# GLUING $p$-PERMUTATION MODULES

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## 1. Introduction

We give a "local" construction of the stable category of $p$-permutation modules : a $p$-permutation $kG$-module gives rise, via the Brauer functor, to a family of $p$-permutation modules for $kN_G(Q)/Q$, where $Q$ runs over the non-trivial $p$-subgroups of $G$, together with certain isomorphisms. Conversely, the data of a compatible family of $kN_G(Q)/Q$-modules comes from a $p$-permutation $kG$-module, unique up to a unique isomorphism in the stable category.

This should be the first half of a paper with a second part devoted to complexes of $p$-permutation modules.

## 2. The Brauer functor

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$.

Let $Q$ be a $p$-subgroup of $G$. We denote by $\mathrm{Br}_Q$ the Brauer functor $\mathrm{Br}_Q : kG-\mathrm{mod} \to kN_G(Q)-\mathrm{mod}$.

For $V$ be a $kG$-module,

$$\mathrm{Br}_Q(V) = V^Q / \left( \sum_{P<Q} \mathrm{Tr}_P^Q V^P \right).$$

We write also $V(Q)$ for $\mathrm{Br}_Q(V)$. For basic results about $p$-permutation modules and the Brauer functor, see [Br] and [Th, §27].

Restriction induces a fully faithful functor $kN_G(Q)/Q - \mathrm{mod} \to kN_G(Q) - \mathrm{mod}$ and we will identify $kN_G(Q)/Q - \mathrm{mod}$ with the full subcategory of $kN_G(Q) - \mathrm{mod}$ of the modules with a trivial action of $Q$.

We denote by $kG-\mathrm{perm}$ the full subcategory of $kG-\mathrm{mod}$ of $p$-permutation modules. This is the smallest full additive subcategory of $kG-\mathrm{mod}$ closed under direct summands and containing the permutation modules.

From now on, we will always consider the restriction of the Brauer functor $\mathrm{Br}_Q : kG-\mathrm{perm} \to kN_G(Q) - \mathrm{perm}$.

Let $\Omega$ be a $G$-set. The composition $\gamma_\Omega = \gamma_\Omega^Q : k\Omega^Q \hookrightarrow (k\Omega)^Q \twoheadrightarrow (k\Omega)(Q)$ is an isomorphism. It induces an isomorphism of functors $\gamma : k(-)^Q \xrightarrow{\sim} \mathrm{Br}_Q\, k(-)$, *i.e.*, there is a diagram of functors, commutative up to isomorphism :

$$\begin{tikzcd}
G - \mathrm{sets} \arrow[r, "k(-)"] \arrow[d, "(-)^Q"'] & kG - \mathrm{perm} \arrow[d, "\mathrm{Br}_Q"] \\
N_G(Q) - \mathrm{sets} \arrow[r, "k(-)"'] & kN_G(Q) - \mathrm{perm}
\end{tikzcd}$$

The following easy result describes the effect of the Brauer construction on a permutation module. Let $H$ and $L$ be two subgroups of $G$. We put $T_G(L,H) = \{g \in G | L \leq H^g\}$.

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**Lemma 2.1.** *We have*

$$(G/H)^L = \bigcup_{g\in H\backslash T_G(L,H)/N_G(L)} N_G(L)/\left(N_G(L)\cap H^g\right)$$

*as $N_G(L)$-sets.*

2.1. **Tensor product and duality.** Let us describe the effect of the functor $\mathrm{Br}_Q$ on tensor products and duality.

Let $V$ and $W$ be two $p$-permutation $kG$-modules. The inclusion $V^Q\otimes W^Q\to (V\otimes W)^Q$ induces a map $\alpha_{V,W}: V(Q)\otimes W(Q)\to (V\otimes W)(Q)$.

Restriction $(V^*)^Q=\mathrm{Hom}_{kQ}(V,k)\to (V^Q)^*=\mathrm{Hom}_k(V^Q,k)$ induces a map $\beta_V: V^*(Q)\to V(Q)^*$.

We put $\Delta G=\{(x,x)|x\in G\}\subseteq G\times G$.

**Lemma 2.2.** *The maps $\alpha_{V,W}$ and $\beta_V$ are isomorphisms and induce isomorphisms of functors from $kG-\mathrm{perm}\times kG-\mathrm{perm}$ to $kN_G(Q)-\mathrm{perm}$*

$$\alpha : \mathrm{Res}^{N_G(Q)\times N_G(Q)}_{\Delta N_G(Q)} \mathrm{Br}_{Q\times Q} \xrightarrow{\sim} \mathrm{Br}_Q \mathrm{Res}^{G\times G}_{\Delta G}$$

*and from $kG-\mathrm{perm}$ to $kN_G(Q)-\mathrm{perm}$*

$$\beta : \mathrm{Br}_Q(-)^* \xrightarrow{\sim} (-)^*\,\mathrm{Br}_Q\,.$$

*Proof.* In order to prove that $\alpha_{V,W}$ and $\beta_V$ are isomorphisms, it is enough to consider permutation modules. So, let $V=k\Omega$ and $W=k\Psi$, where $\Omega$ and $\Psi$ are $G$-sets.

We have an isomorphism $t_{\Omega,\Psi}: k\Omega\otimes k\Psi\xrightarrow{\sim} k(\Omega\times\Psi)$ given by $\omega\otimes\psi\mapsto(\omega\times\psi)$ for $\omega\in\Omega$ and $\psi\in\Psi$.

There is a commutative diagram :

$$\begin{tikzcd}
(k\Omega^Q)\otimes(k\Psi^Q) \arrow[r, hook] \arrow[d, "t_{\Omega^Q,\Psi^Q}"', "\simeq"] \arrow[rr, bend left, "\simeq"] & (k\Omega)^Q\otimes(k\Psi)^Q \arrow[r, two heads] \arrow[d, "t_{\Omega,\Psi}"'] & (k\Omega)(Q)\otimes(k\Psi)(Q) \arrow[d, "\alpha_{V,W}"'] \\
k(\Omega\times\Psi)^Q \arrow[r, hook] \arrow[rr, bend right, "\simeq"'] & (k(\Omega\times\Psi))^Q \arrow[r, two heads] & k(\Omega\times\Psi)(Q)
\end{tikzcd}$$

Hence, $\alpha_{V,W}$ is an isomorphism.

We have an isomorphism $d_\Omega: k\Omega\xrightarrow{\sim}(k\Omega)^*$ given by $\omega\mapsto\left(\sum_{\omega'\in\Omega}a_{\omega'}\omega'\mapsto a_\omega\right)$ for $\omega\in\Omega$. The composition

$$k\Omega^Q\hookrightarrow(k\Omega)^Q\xrightarrow{d_\Omega}((k\Omega)^*)^Q\twoheadrightarrow(k\Omega)^*(Q)$$

is an isomorphism. Since the following diagram is commutative

$$\begin{tikzcd}
(k\Omega)^Q \arrow[r, "d_\Omega"] & ((k\Omega)^*)^Q \arrow[r, two heads] \arrow[d] & (k\Omega)^*(Q) \arrow[d, "\beta_V"] \\
k\Omega^Q \arrow[u, hook] \arrow[r, "\simeq", "d_{\Omega^Q}"'] & (k\Omega^Q)^* & (k\Omega)(Q)^* \arrow[l, "\simeq"', "\gamma^*_\Omega"]
\end{tikzcd}$$

we deduce that $\beta_V$ is an isomorphism.

Let us now check that $\alpha_{V,W}$ and $\beta_V$ induce natural transformations of functors.

Let $V_1$, $V_2$, $W_1$ and $W_2$ be $p$-permutation $kG$-modules and $f_i \in \mathrm{Hom}(V_i, W_i)$. The following diagram is commutative :

$$\begin{tikzcd}
V_1(Q) \otimes V_2(Q) \arrow[rrr, "f_1(Q)\otimes f_2(Q)"] \arrow[ddd, "\alpha_{V_1,V_2}"'] & & & W_1(Q) \otimes W_2(Q) \arrow[ddd, "\alpha_{W_1,W_2}"] \\
& V_1^Q \otimes V_2^Q \arrow[r, "f_1\otimes f_2"] \arrow[ul] \arrow[d, hook] & W_1^Q \otimes W_2^Q \arrow[ur, two heads] \arrow[d, hook] & \\
& (V_1 \otimes V_2)^Q \arrow[r, "f_1\otimes f_2"] \arrow[dl, two heads] & (W_1 \otimes W_2)^Q \arrow[dr, two heads] & \\
(V_1 \otimes V_2)(Q) \arrow[rrr, "(f_1\otimes f_2)(Q)"] & & & (W_1 \otimes W_2)(Q)
\end{tikzcd}$$

Hence, $\alpha_{V,W}$ induces a morphism of functors : $\mathrm{Res}^{N_G(Q)\times N_G(Q)}_{\Delta N_G(Q)} \mathrm{Br}_{Q\times Q} \to \mathrm{Br}_Q \mathrm{Res}^{G\times G}_{\Delta G}$.
The commutativity of the diagram :

$$\begin{tikzcd}
W_1^*(Q) \arrow[rrr, "f_1^*(Q)"] \arrow[ddd, "\beta_{W_1}"'] & & & V_1^*(Q) \arrow[ddd, "\beta_{V_1}"] \\
& (W_1^*)^Q \arrow[r, "f_1^*"] \arrow[ul] \arrow[d] & (V_1^*)^Q \arrow[ur] \arrow[d] & \\
& (W_1^Q)^* \arrow[r, "f_1^*"] & (V_1^Q)^* & \\
W_1(Q)^* \arrow[rrr, "f_1(Q)^*"] \arrow[ur, hook] & & & V_1(Q)^* \arrow[ul, hook']
\end{tikzcd}$$

shows finally that $\beta_V$ induces a morphism of functors : $\mathrm{Br}_Q(-)^* \xrightarrow{\sim} (-)^* \mathrm{Br}_Q$.

$\square$

**Proposition 2.3.** *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}(V, W) \arrow[r, "\simeq"] \arrow[dd, "\mathrm{Br}_Q"'] & \mathrm{Hom}(V \otimes W^*, k) \arrow[d, "\mathrm{Br}_Q"] \\
& \mathrm{Hom}((V \otimes W^*)(Q), k) \arrow[d, "\beta\alpha^{-1}", "\simeq"'] \\
\mathrm{Hom}(V(Q), W(Q)) \arrow[r, "\simeq"] & \mathrm{Hom}(V(Q) \otimes W(Q)^*, k)
\end{tikzcd}$$

*where the horizontal maps are isomorphisms provided by the adjoint pairs* $(- \otimes W^*, - \otimes W)$ *and* $(- \otimes W(Q)^*, - \otimes W(Q))$.

*Proof.* More explicitely, the first horizontal map is the composition

$$\mathrm{Hom}(V, W) \xrightarrow{-\otimes W^*} \mathrm{Hom}(V \otimes W^*, W \otimes W^*) \xrightarrow{tr(W)_*} \mathrm{Hom}(V \otimes W^*, k)$$

where $tr(W) : W \otimes W^* \to k$ is the trace map.

Thanks to Lemma 2.2, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(V,W) \arrow[r, "-\otimes W^*"] \arrow[ddd, "\mathrm{Br}_Q"'] & \mathrm{Hom}(V\otimes W^*, W\otimes W^*) \arrow[d, "\mathrm{Br}_Q"] \\
& \mathrm{Hom}((V\otimes W^*)(Q),(W\otimes W^*)(Q)) \arrow[d, "\alpha^{-1}"] \\
& \mathrm{Hom}(V(Q)\otimes W^*(Q), W(Q)\otimes W^*(Q)) \arrow[d, "\beta"] \\
\mathrm{Hom}(V(Q),W(Q)) \arrow[ur, "-\otimes W^*(Q)"] \arrow[r, "-\otimes W(Q)^*"'] & \mathrm{Hom}(V(Q)\otimes W(Q)^*, W(Q)\otimes W(Q)^*)
\end{tikzcd}$$

We will be done if we prove that the image of $tr(W) : W\otimes W^* \to k$ under the morphism

$$\mathrm{Hom}(W\otimes W^*,k) \xrightarrow{\beta\alpha^{-1}\mathrm{Br}_Q} \mathrm{Hom}(W(Q)\otimes W(Q)^*,k)$$

is $tr(W(Q))$. It is enough to prove this for $W$ a permutation module.

Let $W = k\Omega$, $\Omega$ a $G$-set. The claim follows from the commutativity of the diagram

$$\begin{tikzcd}
& (k\Omega\otimes k\Omega)^Q \arrow[r, "1\otimes d_\Omega"] & (k\Omega\otimes (k\Omega)^*)^Q \arrow[d, two heads] \\
k\Omega^Q\otimes k\Omega^Q \arrow[ur, hook] \arrow[dr, "1\otimes d_{\Omega^Q}"'] & & (k\Omega\otimes (k\Omega)^*)(Q) \arrow[d, "\beta\alpha^{-1}"'] \\
& k\Omega^Q\otimes (k\Omega^Q)^* \arrow[r, "\gamma_\Omega\otimes(\gamma_\Omega^{-1})^*"] & k\Omega(Q)\otimes k\Omega(Q)^*
\end{tikzcd}$$

□

2.2. **Compatibilities.** Let us define a category $\mathcal{T}_G$. Its objects are the non-trivial $p$-subgroups of $G$. Let $P$ and $Q$ be two non-trivial $p$-subgroups of $G$. Then, the set of maps between $P$ and $Q$ in $\mathcal{T}_G$ is $\{{}_Pg_Q | g \in T_G(P,Q)\}$. The composition of maps is the product in $G$ : $({}_Qh_R)\cdot({}_Pg_Q) = {}_P(hg)_R$. We put $\bar{\phi} = g$ for $\phi = {}_Pg_Q$ and $\phi(P) = {}^gP$.

We call a map $\phi = {}_Pg_Q$ in $\mathrm{Hom}_{\mathcal{T}_G}(P,Q)$ *normal* if ${}^gP$ is normal in $Q$. Every normal map can be expressed uniquely as the composition of an isomorphism with a normal inclusion $\phi = \phi_\lhd\phi_\sim$ where $\phi_\sim = {}_Pg_{{}^gP}$ and $\phi_\lhd = {}_{{}^gP}1_Q$.

An important property of the normal maps is that they generate the category $\mathcal{T}_G$, *i.e.*, every map in $\mathcal{T}_G$ is a composition of normal maps.

For $g \in G$ and $H$ a subgroup of $G$, we denote by

$$g_* : kH-\mathrm{mod} \xrightarrow{\sim} k^gH-\mathrm{mod}$$

the isomorphism of categories induced by the group isomorphism $H \xrightarrow{\sim} {}^gH,\ \ x\mapsto {}^gx$. We also denote by

$$g_* : H-\mathrm{sets} \xrightarrow{\sim} {}^gH-\mathrm{sets}$$

the isomorphism of categories induced by this group isomorphism. We have the obvious compatiblity with the previous isomorphism of categories.

Let $V$ be a $p$-permutation $kH$-module and $\phi\in\mathrm{Hom}_{\mathcal{T}_G}(P,Q)$ invertible. Then, the isomorphism of $N_{{}^gH}(Q)$-modules $\bar{\phi}_*(V^P) \xrightarrow{\sim} (\bar{\phi}_*V)^Q,\ v\mapsto v$, induces an isomorphism

$$\langle\phi\rangle_V^0 : \bar{\phi}_*(V(P)) \xrightarrow{\sim} (\bar{\phi}_*V)(Q).$$

If $H = G$, then we have an isomorphism of $kG$-modules

$$\iota_{\bar{\phi},V} : V \mapsto \bar{\phi}_* V,\ \ v \mapsto \bar{\phi} v$$

and an isomorphism of $kN_G(Q)$-modules

$$\langle \phi \rangle_V = \mathrm{Br}_Q(\iota^{-1}_{\bar{\phi},V}) \cdot \langle \phi \rangle^0_V : \bar{\phi}_*(V(P)) \xrightarrow{\sim} V(Q).$$

Let now $P \trianglelefteq Q$ and $\phi = {}_P 1_Q$. The canonical map $V^Q \hookrightarrow V^P \twoheadrightarrow V(P)$ factors through the inclusion $V(P)^Q \hookrightarrow V(P)$ to give a map $V^Q \to V(P)^Q$. Composing with the canonical map $V(P)^Q \twoheadrightarrow V(P)(Q)$, we get a map $V^Q \to V(P)(Q)$ which factors through the canonical map $V^Q \twoheadrightarrow V(Q)$. The induced map $V(Q) \to V(P)(Q)$ is an isomorphism and we denote the inverse isomorphism by $\langle \phi \rangle_V$

$$\langle \phi \rangle_V : V(P)(Q) \xrightarrow{\sim} V(Q)$$

Let us summarize the construction by the following commutative diagram :

$$\begin{tikzcd}
V^P \arrow[r, two heads] & V(P) \\
V^Q \arrow[u, hook] \arrow[r, dotted] \arrow[d, two heads] & V(P)^Q \arrow[u, hook] \arrow[d, two heads] \\
V(Q) \arrow[r, dotted, "\simeq"] & V(P)(Q)
\end{tikzcd}$$

For $\phi \in \mathrm{Hom}_{\mathcal{T}_G}(P, Q)$ a normal map with $P \leq H$, $Q \leq {}^{\bar{\phi}}H$ and $V$ a $p$-permutation $kH$-module, we put

$$\langle \phi \rangle^0_V = \langle \phi_{\lhd} \rangle_{\bar{\phi}_* V} \cdot \mathrm{Br}_Q(\langle \phi_{\sim} \rangle^0_V) : \left(\bar{\phi}_*(V(P))\right)(Q) \xrightarrow{\sim} (\bar{\phi}_* V)(Q).$$

If $V$ is a $p$-permutation $kG$-module, we put

$$\langle \phi \rangle_V = \langle \phi_{\lhd} \rangle_V \cdot \mathrm{Br}_Q(\langle \phi_{\sim} \rangle_V) : \left(\bar{\phi}_*(V(P))\right)(Q) \xrightarrow{\sim} V(Q).$$

This gives an isomorphism of functors from $kH - \mathrm{perm}$ to $kN_{\bar{\phi}H}(\phi(P), Q) - \mathrm{perm}$

$$\langle \phi \rangle^0 : \mathrm{Br}_Q \bar{\phi}_* \mathrm{Br}_P \xrightarrow{\sim} \mathrm{Res}^{N_{\bar{\phi}H}(Q)}_{N_{\bar{\phi}H}(\phi(P),Q)} \mathrm{Br}_Q \bar{\phi}_*.$$

and an isomorphism of functors from $kG - \mathrm{perm}$ to $kN_G(\phi(P), Q) - \mathrm{perm}$

$$\langle \phi \rangle : \mathrm{Br}_Q \bar{\phi}_* \mathrm{Br}_P \xrightarrow{\sim} \mathrm{Res}^{N_G(Q)}_{N_G(\phi(P),Q)} \mathrm{Br}_Q .$$

When $V = k\Omega$, $\Omega$ a $G$-set, we have a commutative diagram

$$\begin{tikzcd}
 & k(\bar{\phi}_* \Omega^P)(Q) \arrow[r, "\mathrm{Br}_Q \bar{\phi}_* \gamma^P_\Omega"] & (\bar{\phi}_*(k\Omega)(P))(Q) \arrow[dd, "\langle \phi \rangle_V"] \\
k(\bar{\phi}_* \Omega^P)^Q \arrow[ur, "\gamma^Q_{\bar{\phi}_* \Omega^P}"] \arrow[dr, "k\langle \phi \rangle_\Omega"'] & & \\
 & k\Omega^Q \arrow[r, "\gamma^Q_\Omega"] & (k\Omega)(Q)
\end{tikzcd}$$

where

$$\langle \phi \rangle_\Omega : (\bar{\phi}_* \Omega^P)^Q \xrightarrow{\sim} \Omega^Q,\ \ \ \omega \mapsto \bar{\phi}^{-1} \omega.$$

Note that this justifies the claim that $\langle\phi\rangle_V$ is an isomorphism for $V$ a permutation module and consequently for $V$ an arbitrary $p$-permutation module.

We will now check a transitivity property of the isomorphisms constructed above.

**Lemma 2.4.** *Let $\phi \in \operatorname{Hom}_{\mathcal{T}_G}(P,Q)$ and $\psi \in \operatorname{Hom}_{\mathcal{T}_G}(Q,R)$ such that $\phi$, $\psi$ and $\psi\phi$ are normal maps and let $V$ be a $p$-permutation $kG$-module. Then, the following diagram is commutative*

$$\begin{tikzcd}
\bar\psi_*\big((\bar\phi_*(V(P)))(Q)\big)(R) \arrow[r, "\mathrm{Br}_R\,\bar\psi_*\langle\phi\rangle_V"] \arrow[d, "\langle\psi\rangle^0_{\bar\phi_*(V(P))}"'] & (\bar\psi_*(V(Q)))(R) \arrow[d, "\langle\psi\rangle_V"] \\
((\bar\psi\bar\phi)_*(V(P)))(R) \arrow[r, "\langle\psi\phi\rangle_V"] & V(R)
\end{tikzcd}$$

*Proof.* Indeed, it is enough to check commutativity for $V = k\Omega$ a permutation module. It follows from the commutativity of the following diagram

$$\begin{tikzcd}
\bar\psi_*\big((\bar\phi_*(\Omega^P))^Q\big)^R \arrow[r, "\omega\mapsto\bar\psi\bar\phi^{-1}\bar\psi^{-1}\omega"] \arrow[d, "\omega\mapsto\omega"'] & (\bar\psi_*(\Omega^Q))^R \arrow[d, "\omega\mapsto\bar\psi^{-1}\omega"] \\
((\bar\psi\bar\phi)_*(\Omega^P))^R \arrow[r, "\omega\mapsto\bar\psi\bar\phi^{-1}\omega"] & \Omega^R
\end{tikzcd}$$

□

The difference between $\langle\phi\rangle$ and $\langle\phi\rangle^0$ is is given by the following lemma :

**Lemma 2.5.** *Let $\psi \in \operatorname{Hom}_{\mathcal{T}_G}(Q,R)$ be a normal map with $\bar\psi = 1$, $g \in G$ and $V$ a $p$-permutation $kG$-module. Then, the following diagram is commutative*

$$\begin{tikzcd}
(V(Q))(R) \arrow[r, "\langle\psi\rangle_V"] & V(R) \\
((g_*V)(Q))(R) \arrow[u, "\mathrm{Br}_R\,\mathrm{Br}_Q(\iota^{-1}_{g,V})"] \arrow[r, "\langle\psi\rangle_{g_*V}"'] & (g_*V)(R) \arrow[u, "\mathrm{Br}_R(\iota^{-1}_{g,V})"']
\end{tikzcd}$$

*In particular, if $\phi \in \operatorname{Hom}_{\mathcal{T}_G}(P,Q)$ is any normal map, then $\langle\phi\rangle_V = \mathrm{Br}_Q(\iota^{-1}_{\bar\phi,V})\cdot\langle\phi\rangle^0_V$.*

*Proof.* Again, it is enough to deal with $V = k\Omega$ a permutation module. Then, the lemma reduces to the commutativity of

$$\begin{tikzcd}
(\Omega^Q)^R \arrow[r, "\omega\mapsto\omega"] & \Omega^R \\
((g_*\Omega)^Q)^R \arrow[u, "\omega\mapsto g^{-1}\omega"] \arrow[r, "\omega\mapsto\omega"'] & (g_*\Omega)^R \arrow[u, "\omega\mapsto g^{-1}\omega"']
\end{tikzcd}$$

For the second part of the lemma, we take $\psi = \phi_{\lhd}$ and $g = \bar\phi$ in the commutative diagram. □

## 3. A category of sheaves on $p$-subgroups complexes

3.1. **Definition.** Let $\mathcal{F}$ be a subcategory of $\mathcal{T}_G$. We define a category $\mathcal{S}_{\mathcal{F}}$ of "sheaves" on $\mathcal{F}$.

Its objects are families $\{V_Q, [\phi]\}_{Q,\phi}$ where $Q$ runs over the objects of $\mathcal{F}$ and $\phi$ over the normal maps of $\mathcal{F}$. Here, $V_Q$ is a $p$-permutation $kN_G(Q)/Q$-module and for $\phi \in \operatorname{Hom}_{\mathcal{F}}(P,Q)$ normal, $[\phi]$ is an isomorphism of $kN_G(\phi(P),Q)$-modules

$$[\phi] : (\bar\phi_* V_P)(Q) \xrightarrow{\sim} \operatorname{Res}^{N_G(Q)}_{N_G(\phi(P),Q)} V_Q.$$

We require the following two conditions to be satisfied :
For $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,Q)$, we have

$$[\phi] = \iota^{-1}_{\bar{\phi},V_Q}. \tag{1}$$

Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ such that $\phi$, $\psi$ and $\psi\phi$ are normal maps. Then, the following diagram should be commutative

$$\begin{tikzcd}
\bar{\psi}_*\big((\bar{\phi}_*V_P)(Q)\big)(R) \arrow[r, "\mathrm{Br}_R\,\bar{\psi}_*[\phi]"] \arrow[d, "\langle\psi\rangle^0_{\bar{\phi}_*V_P}"'] & (\bar{\psi}_*V_Q)(R) \arrow[d, "{[\psi]}"] \\
((\bar{\psi}\bar{\phi})_*V_P)(R) \arrow[r, "{[\psi\phi]}"] & V_R
\end{tikzcd} \tag{2}$$

For $\mathcal{V} = \{V_Q, [\phi]\}$ and $\mathcal{V}' = \{V'_Q, [\phi]'\}$ two objects of $\mathcal{S}_{\mathcal{F}}$, $\mathrm{Hom}_{\mathcal{S}_{\mathcal{F}}}(\mathcal{V},\mathcal{V}')$ is the set of families $\Lambda = \{\lambda_Q\}_Q$, where $Q$ runs over the objects of $\mathcal{F}$. Here, $\lambda_Q \in \mathrm{Hom}_{kN_G(Q)}(V_Q, V'_Q)$. Furthermore, $\Lambda$ should have the following property : for every normal map $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, the following diagram is commutative

$$\begin{tikzcd}
(\bar{\phi}_*V_P)(Q) \arrow[r, "\mathrm{Br}_Q\,\bar{\phi}_*\lambda_P"] \arrow[d, "{[\phi]}"'] & (\bar{\phi}_*V'_P)(Q) \arrow[d, "{[\phi]'}"] \\
V_Q \arrow[r, "\lambda_Q"'] & V'_Q
\end{tikzcd} \tag{3}$$

Thanks to the results of §2.2, we have a functor

$$\mathrm{Br} : kG-\mathrm{perm} \to \mathcal{S}, \quad V \mapsto \{V(Q), \langle\phi\rangle_V\}$$

where $\mathcal{S} = \mathcal{S}_{\mathcal{T}_G}$.

We can now state our main result :

**Theorem 3.1.** *The functor* Br *induces an equivalence of categories*

$$kG-\mathrm{perm}\,/kG-\mathrm{proj} \xrightarrow{\sim} \mathcal{S}.$$

3.2. **Some properties of $\mathcal{S}_{\mathcal{F}}$.** Let us give a special case of the commutative diagram (2).

**Lemma 3.2.** *Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ be two normal maps with $\bar{\psi} \in N_G(\phi(P))$. Then, the following diagram is commutative*

$$\begin{tikzcd}
\bar{\psi}_*\big((\bar{\phi}_*V_P)(Q)\big)(R) \arrow[r, "\mathrm{Br}_R\,\bar{\psi}_*[\phi]"] \arrow[d, "\langle\psi\rangle_{\bar{\phi}_*V_P}"'] & (\bar{\psi}_*V_Q)(R) \arrow[d, "{[\psi]}"] \\
(\bar{\phi}_*V_P)(R) \arrow[r, "{[\psi_{\lhd}\phi]}"] & V_R
\end{tikzcd}$$

*Proof.* Let $\psi' = \psi\phi$ and $\phi' = {}_P(\bar{\phi}^{-1}\bar{\psi}^{-1}\bar{\phi})_P$. We have $\psi_{\lhd}\phi = \psi'\phi'$. By (1), we have $[\phi'] = \iota^{-1}_{\bar{\phi}',V_P}$. We have also $\bar{\psi}_*\bar{\phi}_*\iota^{-1}_{\bar{\phi}',V_P} = \iota_{\bar{\psi},\bar{\phi}_*V_P}$. The commutativity of the diagram (2) applied to $\psi'$ and $\phi'$ gives the commutative diagram

$$\begin{tikzcd}
(\bar{\phi}_*V_P)(R) \arrow[r, "\mathrm{Br}_R\,\iota_{\bar{\psi},\bar{\phi}_*V_P}"] \arrow[dr, "{[\psi_{\lhd}\phi]}"'] & ((\bar{\psi}\bar{\phi})_*V_P)(R) \arrow[d, "{[\psi\phi]}"] \\
& V_R
\end{tikzcd}$$

and we obtain the commutativity of the diagram of the lemma since $\langle\psi\rangle_{\bar\phi_* V_P} = \mathrm{Br}_R(\iota^{-1}_{\bar\psi,\bar\phi_* V_P})\cdot\langle\psi\rangle^0_{\bar\phi_* V_P}$ by Lemma 2.5. ☐

The diagram (3) need be checked only on a generating set :

**Lemma 3.3.** *Let $E$ be a set of normal maps in $\mathcal{F}$ such that every normal map of $\mathcal{F}$ is a product of elements of $E$ and of inverses of invertible elements of $E$.*

*Then, the commutativity of the diagram (3) for $\phi\in E$ implies the commutativity for every normal map $\phi$ of $\mathcal{F}$*

*Proof.* Let $\phi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $\psi\in\mathrm{Hom}_{\mathcal{F}}(Q,R)$ such that $\psi$ and $\psi\phi$ are normal maps. Then, $\phi$ is a normal map and the following diagram is commutative

$$\begin{tikzcd}
((\bar\psi\bar\phi)_* V_P)(R) \arrow[rrr, "{[\psi\phi]}"] \arrow[ddd, "\mathrm{Br}_R(\bar\psi\bar\phi)_*\lambda_P"'] & & & V_R \arrow[ddd, "\lambda_R"] \\
& \bar\psi_*\left((\bar\phi_* V_P)(Q)\right)(R) \arrow[ul, "\langle\psi\rangle^0_{\bar\phi_* V_P}"'] \arrow[r, "\mathrm{Br}_R\,\bar\psi_*[\phi]"] \arrow[d, "\mathrm{Br}_R\,\bar\psi_*\,\mathrm{Br}_Q\,\bar\phi_*\lambda_P"'] & (\bar\psi_* V_Q)(R) \arrow[ur, "{[\psi]}"] \arrow[d, "\mathrm{Br}_R\,\bar\psi_*\lambda_Q"] & \\
& \bar\psi_*\left((\bar\phi_* V'_P)(Q)\right)(R) \arrow[dl, "\langle\psi\rangle^0_{\bar\phi_* V'_P}"'] \arrow[r, "\mathrm{Br}_R\,\bar\psi_*[\phi]'"] & (\bar\psi_* V'_Q)(R) \arrow[dr, "{[\psi]'}"] & \\
((\bar\psi\bar\phi)_* V'_P)(R) \arrow[rrr, "{[\psi\phi]'}"] & & & V'_R
\end{tikzcd}$$

The lemma follows. ☐

We now define a restriction functor from $G$ to $N_G(P)/P$.

Let $P$ be an object in $\mathcal{F}$. We assume ${}_P1_Q\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for every $Q$ in $\mathcal{F}$ with $P\lhd Q$. Let $\mathcal{F}(P)$ be the subcategory of $\mathcal{T}_{N_G(P)/P}$ whose objects are the $Q/P$ where $Q$ is in $\mathcal{F}$ and $P\lhd Q$, $P\neq Q$ and where $\mathrm{Hom}_{\mathcal{F}(P)}(Q/P,R/P)$ is given by the image in $T_{N_G(P)/P}(Q/P,R/P)$ of $\mathrm{Hom}_{\mathcal{F}}(Q,R)$.

Let $\mathcal{V}=\{V_Q,[\phi]\}$ be an object of $\mathcal{S}_{\mathcal{F}}$. Let $V'_{Q/P}=V_Q$. For $\phi'\in\mathrm{Hom}_{\mathcal{F}(P)}(Q/P,R/P)$, we put $[\phi']=[\phi]$ where $\phi\in\mathrm{Hom}_{\mathcal{F}}(Q,R)$ has image $\phi'$ in $\mathrm{Hom}_{\mathcal{F}(P)}(Q/P,R/P)$. It follows from (1) (and from the diagram (2)) that this is independent of the choice of $\phi$.

The restriction functor is

$$\mathrm{Res}^{\mathcal{F}}_{\mathcal{F}(P)}:\mathcal{S}_{\mathcal{F}}\to\mathcal{S}_{\mathcal{F}(P)},\quad \{V_Q,[\phi]\}\mapsto\{V'_{Q/P},[\phi']\}.$$

We denote by $E_P:\mathcal{T}_{\mathcal{F}}\to kN_G(P)/P-\mathrm{perm}$ the functor sending $\mathcal{V}$ on $V_P$.

The commutative diagram in Lemma 3.2 says that objects can be "glued locally" :

**Lemma 3.4.** *For $\mathcal{V}$ in $\mathcal{S}_{\mathcal{F}}$, we have an isomorphism $\{[{}_P1_Q]\}_Q:\mathrm{Br}\,E_P(\mathcal{V})\xrightarrow{\sim}\mathrm{Res}^{\mathcal{F}}_{\mathcal{F}(P)}\mathcal{V}$. This induces an isomorphism of functors $\mathrm{Br}\,E_P\xrightarrow{\sim}\mathrm{Res}^{\mathcal{F}}_{\mathcal{F}(P)}$. So, we have a diagram, commutative up to isomorphism :*

$$\begin{tikzcd}
\mathcal{S}_{\mathcal{F}} \arrow[r, "\mathrm{Res}^{\mathcal{F}}_{\mathcal{F}(P)}"] \arrow[dr, "E_P"'] & \mathcal{S}_{\mathcal{F}(P)} \\
& kN_G(P)/P-\mathrm{perm} \arrow[u, "\mathrm{Br}"']
\end{tikzcd}$$

*Proof.* Let $\mathcal{V} = \{V_Q, [\phi]\}$ in $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{V}' = \operatorname{Res}^{\mathcal{F}}_{\mathcal{F}(P)} \mathcal{V}$. We have $\operatorname{Br}(V_P) = \{W_{Q/P}, \langle\psi\rangle_{V_P}\}$ with $W_{Q/P} = V_P(Q)$.

Let $\lambda_{Q/P} = [{}_P1_Q] : W_{Q/P} \xrightarrow{\sim} V'_{Q/P}$ and $\Lambda = \{\lambda_{Q/P}\}$.

Let $\phi = {}_P1_Q$ in $\mathcal{F}$ and $\psi \in \operatorname{Hom}_{\mathcal{F}}(Q, R)$ be two normal maps with $\bar{\psi} \in N_G(P)$.

We have a commutative diagram (Lemma 3.2)

$$\begin{tikzcd}
(\bar{\psi}_* V_P(Q))(R) \arrow[r, "\operatorname{Br}_R \bar{\psi}_*[\phi]"] \arrow[d, "\langle\psi\rangle_{V_P}"'] & (\bar{\psi}_* V_Q)(R) \arrow[d, "{[\psi]}"] \\
V_P(R) \arrow[r, "{[\psi \triangleleft \phi]}"'] & V_R
\end{tikzcd}$$

This shows that $\Lambda$ defines a map $\operatorname{Br}(V_P) \to \mathcal{V}'$. This induces an isomorphism between $\operatorname{Br} \cdot E_P$ and $\operatorname{Res}^{\mathcal{F}}_{\mathcal{F}(P)}$. $\square$

3.3. **Proof of Theorem 3.1.** Let us first note that $\operatorname{Br}(V) = 0$ if $V$ is projective, hence Br induces indeed a functor $\bar{\operatorname{Br}} : kG - \operatorname{perm} / kG - \operatorname{proj} \to \mathcal{S}$.

**Lemma 3.5.** *The functor* $\bar{\operatorname{Br}}$ *is fully faithful.*

*Proof.* We have to prove that $\bar{\operatorname{Br}}$ induces an isomorphism

$$\overline{\operatorname{Hom}}(V, W) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Br} V, \operatorname{Br} W)$$

for $V$ and $W$ any $p$-permutation $kG$-modules.

Thanks to Proposition 2.3, we have a commutative diagram :

$$\begin{tikzcd}
\overline{\operatorname{Hom}}(V, W) \arrow[r, "\simeq"] \arrow[d, "\operatorname{Br}"'] & \overline{\operatorname{Hom}}(V \otimes W^*, k) \arrow[d, "\operatorname{Br}"] \\
\operatorname{Hom}(\operatorname{Br} V, \operatorname{Br} W) \arrow[r, "\simeq"] & \operatorname{Hom}(\operatorname{Br}(V \otimes W^*), \operatorname{Br} k)
\end{tikzcd}$$

So, it is enough to consider the case $W = k$.

Since the modules $k(G/Q)$, $Q$ a $p$-subgroup of $G$, generate $kG - \operatorname{perm}$ as an additive category closed under taking direct summands, we may assume $V = k(G/Q)$. We may take $Q \neq 1$ since otherwise $V$ is projective.

Now, $\operatorname{Hom}(kG/Q, k) \simeq \overline{\operatorname{Hom}}(kG/Q, k)$ is a one-dimensional vector space, generated by the unique map $f$ between the $G$-sets $G/Q$ and $G/G$.

The map $\operatorname{Br}_Q(f) : V(Q) = k(G/Q)^Q \to k$ is induced by the unique map between the sets $N_G(Q)/Q$ and $N_G(Q)/N_G(Q)$. In particular, it is non-zero, hence $\operatorname{Br}(f) \neq 0$.

Let $\Lambda \in \operatorname{Hom}(\operatorname{Br} kG/Q, \operatorname{Br} k)$. Since $\operatorname{Hom}(V(Q), k)$ is one-dimensional, we have $\lambda_Q = \alpha \operatorname{Br}_Q(f)$ for some $\alpha \in k$. So, $\Lambda - \alpha \operatorname{Br}(f)$ vanishes on $V(Q)$.

We assume now $\lambda_Q = 0$. We will prove that $\lambda_P = 0$ for all $P$. This is clear if $P$ is not conjugate to a subgroup of $Q$, since then $V(P) = 0$. We will now prove the result for $P \leq Q$ by induction on $[Q : P]$.

Let $P < Q$ and $g \in T_G(P, Q)$. Let $R$ be a $p$-subgroup of $G$ such that $P \triangleleft R \leq Q^g$, $P \neq R$. Then, $(N_G(P)/N_G(P) \cap Q^g)^R \neq \emptyset$. Now, $\lambda_P$ and $\lambda_R$ have the same restriction to $(N_G(P)/N_G(P) \cap Q^g)^R$. Consequently, $\lambda_P$ is zero on $(N_G(P)/N_G(P) \cap Q^g)^R$, by induction. By Lemma 2.1, we deduce that $\lambda_P = 0$. $\square$

The first part of following lemma is essentially due to Bouc [Bou1, Bou2] (cf also [Li, Lemma 5.4]).

**Lemma 3.6.** (i) *Let $f : V \to W$ be a morphism between two p-permutation kG-modules such that $Br_Q(f)$ is injective for all $Q \neq 1$. Then, there is a morphism $\sigma : W \to V$ such that $\sigma f$ is a stable automorphism of $V$. In particular, if $V$ has no projective direct summand, then $f$ is a split injection.*

(ii) *Let $\Lambda : \mathcal{V} \to \mathcal{W}$ be a morphism between two objects of $\mathcal{S}$. Assume for all $Q$, there is a projective direct summand $L_Q$ of the $kN_G(Q)/Q$-module $V_Q$ such that the restriction of $\lambda_Q$ to $L_Q$ is injective and $V_Q/L_Q$ has no projective direct summand. Then, for all $Q$, $\lambda_Q$ is a split injection.*

*Proof.* Let us prove part (i) of the lemma. We can assume $V$ has no projective direct summand.

Assume $G$ is a $p$-group.

Let us first consider a morphism $f : V = k(G/Q) \to W = k(G/R)$ such that $\mathrm{Br}_Q(f)$ is injective, where $Q$ and $R$ are non-trivial $p$-subgroups of $G$. The module $V(Q)$ is a projective indecomposable $kN_G(Q)/Q$-module. Since $W(Q) \neq 0$, $Q$ is contained in $R$, up to conjugacy. Without changing $W$, we can assume $Q \leq R$.

Assume $Q \neq R$. Then, for any $g \in T_G(Q, R)$, there is $S$ such that $P \lhd S \leq Q^g$, $P \neq S$. So, $kN_G(Q)/(N_G(Q) \cap R^g)$ is not a projective $kN_G(Q)/Q$-module. By Lemma 2.1, $V(Q)$ is not isomorphic to a submodule of $W(Q)$ and we have reached a contradiction.

If $Q = R$, then $f$ becomes invertible in the quotient $\mathrm{End}_{kN_G(Q)/Q}(V(Q))$ of the local ring $\mathrm{End}_{kG}(V)$, hence $f$ is invertible.

Let now $f : V \to W$ be a morphism between two $p$-permutation modules such that $\mathrm{Br}_Q(f)$ is injective for $Q$ non trivial. In order to prove that $f$ is injective, we may assume $V$ is indecomposable, *i.e.*, $V = k(G/Q)$ for some subgroup $Q$ of $G$. We can assume $Q \neq 1$, otherwise $V$ is projective. Since $V(Q)$ is indecomposable, there is an indecomposable direct summand $W'$ of $W$ such that if $f'$ is the composition $V \xrightarrow{f} W \twoheadrightarrow W'$, then $\mathrm{Br}_Q(f')$ is injective. Now, the considerations above show that $f'$ is an isomorphism and we are done.

Take now $G$ an arbitrary finite group. Let $U = \ker f$ and let $S$ be a Sylow $p$-subgroup of $G$. We know that the inclusion $\mathrm{Res}_S^G U \to \mathrm{Res}_S^G V$ is projective. So, the inclusion $U \to V$ is projective, hence $U = 0$ since we assume $V$ has no projective direct summand. Finally, the short exact sequence $0 \to V \to W \to W/V \to 0$ splits, since it splits by restriction to $S$.

Let us come to part (ii) of the lemma.

We prove the result by inverse induction on the order of $Q$. When $Q$ is a Sylow $p$-subgroup of $G$, then $V_Q = L_Q$, so $\lambda_Q$ is a split injection.

Assume now $Q$ is not a Sylow $p$-subgroup of $G$. Consider the restriction $f : M_Q \to W_Q$ of $\lambda_Q$, where $V_Q = L_Q \oplus M_Q$. By induction, $\mathrm{Br}_{R/Q}(f)$ is injective, for all $p$-subgroups $R$ with $Q \lhd R$, $Q \neq R$. By part (i) of the Lemma applied to $N_G(Q)/Q$, we deduce that $f$ is a split injection. Hence, $\lambda_Q$ is a split injection. $\square$

**Lemma 3.7.** *Let $\mathcal{G}$ and $\mathcal{H}$ be two full subcategories of $\mathcal{T}_G$ closed under inverse inclusion with $\mathcal{G} \subseteq \mathcal{H}$. Let $\mathcal{V}, \mathcal{V}' \in \mathcal{S}_{\mathcal{H}}$. Then, the restriction map*

$$\mathrm{Hom}_{\mathcal{S}_{\mathcal{H}}}(\mathcal{V}, \mathcal{V}') \to \mathrm{Hom}_{\mathcal{S}_{\mathcal{G}}}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{H}} \mathcal{V}, \mathrm{Res}_{\mathcal{G}}^{\mathcal{H}} \mathcal{V}')$$

*is surjective.*

*Proof.* It is enough to prove the lemma when $\mathcal{H}$ has one more object, $Q$, than $\mathcal{G}$. Let $\lambda \in \mathrm{Hom}_{\mathcal{S}_{\mathcal{G}}}(\mathrm{Res}_{\mathcal{G}}^{\mathcal{H}} \mathcal{V}, \mathrm{Res}_{\mathcal{G}}^{\mathcal{H}} \mathcal{V}')$.

Assume first there is $g \in G$ such that $Q^g \in \mathcal{G}$ and let $\psi = {}_Q g_{Q^g} \in \mathrm{Hom}_{\mathcal{H}}(Q, Q^g)$. Let $\lambda'_R = \lambda_R$ for $R \neq Q$ and $\lambda'_Q = [\psi^{-1}]'\lambda_{Q^g}[\psi]$. In order to prove that $\{\lambda'_R\}$ gives a map between $\mathcal{V}$ and $\mathcal{V}'$ (extending $\lambda$), it is enough to check commutativity of the diagram (3) for the map $\psi$, thanks to Lemma 3.3. This is immediate.

Assume now $Q^g \notin \mathcal{G}$ for all $g \in G$. Let $f : \mathrm{Br}(V_Q) \to \mathrm{Br}(V'_Q)$ be the restriction of $\lambda$ to $\mathcal{S}_{\mathcal{H}(Q)}$ (cf Lemma 3.4). By the fullness of Br applied to $N_G(Q)/Q$ (Lemma 3.5), there is a map $\lambda'_Q : V_Q \to V'_Q$ such that $\mathrm{Br}(\lambda'_Q) = f$. Let $\lambda'_R = \lambda_R$ for $R \in \mathcal{G}$. Since every map in $\mathcal{H}$ starting from $Q$ is the composition of a map from $Q$ to $R$, with $Q$ a strict normal subgroup of $R$ and of a map in $\mathcal{G}$, Lemma 3.3 shows that $\{\lambda'_R\}$ defines a map between $\mathcal{V}$ and $\mathcal{V}'$, extending $\lambda$. ☐

We now complete the proof of Theorem 3.1 by showing the essential surjectivity of Br. Let $\mathcal{V} \in \mathcal{S}$. We will prove by induction on the cardinality of $\{Q | V_Q \neq 0\}$ that $\mathcal{V}$ is in the image of Br.

For $Q \in \mathcal{T}_G$, let $L_Q$ be a projective direct summand of $V_Q$ such that $V_Q/L_Q$ has no projective direct summand. We denote by $\alpha_Q : V_Q \to L_Q$ the canonical surjection.

Let $M = \mathrm{Ind}_{N_G(Q)}^G L_Q$ and $\mathcal{M} = \mathrm{Br}\, M$. We have $M(Q) \simeq L_Q$ by Green's correspondence. Let $\zeta : V_Q \xrightarrow{\alpha_Q} L_Q \xrightarrow{\sim} M(Q)$. Let $\mathcal{H} = \mathcal{T}_G$ and $\mathcal{G}$ be the full subcategory of $\mathcal{T}_G$ with objects the $p$-subgroups containing $Q$. Let $\zeta'_R = 0$ for $R \in \mathcal{G}$, $R \neq Q$ and $\zeta'_Q = \zeta$. Then, $\{\zeta'_R\} \in \mathrm{Hom}_{\mathcal{S}_\mathcal{G}}(\mathrm{Res}^{\mathcal{H}}_{\mathcal{G}} \mathcal{V}, \mathrm{Res}^{\mathcal{H}}_{\mathcal{G}} \mathcal{M})$. By Lemma 3.7, there is $\chi \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{V}, \mathcal{M})$ extending $\{\zeta'_R\}$.

Let now $V' = \bigoplus_{Q \in \mathcal{T}_G/G} \mathrm{Ind}_{N_G(Q)}^G L_Q$, $\mathcal{V}' = \mathrm{Br}\, V'$ and $\lambda : \mathcal{V} \to \mathcal{V}'$ be the sum of the morphisms constructed for each $Q$ above.

Then, for all $Q$, the restriction of $\lambda_Q$ to $L_Q$ is injective. By Lemma 3.6, (ii), we deduce that $\lambda_Q$ is a split injection for all $Q$. Let then $\mathcal{W}$ be the cokernel of $\lambda$.

Take $R$ with $V_R = 0$. Then, $V_Q = 0$ whenever $R$ is contained up to $G$-conjugation in $Q$. So, $V'$ has no direct summand with vertex $R$, hence $W_R = 0$. Let now $Q$ be maximal such that $V_Q \neq 0$. Then, $\lambda_Q : V_Q = L_Q \to \left(\mathrm{Ind}_{N_G(Q)}^G L_Q\right)(Q)$ is an isomorphism. So, $W_Q = 0$. It follows that $\{Q | W_Q \neq 0\}$ is strictly contained in $\{Q | V_Q \neq 0\}$. By induction, there is a $p$-permutation $kG$-module $W$ (without projective direct summand) such that $\mathcal{W} = \mathrm{Br}\, W$. By fulness of Br (Lemma 3.5), the canonical morphism $\mathcal{V}' \to \mathcal{W}$ comes from a morphism $f : V' \to W$. By Lemma 3.6, (i), dualized, $f$ is a split surjection. Hence, $\ker f$ is a $p$-permutation $kG$-module and we have an isomorphism $\mathcal{V} \xrightarrow{\sim} \mathrm{Br}(\ker f)$. This finishes the proof of Theorem 3.1.

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