since y, x, and s commute with $\mathbb{C}[x_0, y_0]$, we have an isomorphism of algebras $\mathbb{C}[x_0, y_0] \otimes H'_c \xrightarrow{\sim} H_c$, where H'_c is the subalgebra of H_c generated by x, y, and s.

We have

$$eH_c e_{\det} H_c e = eH_c e \iff H_c e_{\det} H_c = H_c \iff c \neq \frac{1}{2},$$

$$e_{\det} H_c e H_c e_{\det} = e_{\det} H_c e_{\det} \iff H_c e H_c = H_c \iff c \neq -\frac{1}{2}.$$

Indeed, the first equivalences follow from the fact that $y e_{\det} x - x e_{\det} y = e[y, x] = (1-2c)e$, and when $c = 1/2$, there is a one-dimensional representation with $x, y \mapsto 0$, $s \mapsto 1$. The second follows from the first by the isomorphism $H_c \simeq H_{-c}$ given by $s \mapsto -s$. It follows that condition (3.12) is satisfied for all $c + n$ ($n \in \mathbb{Z}_{>0}$) if and only if $c \neq -1/2, -3/2, \ldots$.

Note that $x, y \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W, \det}$ and $\mathrm{Hilb} = \{i_d(x) \neq 0\} \cup \{i_d(y) \neq 0\}$ because $\mu_{\mathfrak{X}}^{-1}(0) \cap \{q(A, z) = q(B, z) = 0\} \subset \{(A, B, z, 0) \in \mathfrak{X}\,;\, Az, Bz \in \mathbb{C}z\} = \emptyset$. Quantized symplectic coordinates of $\widetilde{\mathscr{A}_c}$ are given by

$$\Big((ey)(ex)^{-1}, \hbar^{1/2} e x_0; -\frac{\hbar e x^2}{2}, \hbar^{1/2} e y_0\Big) \quad \text{on } \big\{i_d(x) \neq 0\big\}$$

and

$$\Big((ex)(ey)^{-1}, \hbar^{1/2}ex_0; \frac{\hbar ey^2}{2}, \hbar^{1/2}ey_0\Big) \quad \text{on } \big\{i_d(y) \neq 0\big\}.$$

Indeed, we have $[-ex^2/2, (ey)(ex)^{-1}] = e$ because

$$(ey)(ex)^{-1}(ex^2) = (ey)(ex)^{-1}(ex)(e_{\det}x) = eyx$$

and

$$\begin{aligned}(ex^2)(ey)(ex)^{-1} &= (ex^2y)(ex)^{-1} = (eyx^2 - 2ex)(ex)^{-1} \\ &= (eyx)(ex)(ex)^{-1} - 2e = eyx - 2e.\end{aligned}$$

Note that this provides an isomorphism $\mathrm{Hilb} \xrightarrow{\sim} T^*(\mathbb{P}^1 \times \mathbb{C})$. The projection $\mathrm{Hilb} \to \mathbb{P}^1$ is given by $[i_d(x) : i_d(y)]$ with the notation of homogeneous coordinates. By the isomorphism above, we have $E \simeq T^*_{\mathbb{P}^1}\mathbb{P}^1 \times T^*\mathbb{C}$.

Note that $(xe)^{-1}(ye)$ is invertible only on $\{i_s(x^2) \neq 0\} = \{i_d(x) \neq 0\} \setminus E$ for $c \neq -1/2$ because $exyx = ex(xy + 1 - 2cs) = ex^2y + (1 + 2c)ex$ and $(xe)^{-1}(ye) = (x^2e)^{-1}(xye) = (ex^2)^{-1}(exyx)(ex)^{-1} = (ey)(ex)^{-1}+(1+2c)(ex^2)^{-1}$.

Set $(a, \partial_a) = ((ey)(ex)^{-1}, -ex^2/2)$, set $(b, \partial_b) = ((ex)(ey)^{-1}, ey^2/2)$, and set $\lambda = c - 1/2$. Then we have

$$b = a^{-1} \qquad \text{and} \qquad \partial_b = -a(a\partial_a - \lambda). \tag{4.15}$$

Indeed, we have

$$\begin{aligned}-a(a\partial_a - \lambda) &= (ey)(ex)^{-1}\Big(\frac{(ey)(ex)^{-1}(ex^2)}{2} + c - \frac{1}{2}\Big) \\ &= \frac{1}{2}(ey)(ex)^{-1}(eyx + 2c - 1) = \frac{1}{2}(ey)(ex)^{-1}(exy) = \frac{ey^2}{2}.\end{aligned}$$

Recall that $\mathrm{o} \in (\mathfrak{t} \times \mathfrak{t}^*)/W$ is the image of the origin of $\mathfrak{t} \times \mathfrak{t}^*$. The inverse image $\pi^{-1}(\mathrm{o})$ by the Hilbert-Chow morphism π is $T^*_{\mathbb{P}^1}\mathbb{P}^1 \times \{0\} \subset T^*\mathbb{P}^1 \times T^*\mathbb{C}$. We identify it with $\mathbb{P}^1$. Then (4.15) gives an isomorphism

$$\mathscr{E}nd_F(\widetilde{\mathscr{A}_c})|_{\pi^{-1}(\mathrm{o})} \xrightarrow{\sim} \mathscr{D}_{\mathbb{P}^1,\lambda} \otimes \mathbb{C}[x_0, y_0]$$

with $\lambda = c - 1/2$. Here, $\mathscr{D}_{\mathbb{P}^1,\lambda}$ is the twisted ring of differential operators (e.g., see [16, §2]). If λ is an integer, then $\mathscr{D}_{\mathbb{P}^1,\lambda} \simeq \mathscr{O}_{\mathbb{P}^1}(\lambda) \otimes \mathscr{D}_{\mathbb{P}^1} \otimes \mathscr{O}_{\mathbb{P}^1}(-\lambda)$. Hence, we have a ring isomorphism $eH'_ce \simeq \Gamma(\mathbb{P}^1; \mathscr{D}_{\mathbb{P}^1,\lambda})$ and an equivalence $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c}) \simeq \mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_{\mathbb{P}^1,\lambda} \otimes \mathbb{C}[x_0, y_0])$. It is well known (cf., e.g., [16, §7]) that $\mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_{\mathbb{P}^1,\lambda})$ is equivalent to $\mathrm{Mod}_{\mathrm{coh}}(\Gamma(\mathbb{P}^1; \mathscr{D}_{\mathbb{P}^1,\lambda}))$ if and only if $\lambda \neq -1, -2, \ldots$ (i.e., $c \neq -1/2, -3/2, \ldots$).

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