MODULAR REPRESENTATIONS OF FINITE GROUPS AND LIE THEORY

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In memory of Jacques Tits

Abstract. This article discusses the modular representation theory of finite groups of Lie type from the viewpoint of Broué's abelian defect group conjecture. We discuss both the defining characteristic case, the inspiration for Alperin's weight conjecture, and the non-defining case, the inspiration for Broué's conjecture. The modular representation theory of general finite groups is conjectured to behave both like that of finite groups of Lie type in defining characteristic, and in non-defining characteristic, to a large extent.

The expected behaviour of modular representation theory of finite groups of Lie type in defining characteristic is particularly difficult to grasp along the lines of Broué's conjecture, and we raise a new question related to the change of central character.

We introduce a degeneration method in the modular representation theory of finite groups of Lie type in non-defining characteristic. Combined with the rigidity property of perverse equivalences, this provides a setting for two-variable decomposition matrices, for large characteristic. This should help make progress towards finding decomposition matrices, an outstanding problem with few general results beyond the case of general linear groups. This last part is based on joint work with David Craven and Olivier Dudas.

Contents

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1. Introduction

Every finite simple group is a finite group of Lie type, an alternating group, a cyclic group of prime order, or one of the 26 sporadic groups. This provides a central role for finite groups of Lie type in finite group theory.

Conjectures of Alperin and Broué predict that the modular representation theory of general finite groups shares many features with that of finite groups of Lie type. Alperin's prediction is inspired by finite groups of Lie type in defining characteristic. On the other hand, Broué predicts a behaviour similar to that of finite groups of Lie type in non-defining characteristic.

For simple finite groups of Lie type in defining characteristic, the assumptions of Broué's conjecture (abelian defect groups) are only satisfied for groups of type A_1, like $\mathrm{PSL}_2(\mathbf{F}_q)$, outside cases of simple blocks. Broué's conjecture is known to hold in that case, but the combinatorics involved in the proof have so far not been understood within the usual Lie theoretic or geometric framework for SL_2. An important problem is to find a proof of Broué's conjecture for $\mathrm{SL}_2(\mathbf{F}_q)$ that relates to the geometry associated with the group. A major open problem is to find an extension of Broué's conjecture that removes the assumption on Sylow subgroups, and understanding Broué's conjecture for defining characteristic representations of $\mathrm{SL}_2(\mathbf{F}_q)$ could lead to understanding how the conjecture should extend to higher rank groups, and eventually to all finite groups. Broué's conjecture is about the existence of certain equivalences of derived categories.

There are few known equivalences between blocks of finite groups of Lie type in defining characteristic. We could locate two: the derived equivalence between the principal block of $\mathrm{SL}_2(q)$ and the principal block of a Borel subgroup, and a similar result for the non-simple non-principal block when p is odd. In particular, the two non-simple blocks are derived equivalent. We propose to consider a generalization of this situation. A particular case of that extension applies to $G = \mathrm{SL}_r(q)$, r a prime dividing $q-1$: the non-simple blocks all have the same number of simple modules and the corresponding blocks for proper local subgroups are isomorphic.

Progress in the understanding of modular representations of finite groups of Lie type in non-defining characteristic has mostly been achieved by extending some of the work of Lusztig (and Deligne-Lusztig) about characteristic 0 representations to characteristic ℓ. Broué's conjecture in this case has a formulation in terms of Deligne-Lusztig theory. The main difficulty in proving the conjecture has been about obtaining information about individual cohomology groups of

Deligne-Lusztig varieties, rather than about their alternating sum. In particular, a key required vanishing property is still open in general, even for characteristic 0 coefficients.

The introduction of the notion of perverse equivalences and the conjecture that the equivalences expected from Deligne-Lusztig varieties should be perverse (joint work with Joe Chuang) lead to the fact that torus and Weyl group data determine the decomposition matrices for large enough characteristic, using the conjectural combinatorial perversity function of Craven, and a rigidity property of perverse equivalences (joint work with David Craven).

We explain two ways in which toroidal structures appear by degeneration in the modular representation theory of finite groups of Lie type in non-defining characteristic. We give a "global" topological construction using a limit of completed classifying spaces, the starting point being Friedlander's description of the completed classifying space of a finite group of Lie type in terms of homotopy fixed points on the classifying space of the corresponding Lie group. We provide also an explicit local algebraic construction. These lead to conjectural two-variable decomposition matrices for large characteristic (joint work with Olivier Dudas).

In part 2, we consider general finite groups. We review p-local group theory and p-local representation theory and discuss Alperin and Broué's conjectures. We introduce perverse equivalences, a type of derived equivalences between derived categories with filtrations, that induces abelian equivalences up to shifts on the slices of the filtration.

Part 3 introduces finite groups of Lie type as fixed points of a Frobenius endomorphism, or a more general Steinberg endomorphism of a reductive algebraic group. We discuss the p-local structure of finite groups of Lie type, both in defining and non-defining characteristic.

Part 4 is devoted to the modular representation theory of finite groups of Lie type in defining characteristic. We provide an explanation for Alperin's conjecture. In the case of groups of finite Lie rank 1, we discuss the relation between representations of the group and of a Borel subgroup. We analyze next when a block with a given central character has the same number of simple modules as the principal block and raise the question of understanding the relation between the module categories of such blocks.

Parts 5 and 6 are concerned with the modular representation theory of finite groups of Lie type in non-defining characteristic ℓ. We start with a discussion of Deligne-Lusztig varieties and endomorphisms coming from braid groups. We review Lusztig's theory of characteristic 0 representations and describe modular counterparts. We finish §5 with a discussion of the particular form of Broué's conjecture for finite groups of Lie type in non-defining characteristic.

Part 6 discusses two approaches to generic phenomena. The first approach is based on the description of ℓ-completed classifying spaces of finite groups of Lie type in terms of fixed points under unstable Adams operations and we discuss the rigidification of a certain limit of those Adams operation, in relation with classifying spaces of loop groups. The second approach is based on a degeneration of the group algebra of the local block, and the relation with the rigidity of perverse equivalences. For general linear and unitary groups, there is a further relation with Hilbert schemes of points on surfaces.

We gather in the appendix a number of basic facts on representations of algebras and finite groups, in particular in relation with various types of equivalences. We give a very succinct survey of basic constructions involving complex reflection groups, braid groups and Hecke algebras.

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2. Finite groups

2.1. **Group theory.**

2.1.1. *Classification of finite simple groups.* Every finite simple group is one of the following [Asch, §47]

- a cyclic group of prime order
- an alternating group $\mathfrak{A}_n$ for $n \geq 5$
- a finite simple group of Lie type
- one of the 26 sporadic simple groups, with orders ranging from 7920 for the Mathieu group M_{11} discovered in 1861 to about 8×10^{53} for the Fischer-Griess monster discovered in 1973.

Finite groups of Lie type govern to a large extent the structure of general finite groups.

2.1.2. *p-local group theory.* Let G be a finite group, p a prime and P a Sylow p-subgroup of G.

The p-local group theory is the study of G using *p-local subgroups*, i.e. subgroups of the form $N_G(Q)$ for Q a non-trivial p-subgroup.

This was originally developed mostly in the case $p = 2$ and this underlies the proof of the classification of finite simple groups.

Here are some examples of results of p-local group theory. We denote by $O_{p'}(G)$ the largest normal subgroup of G of order prime to p.

- (Burnside 1897) When P is abelian, two elements of P that are conjugate in G are also conjugate in $N_G(P)$ [Go, Chap.7, Theorem 1.1].
- (Frobenius) If $N_G(Q)/C_G(Q)$ is a p-group for every non trivial subgroup Q of P, then $G = O_{p'}(G) \rtimes P$ [Go, Chap. 7, Theorem 4.5].
- (Brauer-Fowler 1955) If G is simple and $s \in G$ is an involution, then $|G| \leq (2|C_G(s)|^2)!$ [Asch, (45.4)].
- (Brauer-Suzuki 1959) If the Sylow 2-subgroups of G are quaternion groups, then G is not simple [Co, §3.3].
- (Glauberman 1966, case $p = 2$) If x is an element of order p of P that is not G-conjugate to any other element of P, then $G = O_{p'}(G)C_G(x)$ [Co, Appendix].
- (Alperin 1967) One can tell if two elements of P are conjugate in G using only p-local subgroups [Asch, (38.1)].

Glauberman's Theorem (which generalizes Brauer-Suzuki's Theorem) holds also for odd primes, but the proof for odd primes uses the classification of finite simple groups. Modular representation theory of finite groups was developed by Brauer as a tool for studying finite groups. For example, the proof of the Brauer-Suzuki Theorem uses representation theory in characteristic 2.

It is hoped that modular representation theory will eventually reach a point where it can be used to obtain a direct proof of Glauberman's Theorem for odd primes and lead to simplifications of the proof of the classification of finite simple groups.

Modular representation theory leads to a generalization of local group theory, where Sylow subgroups are replaced by defect groups of blocks. A major theme of modular representation theory is to relate modular representations of a group and its local subgroups, sometimes leading to versions of "factorization" results like Frobenius's Theorem above replaced by an equivalence between module categories [BrouPu].

2.2. p-local representation theory.

2.2.1. p-local representation theory is concerned with the study of representations of G over $\mathbf{Z}_p$ and $\mathbf{F}_p$ (or finite extensions of those) in relation with p-local subgroups and their representations.

It involves character-theoretic aspects, in particular the value of complex characters of G on elements whose order is divisible by p. It involves also the study of simple and indecomposable representations and mod-p cohomology.

2.2.2. Let p be a prime number and $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbf{Q}_p$. Let k be the residue field of $\mathcal{O}$.

Let G be a finite group. The category kG-mod is not semisimple if p divides $|G|$, but it still splits as direct sum of indecomposable full abelian subcategories. This is induced by a corresponding decomposition of $\mathcal{O}G$-mod. That decomposition comes from a decomposition of 1 as a sum of orthogonal primitive idempotents $1 = \sum_b b$ of $Z(\mathcal{O}G)$, the block idempotents. We have an algebra decomposition into blocks $\mathcal{O}G = \prod_b \mathcal{O}Gb$ and a category decomposition $\mathcal{O}G\text{-mod} = \bigoplus_b \mathcal{O}Gb\text{-mod}$. We will still denote by b the image of the idempotent in $Z(kGb)$ and we have corresponding decompositions of kGb and kGb-mod.

The *principal block* $\mathcal{O}Gb_0$ of $\mathcal{O}G$ is the one such that b_0 does not act by 0 on the trivial representation.

We will always assume that K contains all $|G|$-th roots of unity.

A *defect group* of a block $\mathcal{O}Gb$ is a minimal subgroup D of G such that the restriction functor $D^b(\mathcal{O}Gb) \to D^b(\mathcal{O}D)$ is faithful. A defect group is a p-subgroup of G and all defect groups are conjugate. The defect groups are trivial if and only if $\mathcal{O}Gb$ is a matrix algebra over $\mathcal{O}$ (equivalently, kGb is semisimple). The defect groups of the principal block are the Sylow p-subgroups of G.

There is a unique block idempotent b_D of $\mathcal{O}N_G(D)$, the *Brauer correspondent* of b, such that the functor $b_D\mathcal{O}Gb \otimes_{\mathcal{O}Gb} - : D^b(\mathcal{O}Gb) \to D^b(\mathcal{O}N_G(D)b_D)$ is faithful. The idempotent b_D is actually contained in $\mathcal{O}C_G(D)$.

2.3. Conjectures.

2.3.1. *Alperin's weight conjecture.* Alperin's weight conjecture [Alp] asserts that the number of non-projective simple kG-modules is locally determined.

A *weight* for G is a pair (Q, V), where Q is a p-subgroup of G and V is a projective simple $kN_G(Q)/Q$-module.

Conjecture 2.1 (Alperin)**.** *The number of isomorphism classes of simple kG-modules is the same as the number of conjugacy classes of weights.*

The conjecture has also a blockwise version.

We will explain in §4.2.1 that the conjecture has a bijective proof when G is a finite group of Lie type in defining characteristic, for example $G = \mathrm{GL}_n(\mathbf{F}_{p^r})$. This is the inspiration for the conjecture. In a sense, Alperin's conjecture predicts that all finite groups behave like finite group of Lie type in defining characteristic.

2.3.2. *Broué's conjecture.* Broué's conjecture [Brou1] asserts that the derived category of kG-modules (excluding as above semi-simple parts) is determined locally, when Sylow p-subgroups are abelian. There is a blockwise version which we now state.

Conjecture 2.2 (Broué)**.** *Let $\mathcal{O}Gb$ be a block with abelian defect group D. We have $D^b(\mathcal{O}Gb) \simeq D^b(\mathcal{O}N_G(D)b_D)$.*

When G is a finite group of Lie type in non-defining characteristic, Broué and others have proposed an explicit candidate for a functor realizing an equivalence, using Deligne-Lusztig varieties (cf §5.4.1). In a sense, Broué's conjecture predicts that all finite groups behave like finite groups of Lie type in non-defining characteristic as far as modular representations are concerned (with the abelian defect assumption), even though the sought-after equivalence won't arise from something like a Deligne-Lusztig variety.

Remark 2.3. Broué's conjecture does not extend in an obvious way to blocks with non-abelian defect groups, cf for example §4.2.3.

On the other hand, one can generalize slightly Broué's conjecture to the case where the hyperfocal subgroup of the defect group is abelian [Rou1, Appendix A.2].

2.3.3. *Comparison.* The two conjectures lead to an odd phenomenon: finite groups, with respect to a prime p, behave like finite groups of Lie type in both defining *and* non-defining characteristic!

For blocks with abelian defect groups, Broué's conjecture implies Alperin's conjecture. A major open problem (beyond proving these conjectures) is to find a structural statement like Broué's conjecture for general blocks.

Note that the neighborhood of the trivial representation is determined locally: $H^*(G,k)$ can be recovered as a subalgebra of $H^*(P,k)$. When P is abelian (or when $N_G(P)$ controls fusion), then $H^*(G,k) = H^*(P,k)^{N_G(P)} = H^*(N_G(P),k)$.

A general version of Broué's conjecture should contain both that fact and the information about the number of simple modules. Going beyond the neighborhood of the trivial representation is discussed for finite groups of Lie type in non-defining characteristic in §6.1.4.

Alperin's conjecture can be reformulated in terms of chains of p-subgroups [KnRob]. Consider the poset of non-trivial p-subgroups of G and the associated simplicial complex $\mathcal{P}$ (n-simplices are chains $Q_0 < Q_1 < \cdots < Q_n$).

Alperin's conjecture (for all finite groups) is equivalent to the equality (for all finite groups G)

$$\sum_{c \in \mathcal{P}/G} (-1)^{|c|} l'(k\,\mathrm{Stab}_G(c)) = 0$$

where $l'(kG)$ is the number of non-projective simple modules. The simplicial complex $\mathcal{P}$ can be replaced by its subcomplex given by elementary abelian subgroups or by other complexes using subgroups Q such that $Q = O_p(N_G(Q))$: the sum does not change. Here we denote by $O_p(H)$ the largest normal p-subgroup of a finite group H.

There is also a blockwise version of that reformulation. This reformulation suggests that appropriate categories of representations of local subgroups could be glued to recover some weakened version of the category of representations of G.

2.4. **Perverse equivalences.**

2.4.1. *Definition.* Let A and A' be two finite dimensional algebras over a field k and let $F : D^b(A) \xrightarrow{\sim} D^b(A')$ be an equivalence of triangulated categories. The equivalence F induces an isomorphism of abelian groups $K_0(D^b(A)) = \mathbf{Z}^{\mathrm{Irr}(A)} \xrightarrow{\sim} \mathbf{Z}^{\mathrm{Irr}(A')} = K_0(D^b(A'))$, but no bijection $\mathrm{Irr}(A) \xrightarrow{\sim} \mathrm{Irr}(A')$. So, in the situation of Broué's abelian defect conjecture (Conjecture 2.2), there is no expectation of a bijection between simple modules. We will introduce now a particular type of derived equivalence that induces such a bijection [ChRou].

Fix $\pi : \mathrm{Irr}(A) \to \mathbf{Z}$. An equivalence $F : D^b(A) \xrightarrow{\sim} D^b(A')$ is *perverse* relative to π if there is a bijection $f : \mathrm{Irr}(A) \xrightarrow{\sim} \mathrm{Irr}(A')$ such that

- given $S \in \mathrm{Irr}(A)$, if T is a composition factor of $H^i(F(S))$, then $\pi(f^{-1}(T)) < \pi(S)$ for $i \neq -\pi(S)$
- $H^{-\pi(S)}(F(S))$ admits $f(S)$ as a composition factor with multiplicity one, and all other composition factors T satisfy $\pi(f^{-1}(T)) < \pi(S)$.

When this holds, the map f is determined by F and π.

Given A and π, then A' is unique up to Morita equivalence (if it exists).

Remark 2.4. This is a particular case of a more general definition that involves the additional data of an order on $\mathrm{Irr}(A)$.

Perverse equivalences can be defined more generally for derived categories of abelian categories. A further generalization is the consideration of a filtered triangulated category with two t-structures and the notion of a shift of t-structures with respect to a perversity function: the t-structures are assumed to be compatible with the filtration and the t-structures induced on the slices of the filtration differ by a shift given by the perversity function.

2.4.2. *Examples.* Consider the situation of Broué's conjecture: we have a block $\mathcal{O}Gb$ with defect group D. If there is an equivalence $D^b(\mathcal{O}Gb) \xrightarrow{\sim} D^b(\mathcal{O}N_G(D)b_D)$ such that the induced equivalence over k is perverse, then there is a total order on $\mathrm{Irr}(kGb)$ and on $\mathrm{Irr}(KGb)$ such that the decomposition matrix of $\mathcal{O}Gb$ has the following shape (cf §7.1.3 for the definition of decomposition matrices):

$$\begin{array}{cccc}
1 & & & \\
* & 1 & & \\
\vdots & \ddots & \ddots & \\
* & \dots & * & 1 \\
* & \dots & \dots & * \\
\vdots & & & \vdots \\
* & \dots & \dots & *
\end{array}$$

In the situation of Broué's abelian defect conjecture, perverse equivalences are known to exist in a number of cases: when defect groups are cyclic and conjecturally in the case of groups of Lie type in non-defining characteristic [ChRou] (cf §5.4.1), for principal blocks with defect group of order 4 or 9 of groups with no simple factor $\mathfrak{A}_6$ or M_{22} when $p = 3$ [CrRou1, Theorem 4.36]. Note that there is no perverse equivalence in the situation of Broué's conjecture for principal 3-blocks of $\mathfrak{A}_6$ and M_{22} [CrRou1, §5.3.2.3 and §5.4.3].

3. Finite groups of Lie type

We discuss now finite groups of Lie type (cf [MalTe]).

3.1. **Reductive groups.** Let p be a prime number and $\bar{\mathbf{F}}_p$ be an algebraic closure of the finite field with p elements. Given q a power of p, we denote by $\mathbf{F}_q$ the subfield of $\bar{\mathbf{F}}_p$ with q elements.

Let $\mathbf{G}$ be a (connected) reductive (linear) algebraic group over $\bar{\mathbf{F}}_p$.

Let $\mathbf{T}_0$ be a maximal torus of $\mathbf{G}$ and $\mathbf{B}_0$ a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}_0$. Let $X = X(\mathbf{T}_0) = \mathrm{Hom}(\mathbf{T}_0, \mathbf{G}_m)$ and $Y = Y(\mathbf{T}_0) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}_0)$. Let $\Phi \subset X$ denote the set of roots, Δ the set of simple roots, $\Phi^\vee$ the set of coroots and $\Delta^\vee$ the set of simple coroots. Given $\alpha \in \Delta^\vee$, we denote by ω_α the corresponding fundamental weight. Let ρ be the half sum of the positive roots.

Let $W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group and let $S = (s_\alpha)_{\alpha\in\Delta}$ be its generating set as a Coxeter group.

3.2. **Rational structures.**

3.2.1. *Frobenius endomorphisms.* Let $V_0 = \mathrm{Spec}\, A_0$ be an affine algebraic variety over $\mathbf{F}_q$, where q is a power of p. The endomorphism F of $V = V_0 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q = \mathrm{Spec}(A_0 \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p)$ given on $A_0 \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p$ by $a \mapsto a^q$ is called the (geometric) Frobenius endomorphism of V.

Given V' an affine algebraic variety over $\bar{\mathbf{F}}_p$, an endomorphism F of V' is called a *Frobenius endomorphism* if there is a power q of p and an affine algebraic variety V_0 defined over $\mathbf{F}_q$ with an isomorphism $V' \xrightarrow{\sim} V_0 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$ identifying F with the Frobenius endomorphism coming from V_0. We say that F defines a *rational structure* for V' over $\mathbf{F}_q$.

3.2.2. *Steinberg endomorphisms.* Let F be an endomorphism of $\mathbf{G}$, a power of which is a Frobenius endomorphism (such an F is called a *Steinberg endomorphism* of $\mathbf{G}$). The group $G = \mathbf{G}^F$ is finite: this is a finite group of Lie type. We will put more generally $H = \mathbf{H}^F$ for $\mathbf{H}$ an F-stable subgroup of $\mathbf{G}$.

There exists an F-stable Borel subgroup $\mathbf{B}_0$ and an F-stable maximal torus $\mathbf{T}_0$ contained in $\mathbf{B}_0$.

We say that $(\mathbf{G}, F)$ is *split* if F acts by a multiple of the identity on $X(\mathbf{T}_0)$. If $(\mathbf{G}, F)$ is split then F is a Frobenius endomorphism.

Let δ be the minimal positive integer such that $(\mathbf{G}, F^\delta)$ is split and we define the positive real number q by the requirement that F^δ defines a rational structure over $\mathbf{F}_{q^\delta}$. The automorphism of $X \otimes_{\mathbf{Z}} \mathbf{Q}$ induced by F permutes the lines $\mathbf{Q}\alpha$ for $\alpha \in \Delta$ and this provides an automorphism ϕ of order δ of the Coxeter diagram of $\mathbf{G}$. When $\mathbf{G}$ is simple and simply connected, the endomorphism F of $\mathbf{G}$ depends only on q and on ϕ, up to an inner automorphism.

If F is a Frobenius endomorphism, then $F(\alpha) = q\phi(\alpha)$ for all $\alpha \in \Delta$.

Let $\bar{\delta}$ be the minimal positive integer such that $F^{\bar{\delta}}$ is a Frobenius endomorphism of $\mathbf{G}$.

3.3. **Finite simple groups of Lie type.** The finite simple groups of Lie type are obtained by the following construction. We assume that $\mathbf{G}$ is simple and simply connected. Then, the group $G/Z(G) = \mathbf{G}^F/Z(\mathbf{G})^F$ is simple (with some exceptions described after the classification below) and we obtain in this way all finite simple groups of Lie type. The group is denoted by ${}^{\delta}D(q^{\bar{\delta}})$, where D is the type of the root system of $\mathbf{G}$ (equivalently, the type of the Dynkin diagram). We give now the list of all those finite simple groups, with restrictions on q if any.

- $A_n(q)$, $n \geq 1$
- $B_n(q)$, $n \geq 2$
- $C_n(q)$, $n \geq 3$
- $D_n(q)$, $n \geq 4$
- $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$
- ${}^2A_n(q)$, $n \geq 2$
- ${}^2D_n(q)$, $n \geq 3$
- ${}^3D_4(q)$, ${}^2E_6(q)$
- ${}^2B_2(q^2)$, $q^2 = 2^{2m+1}$ for some $m \geq 1$
- ${}^2G_2(q^2)$, $q^2 = 3^{2m+1}$ for some $m \geq 1$
- ${}^2F_4(q^2)$, $q^2 = 2^{2m+1}$ for some $m \geq 0$

with the following exceptions: $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$ and $B_2(2)$ are not simple and can be removed from the list. The derived subgroup of ${}^2F_4(2)$ is simple and of index 2 in ${}^2F_4(2)$. It does not arise in another construction and that group (the Tits group) needs to be added to the list of finite simple groups of Lie type. Note that some finite simple groups of Lie type occur more than once in the classification.

3.4. **Local structure.** We consider the setting of §3.2.2.

3.4.1. *Defining characteristic.* Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{B}_0$. Then U_0 is a Sylow p-subgroup of G.

An important class of subgroups in local group theory consists of those p-subgroups Q of G such that $Q = O_p(N_G(Q))$. In our setting where G is a finite group of Lie type and p is the defining characteristic, Q satisfies this condition if and only if Q is the group of F-fixed points of the unipotent radical of an F-stable parabolic subgroup of $\mathbf{G}$ [CaEn2, Remark 6.15].

Note that U_0 is abelian if and only if the quotient of $\mathbf{G}$ by its radical is a product of groups of type A_1.

Remark 3.1. Assume $\mathbf{G}$ is simple and simply connected with F-rank 1, *i.e.* of type A_1 (group $\mathrm{SL}_2(q)$), ${}^2A_2$ (group $SU_3(q^2)$), ${}^2B_2$ (Suzuki group) or ${}^2G_2$ (Ree group).

In that case, the Sylow p-subgroups of G have the trivial intersection property: given $g \in G$ with $g \notin N_G(U_0) = B_0$, we have $U_0 \cap gU_0g^{-1} = 1$. This implies that two subgroups of U_0 that are conjugate in G are already conjugate in B_0.

3.4.2. *Non-defining characteristic.* We consider a prime number $\ell \neq p$.

We denote by d the order of q in $\mathbf{F}_\ell^\times$.

We assume here that F is a Frobenius endomorphism. The ℓ-local structure of G has generic aspects explained in [BrouMal].

A Φ_d*-subgroup* of $\mathbf{G}$ is an F-stable torus $\mathbf{S}$ such that $\Phi_d(q^{-1}F)$ acts by 0 on $Y(\mathbf{S}) \otimes \mathbf{Q}$. Here, Φ_d is the d-th cyclotomic polynomial.

Let $\mathbf{S}$ be a Φ_d-subgroup of $\mathbf{G}$. There is $g \in \mathbf{G}$ such that $g\mathbf{T}_0g^{-1}$ is F-stable and contains $\mathbf{S}$. Let w be the image of $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$ in W. Given $s \in \mathbf{S}$, we have $wF(g^{-1}sg)w^{-1} = g^{-1}F(s)g$. It follows that the image of $Y(g^{-1}\mathbf{S}g) \otimes \mathbf{Q}$ in $Y(\mathbf{T}_0) \otimes \mathbf{Q}$ is a subspace on which $\Phi_d(w(q^{-1}F))$ acts by 0.

All maximal Φ_d-subgroups of $\mathbf{G}$ are G-conjugate. Let $\mathbf{S}$ be a maximal Φ_d-subgroup of $\mathbf{G}$ and g, w be as above. The image of $Y(\mathbf{S})$ in $Y(\mathbf{T}_0)$ is $Y(\mathbf{T}_0) \cap \ker(\Phi_d(wq^{-1}F)) \mid Y(\mathbf{T}_0) \otimes \mathbf{Q})$.

Assume $\ell > 3$ and F is a Frobenius endomorphism. Let $\mathbf{S}$ be a maximal Φ_d-subgroup of $\mathbf{G}$ and $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$, a Levi subgroup. Let $\mathbf{T}_0$ be a maximal F-stable torus of $\mathbf{L}$ and let $W_{\mathbf{L}} = N_{\mathbf{L}}(\mathbf{T}_0)/\mathbf{T}_0$. Let W' be the subgroup of W generated by roots orthogonal to all the roots corresponding to $\mathbf{L}$. There exists an ℓ-subgroup D' of $N_{\mathbf{G}}(\mathbf{T}_0)^F$ whose image in W is a Sylow ℓ-subgroup of W' and such that $D' \cap T_0 = 1$. Then $D = Z(\mathbf{L})_\ell^F \rtimes D'$ is a Sylow ℓ-subgroup of G [Ca2] (cf also [CaEn2, Exercice 22.6]). Furthermore, D is abelian if and only if $D' = 1$.

When D is abelian, we have an isomorphism $N_G(D)/C_G(D) \xrightarrow{\sim} (N_W(W_{\mathbf{L}})/W_{\mathbf{L}})^{wF}$, where w is defined as above from $\mathbf{S}$. The group $N_G(D)/C_G(D)$ is a complex reflection group [LeSp, Theorem 3.4] (cf [Brou3, Theorem 5.7]). When $\mathbf{L}$ is a torus, then $q^{-1}wF$ is d-regular and this is explained in §7.2.3.

Remark 3.2. It is a remarkable fact that when D is abelian, then $N_G(D)/C_G(D)$ is a reflection group. We showed in [Rou3] that a suitable version of this property actually holds for all finite simple groups. We consider G_0 a simple group with an abelian Sylow ℓ-subgroup D such that $H^2(G, \mathbf{F}_\ell) = 0$. Let G be a subgroup of $\mathrm{Aut}(G_0)$ containing G_0 and such that G/G_0 is a Hall ℓ'-subgroup of $\mathrm{Out}(G_0)$. Then there exists

- a field extension K of $\mathbf{F}_\ell$,
- an extension of the structure of $\mathbf{F}_\ell$-vector space on the largest elementary abelian subgroup $\Omega_1(D)$ of D to a structure of K-vector space
- a subgroup N of $\mathrm{GL}_K(\Omega_1(D))$
- a subgroup Γ of $\mathrm{Aut}(K)$

such that $N_G(D)/C_G(D) = N \rtimes \Gamma$ and the normal subgroup of N generated by reflections acts irreducibly on $\Omega_1(D)$.

For example, when $G_0 = \mathrm{PSL}_2(\ell^n)$, we view $D \simeq (\mathbf{Z}/\ell)^n$ as a one-dimensional vector space over $K = \mathbf{F}_{\ell^n}$, we have $N = K^\times$ and Γ is a Hall $2'$-subgroup of $\mathrm{Gal}(K/\mathbf{F}_\ell)$.

It would be very interesting to find a role for reflection groups in Broué's abelian defect group conjecture (Conjecture 2.2).

4. Defining characteristic

As we will see, it is much easier to parametrize irreducible representations of G in characteristic p than in characteristic 0 (cf [Hu]).

4.1. Simple modules and blocks.

4.1.1. *Rational representations.* Let $\mathbf{G}$ be a reductive connected algebraic group over an algebraic closure $\bar{\mathbf{F}}_p$ of a finite field with p elements, where p is a prime number. We consider $\mathbf{T}_0$, $\mathbf{B}_0$, etc as in §3.1.

Let $X_+(\mathbf{T}_0) = \{\lambda \in X(\mathbf{T}_0) \mid \langle \lambda, \alpha^\vee \rangle \geq 0\ \forall \alpha \in \Delta\}$ be the set of dominant weights. Given $\lambda \in X_+(\mathbf{T}_0)$, let $\mathcal{L}_\lambda = \mathbf{G} \times_{\mathbf{B}_0} (\bar{\mathbf{F}}_p)_\lambda$ be the associated line bundle on the flag variety $\mathbf{G}/\mathbf{B}_0$. The rational $\mathbf{G}$-module $H^0(\mathbf{G}/\mathbf{B}_0, \mathcal{L}_\lambda)$ has a unique simple submodule $\mathbf{L}(\lambda)$ and $\{\mathbf{L}(\lambda)\}_{\lambda\in X_+(\mathbf{T}_0)}$ is a full set of representatives of isomorphism classes of simple rational $\mathbf{G}$-modules.

4.1.2. *Representations of the finite group.* We consider now the setting of §3.2.2.

Given $r \geq 0$, we put $X_r = \{\lambda \in X_+(\mathbf{T}_0) \mid \langle\lambda, \alpha^\vee\rangle < r \ \forall\alpha \in \Delta\}$.

Given $\lambda \in X_+$, let $L(\lambda)$ be the restriction of $\mathbf{L}(\lambda)$ to G.

Given A an abelian group, we put $A^\vee = \mathrm{Hom}(A, \bar{\mathbf{F}}_p^\times)$.

We assume in §4.1.2 that $\mathbf{G}$ is simply connected and simple and that F is a Frobenius endomorphism.

We have the following description of simple modules (Steinberg) [Hu, Theorem 2.11 and §20.2] and of their blocks and defect groups (Dagger and Humphreys) [Hu, Theorem 8.5 and §20.3].

Theorem 4.1. *The set $\{L(\lambda)\}_{\lambda\in X_q}$ is a complete set of representatives of isomorphism classes of simple kG-modules.*

There is one block of defect zero, with simple module the Steinberg module $L((q-1)\rho)$. The other blocks have maximal defect, they are parametrized by $Z(G)^\vee$. The simple modules in the block kGb_ζ corresponding to $\zeta \in Z(G)^\vee$ are the $L(\lambda)$ with $\lambda_{|Z(G)} = \zeta$ and $\lambda \in X_q \setminus \{(q-1)\rho\}$.

Remark 4.2. Note that the set of simple kG-modules and their dimensions depend only on q, not on the Frobenius endomorphism. For example, the sets of simple modules for $\mathrm{SU}_n(q)$ and $\mathrm{SL}_n(q)$ are obtained by restricting the same set of simple rational representations of SL_n.

Remark 4.3. When F is not a Frobenius endomorphism, Theorem 4.1 needs to be modified as follows [Hu, §20]. We assume F is not a Frobenius endomorphism. Note that F^2 is a Frobenius endomorphism defining a rational structure over $\mathbf{F}_{q^2}$. Note also that $Z(G) = 1$. Define now $X_{q^2,S}$ as the set of $\lambda \in X_{q^2}$ such that $\langle\lambda, \alpha^\vee\rangle = 0$ for every long simple root α. The set $\{L(\lambda)\}_{\lambda\in X_{q^2,S}}$ is a complete set of representatives of isomorphism classes of simple kG-modules.

The Steinberg module is $L((q^2-1)\rho_S)$, where

$$\rho_S = \sum_{\substack{\alpha\in S\\ \alpha \text{ short}}} \omega_\alpha.$$

It is in a block of defect zero and the principal block is the unique other block.

4.2. **Alperin's conjecture.**

4.2.1. *Bijective proof.* We assume in §4.2.1 that $\mathbf{G}$ is simply connected and F is a Frobenius endomorphism. We follow [Ca1] and [CaEn2, §6.3].

Given I an F-stable subset of S, let $\mathbf{L}_I$ be the corresponding standard Levi subgroup of $\mathbf{G}$ and let X_I be the set of $\lambda \in X_q$ such that given $\alpha \in \Delta$, we have $\langle\lambda, \alpha^\vee\rangle = q-1$ if and only if $\alpha \in I$. We have $X_q = \coprod_{I\subset S} X_I$.

- Restriction from $\mathbf{T}_0$ to T_0 induces a bijection $X_I \xrightarrow{\sim} \mathrm{Irr}_k(T_0/(T_0 \cap [L_I, L_I])) \xrightarrow{\sim} L_I^\vee$.
- Given $\zeta \in L_I^\vee$, the kL_I-module $\mathrm{St}_{L_I} \otimes \zeta$ is simple and projective. This provides a bijection from $L_I^\vee$ to the set of isomorphism classes of projective simple kL_I-modules.
- Let Q be a p-subgroup of G such that $kN_G(Q)/Q$ has a simple projective module. Then $O_p(N_G(Q)/Q) = 1$ and it follows (cf §3.4.1) that Q is the subgroup of F-fixed points of the

unipotent radical of an F-stable parabolic subgroup of $\mathbf{G}$. So, we have a bijection from the union over I an F-stable subset of S of the sets of isomorphism classes of projective simple kL_I-modules to the set of G-conjugacy classes of pairs (Q, V) where Q is a p-subgroup of G and V a simple projective $kN_G(Q)/Q$-module, taken up to isomorphism.

Together with the bijection from X_q to $\mathrm{Irr}_k(G)$, we obtain a bijection between $\mathrm{Irr}_k(G)$ and the set of G-conjugacy classes of pairs (Q, V) where Q is a p-subgroup of G and V a simple projective $kN_G(Q)/Q$-module, taken up to isomorphism. This confirms Alperin's conjecture for G.

$$\begin{tikzcd}
X^+ \arrow[dd, "\sim"'] & X_q^+ \arrow[l] \arrow[r, "\sim"] \arrow[dd, "\sim"'] & \coprod_{\substack{I\subset S \\ I\ F\text{-stable}}} \mathrm{Irr}_k(T_0/(T_0\cap[L_I,L_I])) \arrow[d, "\sim"] \\
 & & \coprod_{\substack{I\subset S \\ I\ F\text{-stable}}} \{\text{proj simple } kL_I\text{-modules}\} \arrow[d, "\sim"] \\
\mathrm{Irr}(\mathbf{G}) & \mathrm{Irr}_k(G) \arrow[l] \arrow[r, dashed, "\sim"'] & \{(Q,V)\ |Q\le G,\ V \text{ proj simple } k(N_G(Q)/Q)\text{-module}\}/G
\end{tikzcd}$$

4.2.2. *Abelian defect.* If $\mathbf{G}$ is simple and kG has a block with non-trivial abelian defect groups, then $\mathbf{G}$ is of type A_1 (cf §3.4.1 and Theorem 4.1). Broué's abelian defect group conjecture has been solved for $\mathrm{SL}_2(\mathbf{F}_q)$ ([Ch] for $q = p^2$, [Ok] for the principal block, [Yo] for the non-principal block with maximal defect and [Wo2] for the proof that the equivalence is a composition of perverse equivalences). The solution involves some rather complicated combinatorial and algebraic constructions.

Remark 4.4. Assume $\mathbf{G}$ is semisimple and simply connected, split over $\mathbf{F}_p$ and assume that p is larger than the Coxeter number of $\mathbf{G}$. We denote by $R = \bigoplus_{\alpha\in\Delta} \mathbf{Z}\alpha$ the root lattice.

Consider the full subcategory $\mathcal{C}$ of the derived category of bounded complexes of finite-dimensional $\mathbf{B}_0$-modules whose objects are those C such that $H^i(C)$ has weights in pR for all i. The functor $R\,\mathrm{Ind}_{\mathbf{B}_0}^{\mathbf{G}}$ induces an equivalence from $\mathcal{C}$ to the bounded derived category of the principal block of finite-dimensional representations of $\mathbf{G}$ ([ArkBeGi] for the case of quantum groups at a root of unity and [HoKaSc] for an adaption and details of the characteristic p case).

It would be very interesting if this equivalence could be used to relate representations of G and B_0 over k, particularly in the case of SL_2. This would possibly shed light on how Broué's conjecture could be generalized to non-abelian defect groups.

4.2.3. *Groups of F-rank* 1. When $\mathbf{G}$ has F-rank 1, the induction and restriction functors provide inverse stable equivalences (cf §7.1.2) between kB_0 and kG because the Sylow p-subgroups of G have the trivial intersection property (cf §3.4.1).

The principal blocks of kB_0 and kG are actually derived equivalent when $G = \mathrm{SL}_2(q)$, but this is known not to generalize to all groups of F-rank 1. It is known in a number of cases that the principal blocks of kG and kB_0 are not derived equivalent because the centers are not isomorphic: for ${}^2G_2(q^2)$ when $q^2 \geq 27$ [BrougSchw] and for $G = \mathrm{SU}_3(q)$ when $3 \leq q \leq 8$ [BouZi]. It is also known that the principal blocks of $\mathcal{O}G$ and $\mathcal{O}B_0$ are not derived equivalent because the centers are not isomorphic for $G = {}^2B_2(q^2)$ when $q^2 \geq 8$ [Cl]. Not that in this last case the centers of the principal blocks of kG and kB_0 are isomorphic. It is expected that the principal blocks of kG and kB_0 are not derived equivalent in that case.

4.3. **Change of central character.**

4.3.1. *Number of simple modules in a block.* We assume in §4.3 that F is a Frobenius endomorphism.

Let $\bar{\gamma} = \mathrm{Res}^{\mathbf{T}_0}_{Z(G)} : X \to Z(G)^\vee$.

Lemma 4.5. *Let $\zeta \in Z(G)^\vee$. The number of simple modules in kGb_ζ is less than or equal to that for the principal block kGb_0. There is equality if and only if $\zeta \in \bigcap_{\alpha\in\Delta^\vee} \bar{\gamma}(\mathbf{Z}\omega_\alpha)$.*

Proof. Let $\gamma : X \to T_0^\vee$ be the restriction map. It induces an isomorphism $X/(F-1)X \xrightarrow{\sim} T_0^\vee$. Let I be a ϕ-stable subset of Δ. Let X'_I be the set of $\lambda \in X_q$ such that

- given $\alpha \notin I$, we have $\langle \lambda, \alpha^\vee \rangle = q-1$
- given $\alpha \in I$, there is $i \geq 0$ such that $\langle \lambda, \phi^i(\alpha^\vee)\rangle \neq q-1$.

The map γ restricts to a bijection $X'_I \xrightarrow{\sim} \gamma(\bigoplus_{\alpha\in I} \mathbf{Z}\omega_\alpha)$.

The restriction $\bar{\gamma}_{|X'_I}$ factors as

$$X'_I \xrightarrow[\sim]{\gamma_{|X'_I}} \gamma(\bigoplus_{\alpha\in I} \mathbf{Z}\omega_\alpha) \xrightarrow{\gamma_I} Z(G)^\vee$$

where γ_I is given by restricting from T_0 to $Z(G)$.

Let $\zeta \in Z(G)^\vee$. Put $\delta_{I,\zeta} = 1$ if $\gamma_I^{-1}(\zeta) \neq \emptyset$ and $\delta_{I,\zeta} = 0$ otherwise.

We have

$$|\operatorname{Irr}_k(kGb_\zeta)| = \sum_I |\bar{\gamma}^{-1}(\zeta) \cap X'_I| = \sum_I |\gamma_I^{-1}(\zeta)| = \sum_I \delta_{I,\zeta} |\gamma_I^{-1}(0)|$$

where I runs over non-empty ϕ-stable subsets of Δ. This shows the requested inequality.

We have $\delta_{I,\zeta} = 1$ if and only if $\zeta \in \bar{\gamma}(\bigoplus_{\alpha\in I} \mathbf{Z}\omega_\alpha)$. Note that $\bar{\gamma}(\omega_\alpha) = \bar{\gamma}(\omega_{\phi(\alpha)})$ for all α. The equivalence of the lemma follows. □

The tables of [Bki, Lie 4,5,6] show that outside type A, there is a fundamental weight in the root lattice, hence $\bigcap_{\alpha\in\Delta^\vee} \bar{\gamma}(\mathbf{Z}\omega_\alpha) = 0$.

Assume $G = \mathrm{SL}_n(q)$ (in which case we put $\varepsilon = 1$) or $G = \mathrm{SU}_n(q)$ (in which case we put $\varepsilon = -1$). We have $\bigcap_{\alpha\in\Delta^\vee} \bar{\gamma}(\mathbf{Z}\omega_\alpha) \neq 0$ if and only if $n = \ell^r$ for some prime $\ell | (q-\varepsilon)$ and $r \geq 1$. In that case $Z(G) \simeq \mathbf{Z}/(\gcd(\ell^r, q-\varepsilon)\mathbf{Z})$ and the non-trivial characters of $Z(G)^\vee$ with order ℓ are those non-trivial characters ζ such that kGb_ζ has the same number of simple modules as kGb_0.

4.3.2. *Equivalences.* We consider the setting above where $G = \mathrm{SL}_{\ell^r}(q)$ or $G = \mathrm{SU}_{\ell^r}(q)$ with ℓ a prime dividing $q - \varepsilon$.

Question 4.6. *Let ζ be a character of order ℓ of $Z(G)^\vee$. What is the relation between kGb_0*-mod *and kGb_ζ*-mod*? Are the blocks kGb_0 and kGb_ζ stably equivalent?*

Note that if $\mathbf{P}$ is a proper F-stable parabolic subgroup of $\mathbf{G}$, then the character ζ of $Z(G)$ extends to P. Given Q a non-trivial p-subgroup of G, there is a proper F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $N_G(Q) \subset P$ (cf [CaEn2, Remark 6.15]). It follows that ζ extends to $N_G(Q)$. As a consequence (cf Remark 4.8 below), the blocks of proper local subgroups corresponding to kGb_0 and kGb_ζ are isomorphic. If those local equivalences could be glued (cf [Rou1, §7.3] for a setting for gluing), then we would obtain a stable equivalence answering positively the question.

When $\ell^r = 2$ and $\varepsilon = 1$, the question has a positive answer. The blocks are actually derived equivalent, since they are both derived equivalent to the corresponding blocks of kB_0 (cf §4.2.2), and those blocks are isomorphic. Cf also [Wo1] for a direct construction as a composition of perverse equivalences.

When $\ell^r = 3$ and $\varepsilon = -1$, the question has also a positive answer since the blocks are both stably equivalent to the corresponding blocks of kB_0 (cf §4.2.3), and those blocks are isomorphic.

Remark 4.7. Note that we do not expect the blocks to be derived equivalent in general. Consider the case $G = \mathrm{SL}_3(4)$. Let us show that there are no perfect isometries between the principal 2-block and a non-principal 2-block of $\mathrm{SL}_3(4)$. In particular, those blocks are not derived equivalent over $\mathcal{O}$.

The principal block of G has 9 irreducible characters: 1, 20, 35, $\overline{35}$, $35'$, 45, $\overline{45}$, 63 and $\overline{63}$. Let b be one of the non-principal block idempotents with positive defect. The irreducible characters of KGb are those with central character a given primitive cubic root of unity. They are 15_1, 15_2, 15_3, 21, 45, $\overline{45}$, 63, $\overline{63}$ and 84.

Let I be a perfect isometry from $\mathcal{O}Gb_0$ to $\mathcal{O}Gb$. Let $\eta = \sum_{\chi \in \mathrm{Irr}(KGb_0)} \chi \otimes I(\chi)$. Let g be an element of order 5 and h an element of even order. We have

$$0 = \eta(g,h) = I(1)(h) + \alpha I(63)(h) + \bar{\alpha} I(\overline{63})(h)$$

with $\alpha = \frac{1-\sqrt{5}}{2}$ and $\bar{\alpha} = \frac{1+\sqrt{5}}{2}$. It follows that $I(63)(h) = I(\overline{63})(h) = -I(1)(h)$. Taking h an involution, we deduce that $\{I(1), I(63), I(\overline{63}\} \subset \{\pm 15_1, \pm 15_2, \pm 15_3, \pm 63, \pm \overline{63}\}$. Taking h of order 4, we obtain that $\{I(1), I(63), I(\overline{63}\} \subset \{\pm 21, \pm 45, \pm \overline{45}, \pm 63, \pm \overline{63}\}$. This is a contradiction.

Remark 4.8. Let G be a finite group and ζ a character of $Z(G)_{p'}$, the largest p'-subgroup of $Z(G)$. Let

$$e_\zeta = \frac{1}{|Z(G)_{p'}|} \sum_{z \in Z(G)_{p'}} \zeta(z)^{-1} z$$

be the associated idempotent.

If ζ extends to a character $\tilde{\zeta}$ of G, then there is an isomorphism of algebras

$$e_1 kG \xrightarrow{\sim} e_\zeta kG, \ \sum_{g \in G} a_g g \mapsto \sum_{g \in G} \tilde{\zeta}(g)^{-1} a_g g.$$

In general, the algebras $e_1 kG$ and $e_\zeta kG$ can have rather different module categories and invariants. When the extension assumption holds for all p-local subgroups, we could hope that the algebras are at least stably equivalent. More precisely, assume that given any non-trivial p-subgroup Q of G, the character ζ extends to $N_G(Q)$. Are the algebras $e_1 kG$ and $e_\zeta kG$ stably equivalent?

One can ask similar questions for two linear characters of $Z(G)$. For example, let d be an integer prime to the order of ζ. What is the relation between $e_\zeta kG$ and $e_{\zeta^d}kG$? If ζ^d can be obtained from ζ by applying a field automorphism of k, then the rings $e_\zeta kG$ and $e_{\zeta^d}kG$ are isomorphic, but they need not be isomorphic as k-algebras (nor even derived equivalent), as shown by Benson and Kessar [BeKe, Example 5.1]. In their examples, $O_p(G) \neq 1$.

5. Non-defining characteristic

We consider a connected reductive algebraic group $\mathbf{G}$ with a Steinberg endomorphism F as in §3.2.2. We fix a prime ℓ distinct from p. We will be discussing mod-ℓ representations of G. We fix K a finite extension of $\mathbf{Q}_\ell$ containing all $|G|$-th roots of unity and denote by $\mathcal{O}$ its ring of integers and by k its residue field.

In §5.1 and §5.2, we recall constructions and results of Deligne-Lusztig and Lusztig [DeLu1, DeLu2, Lu1, Lu2].

5.1. **Deligne-Lusztig varieties.**

5.1.1. *Definition.* Consider the Lang covering $\mathcal{L} : \mathbf{G} \to \mathbf{G},\ g \mapsto g^{-1}F(g)$. This is a surjective étale Galois morphism, with Galois group G.

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{U}$ be its unipotent radical. Assume there is an F-stable Levi subgroup $\mathbf{L}$ with $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$. The associated Deligne-Lusztig variety is $\mathcal{L}^{-1}(F(\mathbf{U}))$. It has a free left (resp. right) action of G (resp. L) by multiplication. We can also consider its quotient $Y_\mathbf{U} = \mathcal{L}^{-1}(F(\mathbf{U}))/(\mathbf{U} \cap F(\mathbf{U}))$, which has the same ℓ-adic cohomology. One can consider further the variety $X_\mathbf{U} = Y_\mathbf{U}/L$. The varieties $Y_\mathbf{U}$ and $X_\mathbf{U}$ are smooth.

Remark 5.1. When $\mathbf{P}$ is F-stable, then $Y_\mathbf{U} = G/U$ is a finite set.

5.1.2. *Case of tori.* A particular role is played by Deligne-Lusztig varieties associated to tori. Let us give another model for those. Fix $\mathbf{B}_0$ an F-stable Borel subgroup of $\mathbf{G}$ and $\mathbf{T}_0$ an F-stable maximal torus contained in $\mathbf{B}_0$. Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{B}_0$ and let $W = N_\mathbf{G}(\mathbf{T}_0)/\mathbf{T}_0$.

Let $\mathcal{B}$ be the variety of Borel subgroups of $\mathbf{G}$. There is a decomposition $\mathcal{B}\times\mathcal{B} = \coprod_{w\in W} \mathcal{O}(w)$ into orbits for the diagonal action of $\mathbf{G}$, where $\mathcal{O}(w)$ is the orbit containing $(\mathbf{B}_0, w\mathbf{B}_0 w^{-1})$.

Given $w \in W$, we put

$$X(w) = \{\mathbf{B} \in \mathcal{B} \mid (\mathbf{B}, F(\mathbf{B})) \in \mathcal{O}(w)\}.$$

Let $\dot{w} \in N_\mathbf{G}(\mathbf{T}_0)$ with image $w \in W$. We put

$$Y(\dot{w}) = \{g\mathbf{U}_0 \in \mathbf{G}/\mathbf{U}_0 \mid g^{-1}F(g) \in \mathbf{U}_0\dot{w}\mathbf{U}_0\}.$$

There is a left action of G on $X(w)$ and $Y(\dot{w})$ by left multiplication and a right action of $\mathbf{T}_0^{wF}$ on $Y(\dot{w})$ by right multiplication. The map $g\mathbf{U}_0 \mapsto g\mathbf{B}_0 g^{-1}$ induces an isomorphism of G-varieties $G \setminus Y(\dot{w}) \xrightarrow{\sim} X(w)$. The varieties $X(w)$ and $Y(\dot{w})$ have pure dimension $l(w)$, the length of w (cf §7.2.4).

Let $h \in \mathbf{G}$ such that $h^{-1}F(h) = \dot{w}$. The maximal torus $\mathbf{T} = h\mathbf{T}_0 h^{-1}$ is F-stable and the isomorphism $\mathbf{T}_0 \xrightarrow{\sim} \mathbf{T},\ t \mapsto hth^{-1}$ restricts to an isomorphism $\mathbf{T}_0^{wF} \xrightarrow{\sim} \mathbf{T}^F$.

There is a commutative diagram

$$\begin{tikzcd}
Y_{h\mathbf{U}_0h^{-1}} \arrow[r, "g\mapsto gh\mathbf{U}_0", "\sim"'] \arrow[d, "g(\mathbf{U}\cap F(\mathbf{U}))\mapsto g(\mathbf{U}\cap F(\mathbf{U}))L"'] & Y(\dot w) \arrow[d, "g\mathbf{U}_0\mapsto g\mathbf{B}_0g^{-1}"] \\
X_{h\mathbf{U}_0h^{-1}} \arrow[r, "\sim", "g\mapsto gh\mathbf{B}_0h^{-1}g^{-1}"'] & X(w)
\end{tikzcd}$$

where the horizontal maps are G-equivariant isomorphisms and the top horizontal map is equivariant for the right action of T, via its identification with $\mathbf{T}_0^{wF}$ above.

Two elements w and w' of W are *F-conjugate* if there is $v\in W$ such that $w'=v^{-1}wF(v)$.

The construction of $\mathbf{T}$ from w induces a bijection from the set of F-conjugacy classes of W to the set of G-conjugacy classes of F-stable maximal tori of $\mathbf{G}$.

Remark 5.2. The varieties $X(w)$ are known to be affine in many cases, but it is not known if they are affine in general. The affinity is known when q is larger than the Coxeter number of $\mathbf{G}$ [DeLu1, Theorem 9.7] and when w has minimal length in its F-conjugacy class ([Lu1, Corollary 2.8], [OrRa, §5], [He, Theorem 1.3] and [BoRou3]).

5.1.3. *Endomorphisms.* We follow [BrouMi2], inspired by an earlier construction of Lusztig [Lu2, pp. 24–25].

Given $w_1,\ldots,w_r\in W$, let

$$X_F(w_1,\ldots,w_r)=\{(\mathbf{B}_1,\ldots,\mathbf{B}_r)\in\mathcal{B}^r\mid(\mathbf{B}_i,\mathbf{B}_{i+1})\in\mathcal{O}(w_i),\ 1\le i<r \text{ and } (\mathbf{B}_r,F(\mathbf{B}_1))\in\mathcal{O}(w_r)\}.$$

The variety $X_F(w_1,\ldots,w_r)$ depends only on the element $b=\lambda(w_1)\cdots\lambda(w_r)$ of B_W^+ (cf §7.2.4 for the notations), up to canonical isomorphism [De] and we denote it by $X_F(b)$.

There is an action of ϕ on B_W by $b_s\mapsto b_{\phi(s)}$ for $s\in S$. Note that given $n>0$, we have a morphism

$$\begin{aligned}\iota_n: X_F(b)&\to X_{F^n}(b\phi(b)\cdots\phi^{n-1}(b))\\ (\mathbf{B}_1,\ldots,\mathbf{B}_r)&\mapsto(\mathbf{B}_1,\ldots,\mathbf{B}_r,F(\mathbf{B}_1),\ldots,F(\mathbf{B}_r),\ldots,F^{n-1}(\mathbf{B}_1),\ldots,F^{n-1}(\mathbf{B}_r)).\end{aligned}$$

Given $0\le i\le r$, the morphism

$$\begin{aligned}D_{w_1,\ldots,w_i}: X_F(w_1,\ldots,w_r)&\to X_F(w_{i+1},\ldots,w_r,\phi(w_1),\ldots,\phi(w_i))\\ (\mathbf{B}_1,\ldots,\mathbf{B}_r)&\mapsto(\mathbf{B}_{i+1},\ldots,\mathbf{B}_r,F(\mathbf{B}_1),\ldots,F(\mathbf{B}_i))\end{aligned}$$

is purely inseparable. Given $b',b''\in B_W^+$, this provides a morphism $D_{b'}:X_F(b'b'')\to X_F(b''\phi(b'))$.

Let $b\in B_W^+$ such that $(b\phi)^d=\pi^r\phi^d$ for some $d,r>0$. We define an action of $C_{B_W^+}(b\phi)$ on $X_F(b)$ as follows.

Let $b'\in C_{B_W^+}(b\phi)$. There are $t>0$ and $b''\in B_W^+$ such that $\pi^{rt}=bb'b''$. We identify $X_F(b)$ with a subvariety of $X_{F^{dt}}(\pi^{rt})$ using the embedding ι_{dt}. The endomorphism $D_{b'}$ of $X_{F^{dt}}(\pi^{rt})$ preserves $X_F(b)$

The constructions above extend to the varieties Y. The composite morphism $B_W\xrightarrow{\mathrm{can}}W=N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ lifts to a morphism $\sigma:B_W\to N_{\mathbf{G}}(\mathbf{T}_0)$. Given $w_1,\ldots,w_r\in W$, we have a variety

$$Y_F(w_1,\ldots,w_r)=\{(g_1\mathbf{U}_0,\ldots,g_r\mathbf{U}_0)\in(\mathbf{G}/\mathbf{U}_0)^r\mid\begin{cases}g_i^{-1}g_{i+1}\in\mathbf{U}_0\sigma(\lambda(w_i))\mathbf{U}_0 \text{ for } 1\le i<r\\ g_r^{-1}F(g_1)\in\mathbf{U}_0\sigma(\lambda(w_r))\mathbf{U}_0\end{cases}\}.$$

It has a left action of G by diagonal left multiplication and a right action of $\mathbf{T}_0^{w_1\cdots w_r F}$ by diagonal right multiplication. We have a G-equivariant morphism corresponding to the quotient by $\mathbf{T}_0^{w_1\cdots w_r F}$

$$Y_F(w_1,\ldots,w_r)\to X_F(w_1,\ldots,w_r),\ (g_1\mathbf{U}_0,\ldots,g_r\mathbf{U}_0)\mapsto(g_1\mathbf{B}_0g_1^{-1},\ldots,g_r\mathbf{B}_0g_r^{-1}).$$

The variety $Y_F(w_1,\ldots,w_r)$ depends only on the element $b=\lambda(w_1)\cdots\lambda(w_r)$ of B_W^+, up to canonical isomorphism and we denote it by $Y_F(b)$. Given $b',b''\in B_W^+$, we obtain a morphism $D_{b'}:Y_F(b'b'')\to Y_F(b''\phi(b'))$ and we have an action of $C_{B_W^+}(b\phi)$ on $Y_F(b)$ compatible with the action on $X_F(b)$.

5.1.4. *Deligne-Lusztig functors.* Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and an F-stable Levi complement $\mathbf{L}$. The complex $\Lambda_c(Y_{\mathbf{U}},\mathbf{Z}_\ell)$ of ℓ-adic cohomology with compact support of $Y_{\mathbf{U}}$ [Ric1, Rou2] is a bounded complex of ℓ-permutation $\mathbf{Z}_\ell(G\times L^{\mathrm{opp}})$-modules. It is well defined up to homotopy. Its cohomology groups are the $H_c^i(Y_{\mathbf{U}},\mathbf{Z}_\ell)$.

Given R a commutative $\mathbf{Z}_\ell$-algebra, we put $\Lambda_c(Y_{\mathbf{U}},R)=\Lambda_c(Y_{\mathbf{U}},\mathbf{Z}_\ell)\otimes_{\mathbf{Z}_\ell}R$. We obtain a functor

$$\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}=\Lambda_c(Y_{\mathbf{U}},R)\otimes_{RL}-:D^b(RL\text{-mod})\to D^b(RG\text{-mod}).$$

When $R=\bar{\mathbf{Q}}_\ell$, the functor $\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ induces a morphism

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}:G_0(\bar{\mathbf{Q}}_\ell L)\to G_0(\bar{\mathbf{Q}}_\ell G).$$

Note that this morphism is expected to depend only on $\mathbf{L}$, and not on $\mathbf{P}$. This is known to hold except possibly when $q=2$, the parabolic subgroup $\mathbf{P}$ is not F-stable and the Dynkin diagram of $\mathbf{G}$ contains a subdiagram of type E_6 (cf [DeLu1, Corollary 4.3] and [DeLu2, BoMi]).

Remark 5.3. When $\mathbf{P}$ is F-stable, then $\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}=R[G/U]\otimes_{RL}-$ is the Harish-Chandra induction functor.

We denote by ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}:G_0(\bar{\mathbf{Q}}_\ell G)\to G_0(\bar{\mathbf{Q}}_\ell L)$ the adjoint of $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$.

5.2. **Characteristic 0 representations.**

5.2.1. *Tori and characters.* Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. Fix M a positive integer multiple of δ such that $(wF)^M(t)=t^{q^M}$ for all $t\in\mathbf{T}$ and $w\in W$ (cf §3.2.2 for the definitions of δ and q). Let ζ (resp. ξ) be a root of unity of order q^M-1 of $\bar{\mathbf{F}}_p$ (resp. $\bar{\mathbf{Q}}_\ell$).

The morphism

$$N:Y(\mathbf{T})\to T,\ y\mapsto y(\zeta)F(y(\zeta))\cdots F^{M-1}(y(\zeta))$$

is surjective and induces an isomorphism $Y(\mathbf{T})/\big((F-1)(Y(\mathbf{T})\big)\xrightarrow{\sim}T$.

The morphism

$$X(\mathbf{T})\to\mathrm{Hom}(Y(\mathbf{T}),\bar{\mathbf{Q}}_\ell^\times),\ \chi\mapsto(y\mapsto\xi^{\langle\chi,y+F(y)+\cdots+F^{M-1}(y)\rangle})$$

factors through $\mathrm{Hom}(N,\bar{\mathbf{Q}}_\ell^\times)$ and gives a surjective morphism $X(\mathbf{T})\to\mathrm{Hom}(T,\bar{\mathbf{Q}}_\ell^\times)$. That induces an isomorphism $X(\mathbf{T})/\big((F-1)(X(\mathbf{T})\big)\xrightarrow{\sim}\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(T)$.

5.2.2. *Tori and dual groups.* Let $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ be a triple dual to $(\mathbf{G}, \mathbf{T}_0, F)$: the group $\mathbf{G}^*$ is a Langlands dual of $\mathbf{G}$, there is a given isomorphism $X(\mathbf{T}_0^*) \xrightarrow{\sim} Y(\mathbf{T}_0)$, and F^* is a Steinberg endomorphism of $\mathbf{G}^*$ stabilizing $\mathbf{T}_0^*$ and dual to F. Furthemore, there is a given isomorphism $W^* = N_{\mathbf{G}^*}(\mathbf{T}_0^*)/\mathbf{T}_0^* \xrightarrow{\sim} W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ and we identify those groups. Note that the action of F^* on W^* corresponds to the action of F^{-1} on W.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. It corresponds to an F-conjugacy class (w) of W (cf §5.1.2). We denote by $\mathbf{T}^*$ an F^*-stable maximal torus of $\mathbf{G}^*$ whose $(\mathbf{G}^*)^{F^*}$-conjugacy class is given by the F^*-conjugacy class (w^{-1}). Furthermore, the identification of $\mathbf{T}_0^*$ with the dual of $\mathbf{T}_0$ provides an isomorphism between $\mathbf{T}^*$ and the dual of $\mathbf{T}$, and that isomorphism is well-defined up to the action of $(N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})^F$. Via the constructions of §5.2.1, this gives an isomorphism $\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(T) \xrightarrow{\sim} (\mathbf{T}^*)^{F^*}$.

This construction provides a bijection from the set of G-conjugacy classes of pairs $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(\mathbf{T}^F)$ to the set of $(\mathbf{G}^*)^{F^*}$-conjugacy classes of pairs $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in (\mathbf{T}^*)^{F^*}$.

5.2.3. *Jordan-Lusztig decomposition.* Let us recall the Jordan decomposition of conjugacy classes. An element $g \in \mathbf{G}$ can be decomposed uniquely as $g = tu$ where t is semi-simple, u is unipotent and $ut = tu$. Denote by $\mathrm{Cl}(G)$ (resp. $\mathrm{Cl}_{ss}(G)$, $\mathrm{Cl}_{unip}(G)$) the set of conjugacy classes of elements (resp. semi-simple, unipotent elements) of G.

The Jordan decomposition induces a bijection

$$\mathrm{Cl}(G) \xrightarrow{\sim} \coprod_{(t)\in \mathrm{Cl}_{ss}(G)} \mathrm{Cl}_{unip}(C_G(t))$$

where t runs over conjugacy classes of semi-simple elements of G.

Given (s) a conjugacy class of semi-simple elements of $(\mathbf{G}^*)^{F^*}$, we denote by $\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, (s))$ the set of irreducible representations of G that occur in the θ-isotypic component of $H_c^*(Y_{\mathbf{U}}, \bar{\mathbf{Q}}_\ell)$ for some Borel subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and containing an F-stable maximal torus $\mathbf{T}$ and $\theta \in \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(T)$ such that $(\mathbf{T}, \theta)$ corresponds to $(\mathbf{T}^*, s)$ by the bijection of §5.2.2 for some F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ containing s.

The *unipotent representations* of G are those in $\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, 1)$. They are the irreducible representations of G that occur in $H_c^*(X(w), \bar{\mathbf{Q}}_\ell)$ for some $w \in W$.

We have the Deligne-Lusztig decomposition (cf [CaEn2, Theorem 8.24])

$$\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G) = \coprod_{(s)\in \mathrm{Cl}_{ss}(\mathbf{G}^*)^{F^*}} \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, (s))$$

where (s) runs over conjugacy classes of semi-simple elements of $(\mathbf{G}^*)^{F^*}$.

Let s be a semi-simple element of $(\mathbf{G}^*)^{F^*}$. When $C_{\mathbf{G}^*}(s)$ is connected, let $(C_{\mathbf{G}^*}(s)^*, F)$ be dual to $(C_{\mathbf{G}^*}(s), F^*)$. Note that $C_{\mathbf{G}^*}(s)^*$ need not occur as a subgroup of $\mathbf{G}$.

Lusztig constructed a bijection (cf [CaEn2, Theorem 15.8])

$$\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(\left(C_{\mathbf{G}^*}(s)^*\right)^F, 1) \xrightarrow{\sim} \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, (s)).$$

When $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$, then $C_{\mathbf{G}^*}(s)^*$ can be realized as an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and the bijection is given by

$$\rho \mapsto \pm R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\rho \otimes \eta) \qquad (1)$$

where η is the one-dimensional representation of L corresponding by duality to $s \in Z(\mathbf{L}^*)^{F^*}$ and $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L}$.

When $Z(\mathbf{G})$ is connected, one obtains the Jordan-Lusztig decomposition of characters

$$\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G) \xrightarrow{\sim} \coprod_{(s)\in \mathrm{Cl}_{ss}(\mathbf{G}^*)^{F^*}} \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}\big(\big(C_{\mathbf{G}^*}(s)^*\big)^F, 1\big).$$

If $Z(\mathbf{G})$ is connected, then $\mathrm{Cl}_{ss}(\mathbf{G}^*)^{F^*}$ can be replaced by $\mathrm{Cl}_{ss}((\mathbf{G}^*)^{F^*})$.

5.2.4. *Unipotent representations.* Lusztig constructed a parametrization of simple unipotent representations of G by a combinatorially defined set $\mathcal{U}(W,\phi)$ that depends only on the Weyl group W and on the finite order automorphism ϕ induced by F of the reflection representation of W. The degrees of the irreducible unipotent representations are polynomials in q (the *generic degrees*). Lusztig also defined a partition of $\mathcal{U}(W,\phi)$ into *families*, and a partial order on the set of families.

When $\mathbf{G} = \mathrm{GL}_n$, the simple unipotent representations are parametrized by partitions of n. When $G = \mathrm{GL}_n(q)$, the simple unipotent representations are the components of $\mathrm{Ind}_{B_0}^G \bar{\mathbf{Q}}_\ell$.

5.3. **Modular representations.**

5.3.1. *Blocks and Lusztig series.* Let t be a semi-simple element of $(\mathbf{G}^*)^{F^*}$ of order prime to ℓ. We put

$$e_{(t)} = e_{(t)}^G = \sum_{\substack{(s)\in \mathrm{Cl}_\ell(C_{\mathbf{G}^*}(t)^{F^*}) \\ \chi\in \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G,(st))}} e_\chi$$

where Cl_ℓ denotes the set of conjugacy classes of ℓ-elements.

This idempotent of $Z(KG)$ is actually in $Z(\mathcal{O}G)$ [BrouMi1], hence it is a sum of (orthogonal) block idempotents. In other terms, $\bigcup_{(s)\in \mathrm{Cl}_\ell(C_{\mathbf{G}^*}(t)^{F^*})} \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G,(st))$ is a union of characters in blocks.

A *unipotent block* is a block kGb such that $be_{(1)} = b$.

Given $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing an F-stable maximal torus $\mathbf{T}$, with unipotent radical $\mathbf{U}$ and given $\theta \in \mathrm{Irr}(T)_{\ell'}$ such that the pair $(\mathbf{T},\theta)$ corresponds to a pair $(\mathbf{T}^*, t)$, then $\Lambda_c(Y_{\mathbf{U}},\mathcal{O})e_\theta$ is an object of $\mathcal{O}Ge_{(t)}$-perf. Furthermore, those complexes (for varying $\mathbf{B}$, $\mathbf{T}$ and θ) generate $\mathcal{O}Ge_{(t)}$-perf (the smallest full thick triangulated subcategory containing those is the whole category) [BoRou1, Theorem A'].

There is a similar statement for derived categories when all elementary abelian ℓ-subgroups of G are contained in tori [BoDaRou, Theorem 1.2].

5.3.2. *Jordan decomposition.* Broué conjectured [Brou2] a modular version of (1): assume $C_{\mathbf{G}^*}(t)$ is a Levi subgroup of $\mathbf{G}^*$, with corresponding dual an F-stable Levi subgroup $\mathbf{L}$ in $\mathbf{G}$, and let η be dual to t. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$. Then $H^{\dim Y_{\mathbf{U}}}(Y_{\mathbf{U}},\mathcal{O}) \otimes \eta$ induces a Morita equivalence between $\mathcal{O}Ge_{(t)}^G$ and $\mathcal{O}Le_{(1)}^L$. This was proven in [Brou2] when $\mathbf{L}$ is a torus, while the general case is [BoRou1, Theorem 11.8]. There is an extension of that result to the case where $C_{\mathbf{G}^*}(t)^\circ$ is a Levi subgroup [BoDaRou].

Remark 5.4. The geometric approach has not enabled us to relate isolated blocks, i.e. corresponding to a semi-simple ℓ'-element t of $(\mathbf{G}^*)^{F^*}$ such that $C_{\mathbf{G}^*}(t)^\circ$ is not contained in a proper Levi subgroup, to unipotent blocks. It is conjectured though that any block is Morita equivalent to a unipotent block, for a possibly non-connected group.

5.3.3. *Unipotent blocks.* Assume p and ℓ are good for $\mathbf{G}$ and $\ell \not| |Z(\mathbf{G})/Z(\mathbf{G})^\circ|\cdot|Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ|$. We also assume for the remainder of §5.3 that F is a Frobenius endomorphism.

There is a "d-Harish Chandra" parametrization of blocks of kG [CaEn1, BrouMalMi].

An F-stable Levi subgroup of $\mathbf{G}$ is *d-split* if it is the centralizer of a Φ_d-subgroup of $\mathbf{G}$. A simple unipotent representation ρ of G is said to be *d-cuspidal* if ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\rho) = 0$ for all proper d-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ with $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L}$.

There is a parametrization of the set of unipotent blocks by the set G-conjugacy classes of pairs $(\mathbf{L}, \lambda)$ where $\mathbf{L}$ is a d-split Levi subgroup of $\mathbf{G}$ and λ is a d-cuspidal unipotent character of G: the simple unipotent representations in the block corresponding to $(\mathbf{L}, \lambda)$ are those that occur in $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda)$ for some $\mathbf{P}$.

The parametrization of classes of pairs $(\mathbf{L}, \lambda)$ above depends only on (W, ϕ) and d, and the corresponding subset of $\mathcal{U}(W, \phi)$ depends only on d [BrouMalMi].

5.3.4. *Unipotent decomposition matrices.* Fix a total order on $\mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, 1)$ such that if ρ_i is in the family $\mathcal{F}_i$ for $i \in \{1, 2\}$ and $\mathcal{F}_1 < \mathcal{F}_2$, then $\rho_1 < \rho_2$ (cf §5.2.4).

The following result was conjectured by Geck [Ge1] and proven by [BruDuTa], following earlier work on basic sets [Ge2, GeHi] and proofs for $\mathrm{GL}_n(q)$ in [DipJa], for $\mathrm{GU}_n(q)$ in [Ge2] and for classical groups and certain ℓ (linear primes, for which the blocks are related to blocks of $\mathrm{GL}_n(q)$) in [GruHi].

Theorem 5.5. *There is a (unique) bijection* $\beta : \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, 1) \xrightarrow{\sim} \mathrm{Irr}(kGe_1)$ *such that* $\mathrm{dec}([\rho]) \in [\beta(\rho)] + \sum_{\rho' > \rho} \mathbf{Z}_{\geq 0}[\beta(\rho')]$ *for any* $\rho \in \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, 1)$.

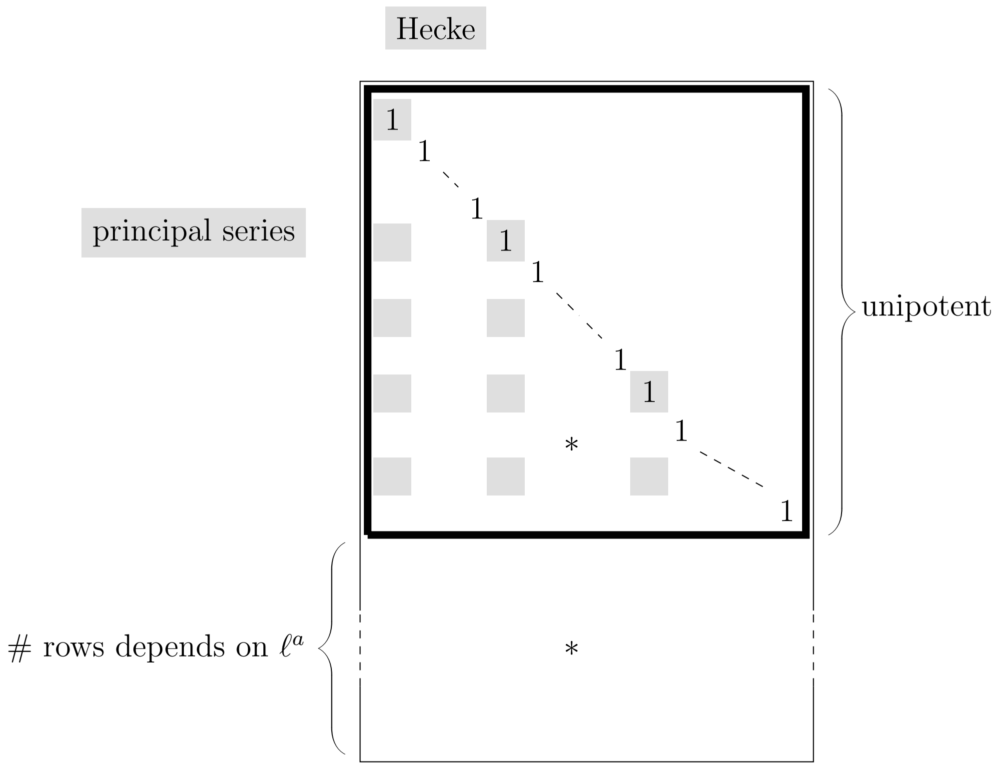
The theorem above together with Lusztig's work (§5.2.4) provides a parametrization of the set of simple kGe_1-modules by a set that depends only on (W, ϕ).

It is conjectured that, given W and d (the order of q in $\mathbf{F}_\ell^\times$), for ℓ large enough, the square part of the decomposition matrix involving unipotent representations depends only on (W, ϕ) and d, i.e., it is independent of ℓ and q. This is known for $\mathrm{GL}_n(q)$ [DipJa] and for linear primes and classical groups [GruHi].

The determination of this "generic" square matrix is a major open problem in the study of decomposition matrices for finite groups of Lie type in non-defining characteristic. The recent [DuMal2] provides a number of new decomposition matrices for groups of low rank.

Assume $\mathbf{G}$ is split. The algebra $\mathrm{End}_{\mathcal{O}G}(\mathrm{Ind}_{B_0}^G \mathcal{O})$ is isomorphic to the Hecke algebra of W over $\mathcal{O}$ (cf §7.2.4), specialized at $x = q$. The decomposition matrix of that specialized Hecke algebra is equal to the submatrix of the decomposition matrix with rows parametrized by simple modules that are direct summands of $\mathrm{Ind}_{B_0}^G K$ (principal series representations) and columns by simple modules that are quotients of $\mathrm{Ind}_{B_0}^G k$ [Dip]. The former depends only on d, if ℓ is large enough, as it is the same as the one for the Hecke algebra at x a primitive d-th root of unity, over $\mathbf{C}$ [Ge3]. This shows the genericity property for a small submatrix. Similar considerations can be used to prove genericity properties for small submatrices along the diagonal corresponding to various Harish-Chandra series using relative Hecke algebras.

Let $a = \nu_\ell(\Phi_d(q))$. Theorem 5.5 asserts that the decomposition matrix has the following shape. Here, the gray entries are on rows corresponding to principal series irreducible characters and columns corresponding to modular simple representations that are quotients of $\mathrm{Ind}_{B_0}^G k$. The gray entries give the decomposition matrix of the Hecke algebra.



Assume $G = \mathrm{GL}_n(q)$. Let $A = \mathcal{O}G/\big(\mathcal{O}G \cap (\bigoplus_{\chi \notin \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G,1)} e_\chi KG)\big)$. The simple unipotent representations of kG are the same as the simple A-modules. The algebra A is Morita equivalent to the q-Schur algebra of $\mathfrak{S}_n$ over $\mathcal{O}$ specialized at $x = q$ [DipJa, Ta].

The q-Schur algebra is the endomorphism ring of the direct sum of induced trivial modules from Hecke algebras of all standard parabolic subgroups. For ℓ large enough, its decomposition matrix is the same as the one obtained for x a primitive d-the root of unity over $\mathbf{C}$. So, the square part of the unipotent decomposition matrix of $\mathrm{GL}_n(q)$ coincides with the decomposition matrix of the q-Schur algebra in characteristic 0, at a primitive d-th root of unity. One deduces the genericity property for decomposition matrices of $\mathrm{GL}_n(q)$. Furthermore, this matrix has a description in terms of the combinatorics of the canonical basis of the Fock space for the quantum group of $\mathfrak{sl}_d$ [LaLeTh, LeTh, Ar].

Remark 5.6. One can define analogs of q-Schur algebras by generalizing the construction to other types of groups, but they do not seem to have good descriptions nor good properties like quasi-heredity, except under particular assumptions making the category of representations look like the one for general linear groups (for example, classical groups and linear primes). The case of unipotent blocks with cyclic defect, fully understood now [CrDuRou], shows already the substantial complications related to the presence of cuspidal representations. We propose in §6 to take a limit $q \to 1$ in the ℓ-adic topology, which makes $q \to \infty$ in the real topology.

5.4. **Broué's conjecture.**

5.4.1. *General version.* Let b be a block idempotent of $\mathcal{O}G$, D a defect group. Assume D is abelian and $\mathbf{L} = C_{\mathbf{G}}(D)$ is a Levi subgroup of $\mathbf{G}$. Let $b_D \in \mathcal{O}L$ be the Brauer correspondent of b and let b'_D be a block idempotent of $\mathcal{O}L$ with $b'_D b_D = b'_D$.

Given $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$, there is a complex of $(\mathcal{O}G, \mathcal{O}L)$-bimodules $\Lambda_c(Y_{\mathbf{U}}, \mathcal{O})b'_D$.

Conjecture 5.7. *There is a choice of* $\mathbf{P}$ *and an extension of the right action of* $C_G(D)$ *on* $\Lambda_c(Y_{\mathbf{U}}, \mathcal{O})b'_D$ *to an action of* $N_G(D, b'_D)$ *such that* $\Lambda_c(Y_{\mathbf{U}}, \mathcal{O})b'_D$ *induces a Rickard equivalence between* $b\mathcal{O}G$ *and* $b'_D\mathcal{O}N_G(D, b'_D)$.

We refer to §7.1.2 for the notion of Rickard equivalences.

The choice of $\mathbf{P}$ and the construction of the extension of the action have been the source of developments involving complex reflection groups, their braid groups and Hecke algebras, regular elements and centralizers, Garside categories and Deligne-Lusztig varieties [Brou3].

With J. Chuang, we conjecture that the derived equivalence will be perverse, with a non-decreasing perversity function (for an order as in §5.3.4) [ChRou]. This would imply the triangularity of the decomposition matrix (cf §2.4.2), a known result (Theorem 5.5). A conjectural perversity function has been proposed by Craven [Cr], cf §5.4.3 below. This implies that the module category of a block with abelian defect groups is determined by Weyl-group type data, toegther with the perversity function.

When $\ell|(q-1)$, the parabolic subgroup $\mathbf{P}$ can be chosen to be F-stable and the conjecture was proven by Puig [Pu] (cf [CaEn2, Theorem 23.12] for a detailed exposition of the principal block case). The difficulty is to construct an extension of the action. As a consequence, for unipotent blocks, the unipotent square part of the decomposition matrix is diagonal, a fact obtained independently by Hiß [Hi, Korollar 3.2].

5.4.2. *Case of a torus.* We assume that $\ell \not| (q-1)$, that b is the principal block idempotent and that $\mathbf{L} = \mathbf{T}$ is a torus. So D is a Sylow ℓ-subgroup of G. Since $C_{\mathbf{G}}(D) = \mathbf{T}$, it follows that this torus corresponds to a regular F-conjugacy class (w) of elements of W (cf §7.2.3). Furthermore, the group $N_G(D)/C_G(D)$ is isomorphic to $C_W(w\phi)$, a complex reflection group. We denote by B_d its braid group (cf §7.2.1).

Let $w \in W$ such that $(\lambda(w)\phi)^d = \pi\phi^d$. As explained in §5.1.3, there is a right action of $T \rtimes C_{B_W^+}(\lambda(w)\phi)$ on $Y(\lambda(w))$, hence a right action on $\Lambda_c(Y(\lambda(w)), \mathcal{O})$ commuting with the action of G.

It is conjectured that there is a representative C of $\Lambda_c(Y(\lambda(w)), \mathcal{O})$ in the quotient of the homotopy category of complexes of $\mathcal{O}(G \times (T \rtimes C_{B_W^+}(\lambda(w)\phi))^{\mathrm{opp}})$-modules by complexes whose restriction to $\mathcal{O}(G \times T)$ is homotopy equivalent to 0 with the following properties:

- the right action of $\mathcal{O}(T \rtimes C_{B_W^+}(\lambda(w)\phi))$ on C factors through an action of $\mathcal{O}N_G(D)b_D$
- the resulting complex of $(\mathcal{O}Gb, \mathcal{O}N_G(D)b_D)$-bimodules induces a Rickard equivalence between the principal blocks of G and $N_G(D)$.

This conjecture is known to hold when $X(w)$ is a curve [Rou2, Corollaire 4.7] and when w is a Coxeter element [BoRou2, Du1, Du3, DuRou]. In those cases the monoid $C_{B_W^+}(\lambda(w)\phi)$ is cyclic and its action on $Y(\lambda(w))$ is given by powers of F.

5.4.3. *Disjunction of cohomology for Deligne-Lusztig varieties.* After extending scalars to $\bar{\mathbf{Q}}_\ell$, one obtains a version of conjecture 5.7 that is also an open problem. Restricting to unipotent representations, the crucial missing fact is the disjunction of the cohomology groups. We state here a conjecture of [BrouMi2] for the case where $\mathbf{L} = \mathbf{T}$ is a torus.

Conjecture 5.8. *Let $w \in W$ such that $(\lambda(w)\phi)^m = \pi\phi^m$ for some $m \geq 1$. Given $i \neq j$, we have* $\mathrm{Hom}_{\bar{\mathbf{Q}}_\ell G}(H_c^i(X(w), \bar{\mathbf{Q}}_\ell), H_c^j(X(w), \bar{\mathbf{Q}}_\ell)) = 0$.

Craven [Cr] has defined a function $C_m : \mathrm{Irr}_{\bar{\mathbf{Q}}_\ell}(G, 1) \to \mathbf{Z}$ depending only on the generic degree of the representation and on m and has conjectured that when ρ occurs in $H_c^i(X(w), \bar{\mathbf{Q}}_\ell)$, it occurs in degree $i = C_m(\rho)$.

Conjecture 5.8 (together with Craven's conjecture) is known to hold when w is a Coxeter element [Lu1] and for groups of rank 2 [DigMiRou]. For GL_n, it is known in general (cf [DigMi1] for $m = n - 1$ and [Du2, Corollary 3.2] and [BoDuRou, Theorem 4.3] in general).

This conjecture is implied by the refined version of Conjecture 5.7 discussed in §5.4.2. In the case where $w = w_0$, Conjecture 5.8 is older and due to Lusztig [Lu2, p.25, line 13]. Also, Conjecture 5.8 was proven earlier by Lusztig when w is a Coxeter element of minimal length in its class [Lu1].

Conjecture 5.8 can be extended to $X(b)$, where $b \in B_W^+$ is such that $(bF)^m = \pi^r F^m$ for some $m, r \geq 1$. The particular case where $b = \pi$ is known to hold [BoDuRou].

Conjecture 5.8 can be extended to the case of Deligne-Lusztig varieties associated to Levi subgroups [DigMi2].

More recently, Lusztig [Lu4, §7] has conjectured that the disjunction property of Conjecture 5.8 should hold (in the split case) for elements $w \in W$ of minimal length in their conjugacy class and such that the trace of the endomorphism of the Hecke algebra given by $h \mapsto T_w h T_{w^{-1}}$ is in $\mathbf{Z}_{\geq 0}[x]$.

6. Degeneration and genericity

We consider here the setting of §3.2.2.

6.1. Classifying spaces and character sheaves.

6.1.1. *Completed classifying spaces.* Consider the ring of Witt vectors $R = W(\bar{\mathbf{F}}_p)$. Let $\mathbf{G}_R$ be a reductive algebraic group over R with a maximal torus $\mathbf{T}_R$ and with an isomorphism $\mathbf{G}_R \times_R \bar{\mathbf{F}}_p \xrightarrow{\sim} \mathbf{G}$ restricting to $\mathbf{T}_R \times_R \bar{\mathbf{F}}_p \xrightarrow{\sim} \mathbf{T}_0$. We fix an embedding of R into $\mathbf{C}$ and we denote by $\mathbf{G}(\mathbf{C}) = \mathbf{G}_R(\mathbf{C})$ the associated complex Lie group. We also put $\mathbf{T}(\mathbf{C}) = \mathbf{T}_R(\mathbf{C})$.

Specialization provides an isomorphism $\mathrm{Aut}(\mathbf{T}_R) \xrightarrow{\sim} \mathrm{Aut}(\mathbf{T}_0)$ and we denote by φ the automorphism of $\mathbf{T}_R$ lifting F.

We denote by φ the automorphism of $\mathbf{T}(\mathbf{C})$ induced by F. The corresponding automorphism $B\varphi$ of the ℓ-completed classifying space $(B\mathbf{T}(\mathbf{C}))^\wedge_\ell$ extends uniquely to an automorphism ψ of $(B\mathbf{G}(\mathbf{C}))^\wedge_\ell$ ([Fr1, Theorem 1.6] and [JaMcOl, Theorem 2.5]).

Furthermore, a theorem of Friedlander [Fr2, Theorem 12.2] (cf also [BrotMoOl, Theorem 3.1]) shows there is an isomorphism

$$(BG)^\wedge_\ell \xrightarrow{\sim} \left((B\mathbf{G}(\mathbf{C}))^\wedge_\ell\right)^{h\psi}$$

where $h\psi$ denotes taking homotopy fixed points by the group $\mathbf{Z}$ acting as powers of ψ.

6.1.2. *Dependence on q.* Consider the group $\mathrm{Out}((B\mathbf{G}(\mathbf{C}))^\wedge_\ell)$ of homotopy classes of homotopy automorphisms of $(B\mathbf{G}(\mathbf{C}))^\wedge_\ell$. There is an isomorphism [AnGro, Theorem 1.2]

$$\mathrm{Out}((B\mathbf{G}(\mathbf{C}))^\wedge_\ell) \xrightarrow{\sim} N_{\mathrm{GL}(Y(\mathbf{T}_0)\otimes\mathbf{Z}_\ell)}(W, \{\mathbf{Z}_\ell\beta\}_{\beta\in\Phi^\vee})/W.$$

Given $\alpha \in \mathrm{Out}((B\mathbf{G}(\mathbf{C}))^\wedge_\ell)$, the space $(B\mathbf{G}(\mathbf{C}))^\wedge_\ell)^{h\alpha}$ depends only on the closed subgroup $\overline{\langle\alpha\rangle}$ of $\mathrm{Out}((B\mathbf{G}(\mathbf{C}))^\wedge_\ell)$ [BrotMoOl, Corollary 2.5], where we use the ℓ-adic topology.

When $(\mathbf{G}, F)$ is split, then φ is the automorphism $x \mapsto x^q$ and ψ is the unstable Adams operation ψ^q. Note that the unstable Adams operation ψ^q is defined more generally for $q \in \mathbf{Z}_\ell^\times$.

In general, when $\mathbf{G}$ is simple and F is a Frobenius endomorphism, then the element of $N_{\mathrm{GL}(Y(\mathbf{T}_0)\otimes\mathbf{Z}_\ell)}(W, \{\mathbf{Z}_\ell\beta\}_{\beta\in\Phi^\vee})/W$ induced by ψ is of the form $\sigma \cdot (q\,\mathrm{id})$ where σ has finite order and $\psi = \sigma\psi^q$ (up to homotopy). In types ${}^2A_n$, ${}^2D_{2n+1}$ and ${}^2E_6$, one has also $\psi = \psi^{-q}$.

The description $\psi = \sigma \cdot (q\,\mathrm{id})$ still works for types ${}^2B_2$ and ${}^2F_4$ (resp. ${}^2G_2$) when 2 (resp. 3) is a square modulo ℓ.

Assume ℓ is odd. The space $(BG)^\wedge_\ell$ depends only on the order d of q in $\mathbf{F}_\ell^\times$ and on $\nu_\ell(q^d - 1)$ [BrotMoOl, §3 and Proposition 3.2].

Also, $B({}^2A_n(q))^\wedge_\ell \simeq B(A_n(q'))^\wedge_\ell$, $B({}^2D_{2n+1}(q))^\wedge_\ell \simeq B(D_{2n+1}(q'))^\wedge_\ell$ and $B({}^2E_6(q))^\wedge_\ell \simeq B(E_6(q'))^\wedge_\ell$ if q and $-q'$ have the same order d in $\mathbf{F}_\ell^\times$ and $\nu_\ell(q^d-1) = \nu_\ell((-q')^d-1)$ [BrotMoOl, Proposition 3.3].

Note that $(BG)^\wedge_\ell$ determines the thick subcategory of $D^b(\mathbf{F}_\ell G)$ generated by the trivial module, so this triangulated category has the same genericity properties.

6.1.3. *Classifying spaces of loop groups.* We assume $(\mathbf{G}, F)$ is split and put $\varepsilon = 1$ or $(\mathbf{G}, F)$ has type ${}^2A_n$, ${}^2D_{2n+1}$ or ${}^2E_6$ and put $\varepsilon = -1$. We assume $\ell | \varepsilon q - 1$.

We have a family of spaces $(B\mathbf{G}(\mathbf{C})^\wedge_\ell)^{h\psi_{1+\ell\bar{h}}}$ over $\mathbf{Z}_\ell$, constant over $\mathbf{Z}_\ell^\times$-orbits:

$$\begin{tikzcd}
(B\mathbf{G}(\mathbf{C})^\wedge_\ell)^{h\psi_{1+\ell\bar{h}}} \arrow[d] & \\
\mathbf{Z}_\ell = \{\bar{h}\} \arrow[r] \arrow[dr, "\nu_\ell"'] & \mathbf{Z}_\ell/\mathbf{Z}_\ell^\times \arrow[d, "\sim"'] \\
& \mathbf{Z}_{\geq 0} \cup \{\infty\}
\end{tikzcd}$$

One could attempt to make sense of this as a continuous family, and then make sense of the limit of those spaces as $\bar{h} \to 0$ to obtain

$$\text{"}\lim_{\bar{h}\to 0}\text{"}(B\mathbf{G}(\mathbf{C})^\wedge_\ell)^{h\psi_{1+\ell\bar{h}}} \simeq (B\mathbf{G}(\mathbf{C})^\wedge_\ell)^{h\mathrm{id}} = L(B\mathbf{G}(\mathbf{C}))^\wedge_\ell \simeq B(L\mathbf{G}(\mathbf{C}))^\wedge_\ell,$$

where $L(X) = \mathrm{Maps}(S^1, X)$ is the free loop space and in particular $L\mathbf{G}(\mathbf{C})$ is the loop group associated to $\mathbf{G}(\mathbf{C})$. So, from the point of view of ℓ-completed classifying spaces, the loop group $L\mathbf{G}(\mathbf{C})$ appears as $\mathbf{G}(\mathbf{F}_1)$.

In other terms,

$$\text{"}\lim_{\nu_l(\varepsilon q-1)\to\infty}\text{"} B\mathbf{G}(\mathbf{F}_q)^\wedge_\ell \simeq B(L\mathbf{G}(\mathbf{C}))^\wedge_\ell.$$

So the space $B(L\mathbf{G}(\mathbf{C}))^\wedge_\ell$ appears as a degeneration of the family of spaces $B\mathbf{G}(\mathbf{F}_q)^\wedge_\ell$ for varying q. Here, we use the abusive notation $\mathbf{G}(\mathbf{F}_q)$ to denote a possibly twisted group in a family.

As a consequence, we have also a description of the limit ("generic version") of the thick subcategory of $D^b(\mathbf{F}_\ell G)$ generated by the trivial module as $\nu_\ell(\varepsilon q-1)\to\infty$: it is the homotopy category of perfect A_∞-modules over the A_∞-algebra $H^*(BL\mathbf{G}(\mathbf{C}),\mathbf{F}_\ell)$, since the thick subcategory of $D^b(\mathbf{F}_\ell G)$ generated by the trivial module is equivalent to perfect A_∞-modules over $\mathrm{Ext}^*_{\mathbf{F}_\ell G}(\mathbf{F}_\ell,\mathbf{F}_\ell)$.

Note that while the family of algebras $H^*(G,\mathbf{F}_\ell)$ stabilizes [KiKo, Theorem 18], the stabilization does not hold when the A_∞-algebra structure is taken into account (cf Remark 6.1 below).

Remark 6.1. When $\mathbf{G}=\mathbf{G}_m$, we have $BL\mathbf{G}(\mathbf{C})=BL\mathbf{C}^\times\simeq S^1\times\mathbf{CP}^\infty$, a space whose mod-$\ell$ cohomology is formal as an algebra. The A_∞-structure on $H^*(B\mathbf{F}_q^\times,\mathbf{F}_\ell)$ can be chosen so that there is a single higher multiplication m_{ℓ^r}, where $r=\nu_\ell(q-1)$ [Mad, Appendix B, Example 2.2]. This higher multiplication disappears in the limit $r\to\infty$.

Remark 6.2. The relation between the cohomology of the finite group and that of the loop group becomes much more subtle for small ℓ (and r). An approach using the string topology is given in [GroLa].

Remark 6.3. Considering the usual topology instead of the ℓ-adic one, we obtain $\lim B\mathbf{G}(\mathbf{F}_q)^\wedge_\ell = B\mathbf{G}(\bar{\mathbf{F}}_q)^\wedge_\ell\simeq B\mathbf{G}(\mathbf{C})^\wedge_\ell$ [FrMi, Theorem 1.4].

6.1.4. *Rigidification and character sheaves.* In the previous section, we explained how to obtain, under some assumptions, a generic version of the modular representation theory of G, in the neighborhood of the trivial representation.

To move away from the neighborhood of the trivial representation, we consider a more rigid version of $B(L\mathbf{G}(\mathbf{C}))^\wedge_\ell$. There is a homotopy equivalence $B(L\mathbf{G}(\mathbf{C}))\simeq\frac{\mathbf{G}(\mathbf{C})}{h\mathbf{G}(\mathbf{C})}$, the homotopy adjoint quotient.

We can now consider the derived category of $\mathcal{D}$-modules on the stack $\frac{\mathbf{G}(\mathbf{C})}{\mathbf{G}(\mathbf{C})}$, or constructible sheaves with k-coefficients. This is the $\mathbf{G}(\mathbf{C})$-equivariant derived category of $\mathbf{G}(\mathbf{C})$, for the adjoint action, and it has a thick subcategory of unipotent objects, also called the derived unipotent character sheaves, providing a non semi-simple enrichment of Lusztig's theory [BeZNa, Definition 6.8]. It is conjectured that the principal series part of this triangulated category (i.e., its principal block) is equivalent, for ℓ not too small, to the derived category of differential graded modules over $k[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$, where $\mathfrak{h}$ is the Lie algebra of $\mathbf{T}_0$ over k. A similar result is known for the adjoint quotient $\frac{\mathrm{Lie}\mathbf{G}}{\mathbf{G}}$ [Rid].

For $\mathbf{G}=\mathrm{GL}_n$, all unipotent character sheaves are in the principal series, so the description coincides with our degeneration approach in §6.2 below.

An important problem is to find a conjecture for an algebraic description of the derived unipotent character sheaves, beyond the principal series, starting with the case $\mathbf{G}=\mathrm{Sp}_4$. This is related to the problem of finding a canonical generic description of the category of unipotent representations in characteristic zero, cf [Lu3].

Note that the category breaks down according to Harish-Chandra series, but it would be desirable to find a description that does not use cuspidal objects.

6.1.5. *General d.* The first constructions of §6.1.3 can be performed without the assumption that $\ell|\varepsilon q-1$. Denote by d the order of εq in $\mathbf{F}_\ell^\times$. Let ζ be a primitive d-th root of unity in $\mathbf{Z}_\ell$.

One can consider the family of spaces $(B\mathbf{G}(\mathbf{C})_\ell^\wedge)^{h\psi_{\zeta+\ell\bar{h}}}$ over $\mathbf{Z}_\ell=\{\bar{h}\}$ and

$$"\lim_{\nu_l(\Phi_d(\varepsilon q))\to\infty}"B\mathbf{G}(\mathbf{F}_q)_\ell^\wedge\simeq"\lim_{\bar{h}\to 0}"(B\mathbf{G}(\mathbf{C})_\ell^\wedge)^{h\psi_{\zeta+\ell\bar{h}}}\simeq L(B\mathbf{G}(\mathbf{C})_\ell^\wedge)^{h\mu_d}.$$

Here μ_d is the cyclic group of order d acting on $B\mathbf{G}(\mathbf{C})_\ell^\wedge$ by ψ_x, $x\in\mu_d(\mathbf{Z}_\ell)$.

The space $(BG(\mathbf{C})_\ell^\wedge)^{h\mu_d}$ is an ℓ-compact group [Gro]. Its "Weyl group" is a complex (or rather ℓ-adic) reflection group, not a Coxeter group in general. To proceed as in §6.1.4 we would need an appropriately rigidified version of the space $L(BG(\mathbf{C})_\ell^\wedge)^{h\mu_d}$.

Remark 6.4. In [KeMalSe], Kessar, Malle and Semeraro explain how to understand Alperin's conjecture in the setting of ℓ-completed classifying spaces. One can expect there is a framework which encompasses both the cohomological aspects, which was our starting point, and the character counts, which they study.

6.2. **Degeneration.**

6.2.1. *Degeneration of group algebras of abelian ℓ-groups.* Let P be an abelian ℓ-group isomorphic to $(\mathbf{Z}/\ell^r)^n$. Let $V=J(\mathbf{F}_\ell P)/J(\mathbf{F}_\ell P)^2$. This is an n-dimensional vector space over $\mathbf{F}_\ell$. Fix a morphism of $\mathbf{F}_\ell$-modules $\sigma: V\to J(\mathbf{F}_\ell P)$ that is a right inverse to the quotient map $J(\mathbf{F}_\ell P)\to V$. The map σ extends uniquely to a morphism of $\mathbf{F}_\ell$-algebras $S(V)\to\mathbf{F}_\ell P$. That morphism induces an isomorphism

$$S(V)/(v^{\ell^r})_{v\in V}\xrightarrow{\sim}\mathbf{F}_\ell P. \tag{2}$$

Consider now a finite ℓ'-group E acting on P. The vector space V is an $\mathbf{F}_\ell E$-module. Since $J(\mathbf{F}_\ell P)$ is a semi-simple $\mathbf{F}_\ell E$-module, there exists a σ as above that is a morphism of $\mathbf{F}_\ell E$-modules. The isomorphism (2) is equivariant for the action of E, hence it extends to an isomorphism of $\mathbf{F}_\ell$-algebras

$$\big(S(V)/(v^{\ell^r})_{v\in V}\big)\rtimes E\xrightarrow{\sim}\mathbf{F}_\ell(P\rtimes E).$$

Consider now a general finite abelian ℓ-group P acted on by a finite ℓ'-group E. There exists an E-stable decomposition $P=P_1\times\cdots\times P_m$ such that $P_i\simeq(\mathbf{Z}/\ell^{r_i})^{n_i}$ for some r_i and n_i. Put $V_i=J(\mathbf{F}_\ell P_i)/J(\mathbf{F}_\ell P_i)^2$. The construction above provides an isomorphism of $\mathbf{F}_\ell$-algebras

$$\big(S(V)/(\bigcup_i\{v^{\ell^{r_i}}\}_{v\in V_i})\big)\rtimes E\xrightarrow{\sim}\mathbf{F}_\ell(P\rtimes E).$$

Consider the graded $\mathbf{F}_\ell[t]$-algebra $A=\mathbf{F}_\ell[t]\otimes\Lambda(V)\otimes S(V)$, where $\mathbf{F}_\ell[t]\otimes\mathbf{F}_\ell\otimes S(V)$ is in degree 0 and $\mathbf{F}_\ell\otimes V\otimes\mathbf{F}_\ell$ is in degree -1.

We define a structure of differential $(\mathbf{F}_\ell[t]\otimes\mathbf{F}_\ell\otimes S(V))$-algebra on A by setting $d(1\otimes v\otimes 1)=t\otimes 1\otimes v^{\ell^{r_i}}$ for $v\in V_i$.

We have $H^i(\mathbf{F}_\ell(t)\otimes_{\mathbf{F}_\ell[t]}A)=0$ for $m\neq 0$ and

$$H^0(\mathbf{F}_\ell(t)\otimes_{\mathbf{F}_\ell[t]}A)=(\mathbf{F}_\ell(t)\otimes S(V))/(\bigcup_i\{tv^{\ell^{r_i}}\}_{v\in V_i}\simeq\mathbf{F}_\ell(t)\otimes S(V)/(\bigcup_i\{v^{\ell^{r_i}}\}_{v\in V_i}).$$

So, the algebra $\mathbf{F}_\ell(P \rtimes E)$ is, up to quasi-isomorphism, a deformation of the graded algebra $\big(\Lambda(V) \otimes S(V)\big) \rtimes E$. The derived category of $\mathbf{F}_\ell(P \rtimes E)$-modules is a deformation of the derived category of dg modules over the graded algebra (with zero differential) $\big(\Lambda(V) \otimes S(V)\big) \rtimes E$.

Koszul duality provides an equivalence from the derived category of finitely generated differential graded modules over the graded algebra (with zero differential) $\big(\Lambda(V) \otimes S(V)\big) \rtimes E$ to the derived category of finitely generated differential graded modules over the graded algebra $S(V^* \oplus V) \rtimes E$ (here V is in degree 0 and V^* in degree 2).

To summarize, $D^b(\mathbf{F}_\ell(P \rtimes E))$ degenerates into the derived category of differential graded coherent sheaves on the orbifold $[(V \times V^*)/E]$.

6.2.2. *Genericity of perverse equivalences.* The discussion here is based on joint work with David Craven [CrRou2]. We consider the setting of §5.4.1 and we assume to simplify that b is the principal block. So D is a Sylow ℓ-subgroup of G and there is an isomorphism of algebras $kN_G(D)b_D \simeq kD \rtimes E$ where $E = N_G(D)/C_G(D)$.

It is conjectured that there is a perverse equivalence between $kD \rtimes E$ and kGb, with a specific perversity function $\pi : \mathrm{Irr}_{\mathbf{C}}(E) \xrightarrow{\sim} \mathrm{Irr}_k(E) \to \mathbf{Z}$. That function depends only on the type of the group G and on d, not on q or ℓ.

As explained in §3.4.2, the group E is a reflection group. We denote by K_E the field of definition of its reflection representation V and by $\mathcal{O}_E$ the ring of integers of K_E. Let $R = \mathcal{O}_E[|W|^{-1}]$ and let V_R be an RE-module, finitely generated and projective over R, such that $V \simeq K \otimes_R V_R$.

We conjecture that the function π defines a t-structure on the derived category of differential graded modules over the graded algebra $(\Lambda(V_R) \otimes S(V_R)) \rtimes E$, where $(R \otimes S(V_R)) \rtimes E$ is in degree 0 and $V_R \otimes R$ in degree -1. The heart $\mathcal{A}$ of that t-structure would be a "generic version" of kGb, i.e., a limit as $\nu_\ell(\Phi_d(\varepsilon q)) \to \infty$. A ridigity property of perverse simple objects would show that the classes of the indecomposable projective objects of $\mathcal{A} \otimes_R K_E$ expressed in terms of the classes of the simple $K_E E$-modules would give the transpose of the square unipotent part of the decomposition matrix of the principal ℓ-block of G for ℓ large enough. Note that the presence of a double grading on $(\Lambda(V) \otimes S(V)) \rtimes E$ leads to a two-variable deformation of the matrix.

Remark 6.5. The discussion generalizes to the case of non-principal blocks. The block of the normalizer is isomorphic to a twist of the group algebra of the semi-direct product by a 2-cocycle but that cocycle is expected to be always trivial.

Remark 6.6. We expect the algebra $S(V \oplus V^*) \rtimes E$ to control generic aspects of the modular representation theory of G. This algebra admits deformations as rational Cherednik algebras and the "$t = 0$" case is expected to relate to unipotent representations [BoRou4, Bo].

6.2.3. *Hilbert schemes.* We discuss here joint work with Olivier Dudas [DuRou].

Assume that $\mathbf{G} = \mathrm{GL}_n$. Let $m = \lfloor \frac{n}{d} \rfloor$. We have $V \simeq K_E^m$ and $W \simeq (\mathbf{Z}/d)^m \rtimes \mathfrak{S}_m$. Let X_d be the minimal resolution of $\mathbf{A}^2_{K_E}/(\mathbf{Z}/d)$, where $\mathbf{Z}/d$ is embedded in $\mathrm{SL}_2(K_E)$.

Let $\mathrm{Hilb}^m(X_d)$ be the Hilbert scheme of m points on X_d and $\pi : \mathrm{Hilb}^n(X_d) \to S^m(X_d)$ be the Hilbert-Chow map. Let $f : \mathbf{A}^{2m} \to S^m(X_d)$ be the quotient map by $(\mathbf{Z}/d)^m \rtimes \mathfrak{S}_m$.

Combining Koszul duality with the derived McKay equivalence, we obtain an equivalence between the derived category of differential graded $(\Lambda(V) \otimes S(V)) \rtimes E$-modules and the derived category of dg coherent sheaves on $\mathrm{Hilb}^m(X_d)$, where we consider the $\mathbf{G}_m$-action on X_d coming from its action on $\mathbf{A}^2$ with weights 0 and -2. The conjecture in §6.2.2 implies the existence of a particular t-structure on that derived category.

When $d = 1$ and $\varepsilon = -1$, the combinatorics of Macdonald polynomials can be used to obtain a conjectural combinatorial formula for the two-parameter deformed decomposition matrix of $U_n(q)$. That conjecture has been checked for $n \leq 11$, using the determination of the decomposition matrices in [DuMal1].

7. Appendix

7.1. **Representations.**

7.1.1. *Categories.* Let A be an algebra over a commutative regular local noetherian ring R and assume A is a free R-module of finite rank. By module, we mean left module. We identify right A-modules with left modules for the opposite algebra A^{opp}.

Given M an A-module, we put $M^* = \mathrm{Hom}_R(M, R)$, a right A-module.

We denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of simple A-modules.

We denote by A-mod the abelian category of finitely generated A-modules. We denote by $G_0(A)$ the Grothendieck group of A-mod.

We denote by $D^b(A)$ (resp. $\mathrm{Ho}^b(A)$) the derived (resp. homotopy) category of bounded complexes of finitely generated A-modules. We denote by A-perf the full subcategory of $D^b(A)$ of complexes quasi-isomorphic to bounded complexes of finitely generated projective A-modules.

Let A-stab be the triangulated category quotient $D^b(A)/A\text{-perf}$. When A is symmetric as an R-algebra, the inclusion $A\text{-mod} \hookrightarrow D^b(A)$ induces an equivalence of categories from the additive category quotient of A-mod by its subcategory of A-modules of the form $A \otimes_R V$ where $V \in R$-mod, to A-stab.

7.1.2. *Equivalences.* Let A and B be two finite-dimensional algebras over a commutative noetherian ring R.

Let C be a bounded complex of (A, B)-bimodules, all of whose terms are finitely generated and projective as left A-modules and as right B-modules. Assume there is a complex L (resp. M) of (A, A)-bimodules (resp. (B, B)-bimodules) such that there are isomorphisms of complexes of (A, A)-bimodules and (B, B)-bimodules

$$C \otimes_B C^* \simeq A \oplus L \text{ and } C^* \otimes_A C \simeq B \oplus M.$$

We say that M induces a

- Morita equivalence if $C^i = 0$ for $i \neq 0$ and $L = M = 0$
- Rickard equivalence if L and M are homotopy equivalent to 0
- derived equivalence if L and M are acyclic
- stable equivalence if L and M are perfect.

These conditions ensure that $C \otimes_B -$ induces an equivalence

- (Morita) $B\text{-mod} \xrightarrow{\sim} A\text{-mod}$
- (Rickard) $\mathrm{Ho}^b(B) \xrightarrow{\sim} \mathrm{Ho}^b(A)$
- (derived) $D^b(B) \xrightarrow{\sim} D^b(A)$
- (stable) $B\text{-stab} \xrightarrow{\sim} A\text{-stab}$

7.1.3. *Finite groups.* Let G be a finite group. We put $\mathrm{Irr}_R(G) = \mathrm{Irr}(RG)$. Consider a prime p and a finite field extension K of $\mathbf{Q}_p$. Let $\mathcal{O}$ be its ring of integers and k the residue field. We assume that K contains all $|G|$-th roots of unity. This ensures that KG is a product of matrix algebras over K and that all simple kG modules are absolutely simple.

Let M be a finitely generated KG-module. There exists an $\mathcal{O}G$-module M' that is free over $\mathcal{O}$ and such that $M' \otimes_{\mathcal{O}} K \simeq M$. Let $M'' = M' \otimes_{\mathcal{O}} k$. The class $[M'']$ in $G_0(kG)$ depends only on $[M] \in G_0(KG)$ and we put $\mathrm{dec}([M]) = [M'']$. This defines a morphism of abelian groups, the *decomposition map*, $\mathrm{dec} : G_0(KG) \to G_0(kG)$. The *decomposition matrix* is the matrix of dec in the bases $\mathrm{Irr}_k(G)$ (columns) and $\mathrm{Irr}_K(G)$ (rows).

7.2. **Braid groups and Hecke algebras.**

7.2.1. *Braid groups.* Let V be a finite dimensional complex vector space. A *reflection* s of V is a finite order automorphism of V such that $\ker(s-1)$ is a hyperplane.

Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections (a *complex reflection group*). Let $\mathcal{R}$ be the set of reflections in W and $\mathcal{A} = \{\ker(s-1)\}_{s\in\mathcal{R}}$ be the set of reflecting hyperplanes.

We put $V^{reg} = V \setminus \bigcup_{H\in\mathcal{A}} H$. The group W acts freely on V^{reg}, i.e., the quotient map $q : V^{reg} \to V^{reg}/W$ is unramified.

Let $x_0 \in V^{reg}$. The *braid group* of W is $B_W = \pi_1(V^{reg}/W, q(x_0))$. The map q gives a bijection from (homotopy classes of) paths in V^{reg} starting at x_0 and ending in $W(x_0)$ to (homotopy classes of) loops in V^{reg}/W based at $q(x_0)$, and we will identify those two types of objects. There is a surjective morphism $B_W \to W$: it sends w to the homotopy class of a path in V^{reg} from x_0 to $w(x_0)$. We denote by $\pi \in B_W$ the homotopy class of the path $t \mapsto \exp(2i\pi t)x_0$. This is a central element of B_W.

7.2.2. *Hecke algebras.* Given $H \in \mathcal{A}$, let e_H be the order of the fixator of H in W. Let $R = \mathbf{Z}[\{q_{H,r}^{\pm 1}\}_{H\in\mathcal{A}/W, 0\le r<e_H}]$.

We define the *Hecke algebra* $\mathcal{H} = \mathcal{H}(W)$ of W as the quotient of the group algebra RB_W by the ideal generated by $\prod_{0\le r<e_H}(\sigma_H - q_{H,r})$, where H runs over $\mathcal{A}$ and σ_H is a generator of the monodromy around the image of H in V/W [BrouMalRou, Definition 4.21].

The specialization $q_{H,r} \mapsto \exp(2i\pi r/e_H)$ of $\mathcal{H}$ is the group algebra $\mathbf{Z}W$.

7.2.3. *Regular elements.* We recall some constructions and results of Springer [Sp].

Let σ be an element of finite order of $N_{\mathrm{GL}(V)}(W)$. Let $w \in W$ and let $v \in V^{reg}$ be an eigenvector of $w\sigma$ with eigenvalue ζ. Let d be the order of ζ. The element $w\sigma$ is said to be *ζ-regular*, or *d-regular*. If $w' \in W$ and $w'\sigma$ is ζ-regular, then $w'\sigma$ is W-conjugate to $w\sigma$.

Let $V_\zeta = \ker(w\sigma - \zeta)$. The group $C_W(w\sigma)$ acting on V_ζ is a reflection group.

The inclusion $V_\zeta \hookrightarrow V$ induces an isomorphism $\iota_\zeta : V_\zeta/C_W(w\sigma) \xrightarrow{\sim} (V/W)^{\mu_d}$, where $\mu_d = \{\zeta^n \,\mathrm{id}_V\}_{n\in\mathbf{Z}/d}$.

Assume $\zeta = \exp(2i\pi/d)$ and $x_0 = v$. There exists $\mathbf{w}_d \in B_W$ such that $(\mathbf{w}_d\sigma)^d = \pi\sigma^d$ [BrouMi2, Proposition 6.5]. When $\sigma = 1$ we can take for $\mathbf{w}_d \in B_W$ the homotopy class of the path $t \mapsto \exp(2i\pi t/d)x_0$.

The map ι_ζ induces a morphism $B_{C_W(w\sigma)} = \pi_1(V_\zeta^{reg}/C_W(w\sigma), q(x_0)) \to B_W = \pi_1(V^{reg}/W, q(x_0))$. Its image is contained in $C_{B_W}(\mathbf{w}_d\sigma)$.

7.2.4. *Real reflection groups.* We assume now that $V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ and W is a subgroup of $\mathrm{GL}(V_{\mathbf{R}})$. All reflections of W have order 2.

Fix a connected component C of the space $V_{\mathbf{R}} \cap V^{reg}$ and let $\bar{C}$ be its closure. Let S be the subset of $\mathcal{R}$ of reflections s such that $\ker(s-1) \cap \bar{C}$ has codimension 1 in $V_{\mathbf{R}}$. Then (W, S) is a Coxeter group. We denote by $l : W \to \mathbf{Z}_{\geq 0}$ its length function: given $w \in W$, the integer $l(w)$ is the minimal m such that $w = s_{i_1} \cdots s_{i_m}$ for some $s_{i_1}, \ldots, s_{i_m} \in S$.

Choose now $x_0 \in C$. Given $s \in S$, let $\sigma_s \in B_W$ be the homotopy class of the path that is the concatenation of $t \mapsto x_0 + tix_0$, $t \mapsto (1-t)x_0 + ts(x_0) + ix_0$ and $t \mapsto s(x_0) + (1-t)ix_0$.

There is an isomorphism

$$\langle (b_s)_{s \in S} \mid \underbrace{b_s b_t b_s \cdots}_{m_{st} \text{ terms}} = \underbrace{b_t b_s b_t \cdots}_{m_{st} \text{ terms}} \forall s, t \in S\} \xrightarrow{\sim} B_W,\ b_s \mapsto \sigma_s \tag{3}$$

where m_{st} is the order of st [Bri]. We identify B_W with the group on the left side of (3) and we denote by B_W^+ its submonoid generated by $(b_s)_{s \in S}$.

There is a map $\lambda : W \to B_W$ given by $\lambda(w) = b_{s_1} \cdots b_{s_r}$ if $w = s_1 \cdots s_r$ is any reduced decomposition of $w \in W$ with $s_i \in S$. Denote by w_0 the longest element of W. We have $\pi = \lambda(w_0)^2$.

Let x be an indeterminate and let $H(W)$ be the "usual" Hecke algebra of W, i.e., the $\mathbf{Z}[x^{\pm 1}]$-algebra generated by $(T_s)_{s \in S}$ with relations

$$(T_s - x)(T_s + 1) = 0,\ \underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}} \text{ for } s, t \in S.$$

The isomorphism (3) induces an isomorphism between $H(W)$ and the specialization of $\mathcal{H}(W)$ at $q_{H,0} \mapsto x,\ q_{H,1} \mapsto -1$.

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