Representations of rational Cherednik algebras

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Abstract. This paper surveys the representation theory of rational Cherednik algebras. We also discuss the representations of the spherical subalgebras. We describe in particular the results on category $\mathcal{O}$. For type A, we explain relations with the Hilbert scheme of points on $\mathbf{C}^2$. We insist on the analogy with the representation theory of complex semi-simple Lie algebras.

Contents

1. Introduction

Let G be a complex reductive algebraic group, T a maximal torus and $W = N_G(T)/T$ the Weyl group. Let $\mathfrak{t} = \operatorname{Lie} T$.

There are several "Hecke" algebras associated to G (or to W):

Degeneration

finite Hecke algebra $\mathcal{H}$	affine Hecke algebra $\mathbf{C}[\mathbf{T}^\vee] \otimes \mathcal{H}$	double affine Hecke algebra $\mathbf{C}[\mathbf{T}] \otimes \mathbf{C}[\mathbf{T}^\vee] \otimes \mathcal{H}$
	degenerate affine Hecke $\mathbf{C}[\mathfrak{t}^*] \otimes \mathbf{C}[W]$	degenerate (trigonometric) daha $\mathbf{C}[\mathbf{T}] \otimes \mathbf{C}[\mathfrak{t}^*] \otimes \mathbf{C}[W]$
		doubly degenerate (rational) daha $\mathbf{C}[\mathfrak{t}] \otimes \mathbf{C}[\mathfrak{t}^*] \otimes \mathbf{C}[W]$

Affinization

Here, "daha" stands for double affine Hecke algebra (also called Cherednik algebra) and the structure is given as a **C**-vector space. These algebras have various incarnations:

• The finite Hecke algebra is a quotient of the group algebra of the braid group, which is the fundamental group of $\mathfrak{t}_{reg}/W$. There is a similar description of the affine Hecke algebra (use the space $\mathbf{T}_{reg}/W$) as well as of the double affine Hecke algebra [**Che5**].

• The finite Hecke algebra appears as a coset algebra for a group of the same type as G, over a finite field. There are similar realizations of the affine Hecke algebra (use a 1-dimensional local field) and of the double affine Hecke algebra (2-dimensional local field, cf [**Kap**]).

• There is a geometric realization of (a quotient of) the daha as the equivariant K-theory of the loop Steinberg variety [**Vas**] (cf also [**GarGr, GiKapVas**]), generalizing the realization of the affine Hecke algebra as the equivariant K-theory of the ordinary Steinberg variety. As pointed out in [**BerEtGi2**, §7.1], it is likely that there is an analogous description of the degenerate daha obtained by using homology instead of K-theory (generalizing the realization of the degenerate affine Hecke algebra as the equivariant homology of the Steinberg variety). There is no hint of existence of a geometric realization of the rational daha.

• When W has type A_{n-1}, there are Schur-Weyl dualities between any of the six types of Hecke algebras and corresponding Lie algebras: $\mathfrak{sl}_n$ in the finite case, quantum $\mathfrak{sl}_n$ and Yangian $\mathfrak{sl}_n$ in the affine and degenerate affine case, toroidal quantum $\mathfrak{sl}_n$ [**VarVas1**] in the daha case, toroidal Yangian $\mathfrak{sl}_n$ in the degenerate daha case, and a subalgebra of the latter in the rational case [**Gu2**].

After suitable completions, the daha and the degenerate daha can be viewed as *trivial* deformations of the rational daha ([**EtGi**, p. 283], [**Che6**, p.65], [**BerEtGi2**, §7.1]). As a consequence, the categories of finite dimensional modules for ordinary, degenerate and rational daha's are equivalent ([**Che6**], [**BerEtGi2**, Proposition 7.1], [**VarVas2**]). The categories $\mathcal{O}$ for the ordinary and degenerate daha's are related [**VarVas2**]. Finally, category $\mathcal{O}$ for the rational daha can be realized as a full subcategory of category $\mathcal{O}$ for the degenerate daha [**Su**].

Double affine Hecke algebras are related to combinatorics and they were introduced by Cherednik as a crucial instrument in the proof of the Macdonald's constant term conjectures. The degenerations (trigonometric and rational) were obtained in a straightforward way.

In this survey, we are concerned with the representation theory of rational daha's. Rational Cherednik algebras are connected to

- Finite Hecke algebras
- Resolutions and deformations of symplectic singularities
- Hilbert schemes of points on surfaces.

There are also the following connections, which we won't discuss in this survey:

- Integrable systems: rings of quasi-invariants and quantum Calogero-Moser systems (cf [**EtSt**] for a survey)
- Analytic representation theory, Bessel functions, unitary representations (cf the book [**Che7**]).

Many of these aspects actually make sense in the framework of symplectic reflection algebras [**EtGi**].

The rational Cherednik algebra is a deformation of the algebra $\mathbf{C}[\mathfrak{t}\times\mathfrak{t}^*]\rtimes W$ depending on parameters t, c.

The main idea in the study of representations of rational Cherednik algebras (at $t=1$) is to handle them like universal enveloping algebras of semi-simple complex Lie algebras and study in particular a "category $\mathcal{O}$". We emphasize this in these notes, by expounding the analogies (cf also the table in §9). The (finite) Hecke algebra controls a large part of the representation theory, via the construction of Knizhnik-Zamolodchikov connections. Another important feature of category $\mathcal{O}$ is that it generalizes, for any Weyl group (or even any complex reflection group), the construction of q-Schur algebras.

The usual interplay between representation theory and geometry is incomplete in general. In type A, there are more geometric objects at hand and the Hilbert scheme of points in $\mathbf{C}^2$ plays the role of the cotangent bundle of the flag variety.

The representation theory is quite different for $t=0$. It is related to (generalized) Calogero-Moser spaces.

Section §2 is independent from the rest of the text. It explains how the rational Cherednik algebras occur naturally in the study of a commuting family of operators deforming the partial derivatives.

Although the theory has been developed quite intensively over the last few years, many problems remain and we have listed a number of them.

I have tried to give detailed references for most results, I apologize in advance for possible omissions.

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2. A motivation via Dunkl operators

2.1. Dimension 1.

2.1.1. Fix $k\in\mathbf{R}$. Given $f:\mathbf{R}\to\mathbf{R}$ a function of class C^1, consider the function

$$T(f): x\mapsto f'(x)+k\frac{f(x)-f(-x)}{x}.$$

The operator T deforms the ordinary derivation, and presents new features for special values of k.

For example, one shows easily that there exists a non-constant polynomial killed by T if and only if $k\in-\frac{1}{2}+\mathbf{Z}_{\leq 0}$.

2.1.2. Let us now study the spectrum of T. We consider the Banach algebra of functions $B=\{f=\sum_{n\geq 0}a_nX^n :]-1,1[\to\mathbf{R},\ \sum_n|a_n|<\infty\}$.

We want to solve the equation

$$T(f)=\lambda f \text{ and } f(0)=1 \tag{1}$$

for some $\lambda\in\mathbf{R}$ and $f\in B$.

Assume $k > 0$. Define

$$\chi : B \to B, \quad f \mapsto (x \mapsto \alpha \int_{-1}^{1} f(xt)(1-t)^{k-1}(1+t)^k dt)$$

where $\alpha = \left(\int_{-1}^{1}(1-t)^{k-1}(1+t)^k dt\right)^{-1}$ (so that $\chi(1) = 1$). Then, one shows that

$$T \circ \chi = \chi \circ \frac{d}{dx}.$$

So, $\chi(\exp(\lambda x))$ is the unique solution of (1) (it can be expressed in terms of Bessel functions).

More classical would be the study of the eigenfunctions of the operator T^2 acting on even functions: given f with $f(-x) = f(x)$, then $T^2(f) = \frac{d^2 f}{dx^2} + \frac{2k}{x}\frac{df}{dx}$.

We refer to [**CheMa**] and [**Che7**, §2] for a more detailed study, in particular of the analytic aspects (Hankel transform, truncated Bessel functions).

2.2. Dimension n.

2.2.1. We are now going to generalize the previous construction to the case of $n \geq 2$ variables. We will focus on the algebraic aspects (polynomial functions) and work with complex coefficients. In particular, we take now $k \in \mathbf{C}$.

Let $V = \bigoplus_n \mathbf{C}\xi_i$ and $V^* = \bigoplus_n \mathbf{C}x_i$ with the dual basis. Let $\mathfrak{S}_n$ be the symmetric group on $\{1, \ldots, n\}$. It acts on V by permutation of the coordinates, hence it acts on functions $V \to \mathbf{C}$. We denote by ρ_{ij} the endomorphism of the space of functions $V \to \mathbf{C}$ given by the transposition (ij).

Given $1 \leq i \leq n$ and $f : V \to \mathbf{C}$ smooth, we define

$$T_i(f) = \frac{\partial f}{\partial \xi_i} + k \sum_{j \neq i} \frac{f - \rho_{ij}(f)}{X_i - X_j}.$$

We have a family of operators (the Dunkl operators) deforming the ordinary partial derivatives. What makes this deformation interesting is Dunkl's result:

$$T_i \circ T_j = T_j \circ T_i \text{ for all } i, j.$$

Note also that T_i sends a polynomial to a polynomial.

Let $\mathcal{E}$ be the set of values of k for which there are non-constant polynomials killed by $T_1, \ldots, T_n$.

The case $n = 2$ here is related to the 1-dimensional case of §2.1 by restricting functions $\mathbf{C}^2 \to \mathbf{C}$ to the subspace $x_1 + x_2 = 0$.

Note that the study of functions, in a space similar to B above, that are simultaneous eigenvectors for all T_i's can be done similarly, when $k \in \mathbf{R}_{>0}$. The endomorphism χ can be constructed first for polynomials, and then extended to B by continuity. Nevertheless, there is no explicit integral form for χ.

2.2.2. We denote by $\mathbf{C}[V] = \mathbf{C}[X_1, \ldots, X_n]$ the algebra of polynomial functions on V. Let H be the subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathbf{C}[V])$ generated by $T_1, \ldots, T_n$, $X_1, \ldots, X_n$ (acting by multiplication), and $\mathfrak{S}_n$ (acting by permutation on the X_i's). This is the rational Cherednik algebra.

One shows that $k \in \mathcal{E}$ if and only if $\mathbf{C}[V]$ is not an irreducible representation of H. Let us now say more about the structure of H.

When $k = 0$, then $H = D(V) \rtimes \mathfrak{S}_n$, where $D(V)$ is the algebra of polynomial differential operators on V. In general, there is a vector space decomposition

$$H = \mathbf{C}[T_1, \ldots, T_n] \otimes \mathbf{C}[\mathfrak{S}_n] \otimes \mathbf{C}[X_1, \ldots, X_n].$$

This shows that H (which depends on the parameter k) is a deformation of $D(V) \rtimes \mathfrak{S}_n$.

This is analogous to the decomposition $\mathfrak{gl}_n(\mathbf{C}) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) are strictly upper (resp. lower) triangular matrices and $\mathfrak{h}$ diagonal matrices, or rather analogous to the decomposition of the enveloping algebra (Poincaré-Birkhoff-Witt Theorem)

$$U(\mathfrak{gl}_n(\mathbf{C})) = U(\mathfrak{n}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^-).$$

This analogy is a guide for the study of the representation theory of H.

First, one defines a category $\mathcal{O}$ of finitely generated H-modules on which the T_i's act locally nilpotently. Given E a complex irreducible representation of $\mathfrak{S}_n$, one gets $\Delta(E) = \mathrm{Ind}_{\mathbf{C}[T_1,\ldots,T_n]\rtimes\mathfrak{S}_n}^H E$, an object of $\mathcal{O}$. It has a unique simple quotient $L(E)$, and one obtains this way all simple objects of $\mathcal{O}$. Note that $\Delta(\mathbf{C}) = \mathbf{C}[V]$ is the original faithful representation. Outside a countable set of values of k, then $\mathcal{O}$ is semi-simple.

2.2.3. We now relate $\mathcal{O}$ to the Hecke algebra $\mathcal{H}$ of $\mathfrak{S}_n$, at $q = \exp(2i\pi k)$.

Let $V_{reg} = \{(z_1, \ldots, z_n) \in V | z_i \neq z_j \text{ for } i \neq j\}$. Note that $T_i \in D(V_{reg}) \rtimes \mathfrak{S}_n$ and one gets an embedding $H \subset D(V_{reg}) \rtimes \mathfrak{S}_n$. This induces an isomorphism of algebras

$$H \otimes_{\mathbf{C}[V]} \mathbf{C}[V_{reg}] \xrightarrow{\sim} D(V_{reg}) \rtimes \mathfrak{S}_n$$

After localization, the deformation becomes trivial!

Let $M \in \mathcal{O}$. Then, $M \otimes_{\mathbf{C}[V]} \mathbf{C}[V_{reg}]$ is an $\mathfrak{S}_n$-equivariant vector bundle on V_{reg} with a flat connection, that is shown to have regular singularities (along the hyperplanes and at infinity). This provides us with a system of differential equations, and taking solutions we obtain an $\mathfrak{S}_n$-equivariant local system on V_{reg}, *i.e.*, a local system on $V_{reg}/\mathfrak{S}_n$. This corresponds to a finite dimensional representation of the braid group $B_n = \pi_1(V_{reg}/\mathfrak{S}_n, (1, 2, \ldots, n))$. That representation is shown to come from a representation of $\mathcal{H}$ and this defines a functor

$$\mathrm{KZ} : \mathcal{O} \to \mathcal{H}\text{-mod}$$

(actually, a contravariant functor; one needs to dualize or equivalently use the de Rham functor instead of the solution functor in order to have a covariant functor). This functor has good homological properties and there is an equivalence $\mathcal{O} \simeq \mathrm{End}_{\mathcal{H}}(P)$-mod, where P is a certain $\mathcal{H}$-module.

When $k \notin \frac{1}{2} + \mathbf{Z}$, P can be identified as the q-tensor space $L^{\otimes_q n}$, where L is an n-dimensional vector space, and this identifies $\mathcal{O}$ with the category of modules over the q-Schur algebra, *i.e.*, a full subcategory of the category of modules over the quantum general linear group $U_q(\mathfrak{gl}_n)$. The knowledge of character formulas for simple modules in that setting allows to deduce the multiplicities $[\Delta(E) : L(F)]$ for the algebra H. This is nicely described in terms of canonical basis for the Fock space, under the action of $\hat{\mathfrak{sl}}_d$, where d is the order of k in $\mathbf{C}/\mathbf{Z}$.

Let us finally describe the set $\mathcal{E}$:

$$\mathcal{E} = \{-\frac{r}{s} | 2 \leq s \leq n,\ r \in \mathbf{Z}_{>0} \text{ and } (s, r) = 1\}.$$

The space of polynomials killed by the T_i's and its structure as a representation of $\mathfrak{S}_n$ can be determined (cf [**DuDeJOp, Du4**]).

The analytic aspects, which we are not considering here, are very interesting: multi-dimensional Bessel functions, Hankel transform [**Op, DuDeJOp**], and unitary representations of rational Cherenik algebras. It sheds new light on a number of classical results (cf [**Du2**], [**Che7**, Chapter 2]).

3. Structure

Let us start with the definition of the rational Cherednik algebras and some important subalgebras, and their main properties, following [**EtGi**, Part 1].

3.1. The rational Cherednik algebra.

3.1.1. Let W be a finite reflection group on a finite dimensional real vector space $V_{\mathbf{R}}$ and let $V=\mathbf{C}\otimes_{\mathbf{R}}V_{\mathbf{R}}$. Let $n=\dim_{\mathbf{C}}V$. Let $\mathcal{S}$ be the set of reflections of W and $\bar{\mathcal{S}}=\mathcal{S}/W$. Given $s\in\mathcal{S}$, let $v_s\in V$ (resp. $\alpha_s\in V^*$) be a -1 eigenvector for s acting on V (resp. V^*).

Let $\mathbf{c}=\{\mathbf{c}_s\}_{s\in\bar{\mathcal{S}}}$ be a family of variables and $\mathbf{A}=\mathbf{C}[\{\mathbf{c}_s\},\mathbf{t}]$. Note that for types ADE, we have $|\bar{\mathcal{S}}|=1$.

The *rational Cherednik algebra* $\mathbf{H}$ associated to (W,V) is the quotient of $\mathbf{A}\otimes_{\mathbf{C}}(T(V\oplus V^*)\rtimes W)$ by the relations[1]

$$[\xi,\eta]=0 \text{ for } \xi,\eta\in V,\quad [x,y]=0 \text{ for } x,y\in V^*$$

$$[\xi,x]=\mathbf{t}\langle\xi,x\rangle-2\sum_{s\in\mathcal{S}}\frac{\langle\xi,\alpha_s\rangle\langle v_s,x\rangle}{\langle v_s,\alpha_s\rangle}\mathbf{c}_s s$$

There is a filtration on $\mathbf{H}$ given by

$$F^0\mathbf{H}=\mathbf{A}[W],\ F^1\mathbf{H}=(V\oplus V^*)\otimes_{\mathbf{C}}\mathbf{A}[W]\oplus\mathbf{A}[W],\text{ and } F^i\mathbf{H}=(F^1\mathbf{H})^i \text{ for } i\geq 2.$$

Let $\mathrm{gr}\mathbf{H}=\bigoplus_{i\geq 0}F^i\mathbf{H}/F^{i-1}\mathbf{H}$. The canonical morphism of $\mathbf{A}$-modules $(V\oplus V^*)\otimes_{\mathbf{C}}\mathbf{A}[W]\to\mathrm{gr}\mathbf{H}$ extends to a surjective morphism of $\mathbf{A}$-algebras $\mathbf{A}\otimes_{\mathbf{C}}(S(V)\otimes S(V^*))\rtimes W\to\mathrm{gr}\mathbf{H}$. The following result asserts it is actually an isomorphism. This gives a triangular decomposition of $\mathbf{H}$ (Cherednik, [**EtGi**, Theorem 1.3]):

Theorem 3.1. *We have a canonical isomorphism of* $\mathbf{A}$*-modules* $S(V)\otimes_{\mathbf{C}}\mathbf{A}[W]\otimes_{\mathbf{C}}S(V^*)\xrightarrow{\sim}\mathbf{H}$. *In particular, the canonical map of* $\mathbf{A}$*-algebras*

$$\mathbf{A}\otimes_{\mathbf{C}}(S(V)\otimes S(V^*))\rtimes W\xrightarrow{\sim}\mathrm{gr}\mathbf{H}$$

is an isomorphism.

About the proof. Note that it is enough to prove the isomorphism after applying $-\otimes_{\mathbf{A}}\mathbf{C}$ for any morphism $\mathbf{A}\to\mathbf{C}$, *i.e.*, for specialized parameters. Then, one can use the faithful representation by Dunkl operators (cf §4.2.1 and 5.1.1). □

Given $t\in\mathbf{C}$ and $c=\{c_s\}_{s\in\bar{\mathcal{S}}}\in\mathbf{C}^{\bar{\mathcal{S}}}$, we put $H_{t,c}=\mathbf{H}\otimes_{\mathbf{A}}\mathbf{C}$, where the morphism $\mathbf{A}\to\mathbf{C}$ is given by $\mathbf{t}\mapsto t$ and $\mathbf{c}_s\mapsto c_s$. Theorem 3.1 shows that $\mathbf{H}$ is a deformation of $H_{t,c}$.

Analogy 1. Let G be a semi-simple complex algebraic group, T a maximal torus, B a Borel subgroup containing T, U^+ its unipotent radical and U^- the opposite unipotent subgroup. Let $\mathfrak{g}=\mathrm{Lie}\,G$, $\mathfrak{h}=\mathrm{Lie}\,T$, $\mathfrak{n}^+=\mathrm{Lie}\,U^+$ and $\mathfrak{n}^-=\mathrm{Lie}\,U^-$. We have (Poincaré-Birkhoff-Witt Theorem)

$$U(\mathfrak{g})=U(\mathfrak{n}^+)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}^-).$$

We have a filtration of $U(\mathfrak{g})$ given by $F^0U(\mathfrak{g})=\mathbf{C}$, $F^1U(\mathfrak{g})=\mathfrak{g}\oplus\mathbf{C}$ and $F^iU(\mathfrak{g})=(F^1U(\mathfrak{g}))^i$. There is a canonical isomorphism $S(\mathfrak{g})\xrightarrow{\sim}\mathrm{gr}U(\mathfrak{g})$.

Remark 3.2. If $W=W_1\times W_2$ and $V=V_1\oplus V_2$ are compatible decompositions, then there are canonical isomorphisms $\mathbf{H}_1\otimes\mathbf{H}_2\xrightarrow{\sim}\mathbf{H}$, where $\mathbf{H}_i$ is the rational Cherednik algebra of (W_i,V_i).

Let $\mathcal{S}'$ be a W-invariant subset of $\mathcal{S}$ such that $c_s=0$ for $s\in\mathcal{S}-\mathcal{S}'$. Let W' be the reflection subgroup of W generated by $\mathcal{S}'$. Then, there is an embedding $\mathbf{H}'\subset\mathbf{H}$ and $\mathbf{H}$ is a twisted group algebra of W/W' over $\mathbf{H}'$.

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[1] In [**GGOR**], we use $\gamma_H=-2\mathbf{c}_s s$ and $\mathbf{k}_{H,1}=-\mathbf{c}_s$, where H is the reflecting hyperplane of s

3.1.2. *Specializations.* For $t \neq 0$, we have $H_{t,c} \xrightarrow{\sim} H_{1,t^{-1}c}$. We put $H_c = H_{1,c}$.

Consider the case $c = 0$:

- We have $H_{0,0} = S(V \oplus V^*) \rtimes W$. So, $H_{0,\mathbf{c}}$ is a deformation of $S(V \oplus V^*) \rtimes W$.
- We have $H_{1,0} = D(V) \rtimes W$, where $D(V)$ is the Weyl algebra of V (algebra of algebraic differential operators on V). So, $H_{1,\mathbf{c}}$ is a deformation of $D(V) \rtimes W$.

Analogy 2. The parameter space $\mathbf{C}^{\bar{\mathcal{S}}}$ corresponds to $\mathfrak{h}^*/W$ and the analog of $H_{1,c}$ is $\bar{U}_\lambda(\mathfrak{g}) = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\lambda$, where $\lambda \in \mathfrak{h}^*/W$, and $\mathfrak{m}_\lambda$ is the maximal ideal of $Z(U(\mathfrak{g}))$ image of λ by the canonical isomorphism $\mathfrak{h}^*/W \xrightarrow{\sim} \operatorname{Spec} Z(U(\mathfrak{g}))$.

Consider the induced filtration on $\bar{U}_\lambda(\mathfrak{g})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. Then, there is a canonical isomorphism $\mathbf{C}[\mathcal{N}] \xrightarrow{\sim} \mathrm{gr}\bar{U}_\lambda(\mathfrak{g})$.

3.1.3. *Fourier transform.* Cf [**EtGi**, §4,5].

Fix an isomorphism of $\mathbf{C}[W]$-modules $F : V \xrightarrow{\sim} V^*$. This extends to an automorphism F of $\mathbf{H_c}$ given by

$$V \ni \xi \mapsto F(\xi), \quad V^* \ni x \mapsto -F^{-1}(x) \quad \text{and} \quad W \ni w \mapsto w.$$

More generally, there is an action of $\mathrm{SL}_2(\mathbf{A})$ on $\mathbf{H}$. The action of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is given by

$$V \ni \xi \mapsto a_{22}\xi + a_{21}F(\xi), \quad V^* \ni x \mapsto a_{11}x + a_{12}F^{-1}(x) \quad \text{and} \quad W \ni w \mapsto w.$$

3.1.4. *Twist by characters.* Let $c \in \mathbf{C}^{\bar{\mathcal{S}}}$ and $t \in \mathbf{C}$. Let $\zeta : W \to \{\pm 1\}$ be a character. There is an isomorphism of $\mathbf{C}$-algebras

$$H_{t,c} \xrightarrow{\sim} H_{t,c\zeta}, \; V \ni \xi \mapsto \xi, \quad V^* \ni x \mapsto x, \quad W \ni w \mapsto \zeta(w)w.$$

3.1.5. *Deformed Euler vector field and canonical grading.* We consider in the remaining part of §3.1 the algebra $\mathbf{H_c} = \mathbf{H}_{1,\mathbf{c}} = \mathbf{H} \otimes_{\mathbf{C}[\mathbf{t}]} \mathbf{C}[\mathbf{t}]/(\mathbf{t}-1)$.

Let B be a basis of V and $(b^\vee)_{b\in B}$ be the dual basis. Let $\mathrm{eu}' = \sum_{b\in B} b^\vee b$ be the "deformed" Euler vector field, $z = \sum_{s\in\mathcal{S}} \mathbf{c}_s(s-1)$ and $\mathrm{eu} = \mathrm{eu}' - z$. Let $h = \frac{1}{2}\sum_{b\in B}(bb^\vee + b^\vee b)$. Then, $h = \mathrm{eu} + \frac{1}{2}\dim V - \sum_{s\in\mathcal{S}} \mathbf{c}_s$. An easy computation in $\mathbf{H_c}$ shows that

$$[\mathrm{eu}, \xi] = -\xi, \; [\mathrm{eu}, x] = x \text{ and } [\mathrm{eu}, w] = 0 \text{ for } \xi \in V, \; x \in V^* \text{ and } w \in W.$$

So, the eigenspace decomposition of $\mathbf{H_c}$ under the action of $[\mathrm{eu}, -]$ puts a grading on $\mathbf{H_c}$: W is in degree 0, V^* in degree 1 and V in degree -1 (note that this defines a grading on $\mathbf{H}$ as well, before specializing $\mathbf{t}$ to 1).

Let M be an $\mathbf{H_c}$-module. We denote by M_α is the generalized α-eigenspace for eu acting on M. For certain modules, we have a decomposition $M = \bigoplus_{\alpha\in\mathbf{C}} M_\alpha$ and this gives a *canonical* $\mathbf{C}$*-grading* on M.

3.1.6. $\mathfrak{sl}_2$*-triple.* Cf [**De**, §4], [**BerEtGi3**, §3].

Let $p : V \times V \to \mathbf{C}$ and $q : V^* \times V^* \to \mathbf{C}$ be the W-equivariant perfect pairings induced by F. If $\mathcal{B}$ is orthonormal for p, then $p = \sum_b (b^\vee)^2$ and $q = \sum_b b^2$. An easy computation shows that $\langle \frac{1}{2}p, h, -\frac{1}{2}q \rangle$ is an $\mathfrak{sl}_2$-triple in $\mathbf{H_c}$.

3.2. The spherical subalgebra.

3.2.1. Cf [**EtGi**, §2].

Let $e = \frac{1}{|W|}\sum_{w\in W} w$, an idempotent of $Z(\mathbf{C}[W])$. Let $\mathbf{B} = e\mathbf{H}e$ be the *spherical subalgebra* of $\mathbf{H}$. Theorem 3.1 gives a canonical isomorphism $\mathbf{A} \otimes_{\mathbf{C}} \mathbf{C}[(V^* \times V)/W] \xrightarrow{\sim} \mathrm{gr}\mathbf{B}$ (for the induced filtration on $\mathbf{B}$).

We have canonical isomorphisms $B_{1,0} \xrightarrow{\sim} D(V)^W$ and $B_{0,0} \xrightarrow{\sim} \mathbf{C}[(V^* \times V)/W]$. So, $\mathbf{B}$ is a deformation of $D(V)^W$ and of $\mathbf{C}[(V^* \times V)/W]$.

Analogy 3. The analogy between $\mathbf{B}$ and $U(\mathfrak{g})$ is more accurate than that with $\mathbf{H}$: instead of the smooth orbifold $[(V^* \times V)/W]$, one gets the singular variety $(V^* \times V)/W$ as corresponding to the nilpotent cone. Also, for a Cherednik algebra of type A_1, then $B_{1,c}$ is isomorphic to an algebra $\bar{U}_\lambda(\mathfrak{sl}_2)$, cf §7.3.3.

3.2.2. The center and the spherical subalgebra have different behaviours, depending on t.

We have $Z(H_{t,c}) = \mathbf{C}$ if $t \neq 0$ [**BrGo**, Proposition 7.2].

The algebra $B_{t,c}$ is commutative if and only if $t = 0$ [**EtGi**, Theorem 1.6]. In particular, we get a structure of Poisson algebra on $B_{0,c}$.

We have [**EtGi**, Theorem 3.1]:

Theorem 3.3 ("Satake isomorphism"). *We have an isomorphism* $Z(H_{0,c}) \xrightarrow{\sim} B_{0,c},\ z \mapsto ze$.

The *Calogero-Moser space* associated to W is $\mathcal{CM}_c = \operatorname{Spec} Z(H_{0,c})$. It is a Gorenstein normal Poisson variety [**EtGi**, Theorem 1.5 and Lemma 3.5] and a symplectic variety when smooth [**EtGi**, Theorem 1.8].

There is an inclusion $S(V)^W \otimes S(V^*)^W \subset Z(H_{0,c})$ and $Z(H_{0,c})$ is a free $(S(V)^W \otimes S(V^*)^W)$-module of rank $|W|$ [**EtGi**, Proposition 4.15]. This gives a finite surjective map $\Upsilon : \mathcal{CM}_c \to V^*/W \times V/W$.

3.2.3. There is a "double centralizer Theorem":

Theorem 3.4 ([**EtGi**, Theorem 1.5]). *We have canonical isomorphisms* $\mathbf{B} \xrightarrow{\sim} \operatorname{End}_{\mathbf{H}}(\mathbf{H}e)$ *and* $\mathbf{H} \xrightarrow{\sim} \operatorname{End}_{\mathbf{B}^\circ}(\mathbf{H}e)$.

The bimodule $H_{t,c}e$ induces a Morita equivalence between $H_{t,c}$ and $B_{t,c}$ (*i.e.*, $H_{t,c}e \otimes_{B_{t,c}} - : B_{t,c}\text{-mod} \to H_{t,c}\text{-mod}$ is an equivalence) if and only if $H_{t,c} = H_{t,c}eH_{t,c}$. Cf Theorems 4.14 and 6.6 for cases of Morita equivalence. Note that $H_{0,0}$ is not Morita equivalent to $B_{0,0}$ for $W \neq 1$ (the first algebra has finite global dimension while the second one doesn't).

4. Representation theory at $t \neq 0$

4.1. Category $\mathcal{O}$.

Cf [**DuOp**, §2], [**Gu1**], [**GGOR**, §2,3], [**BerEtGi3**, §2], [**Gi**, Corollary 6.7].

4.1.1. *Decomposition.* Fix a specialization $H = H_c$ (*i.e.*, $t = 1$). Let $\mathcal{O}'$ be the category of finitely generated H-modules that are locally finite for $S = S(V)$.

Let $\bar{\lambda} \in V^*/W$, *i.e.*, $\bar{\lambda}$ is a morphism of algebras $S^W \to \mathbf{C}$. Let $\mathcal{O}_{\bar{\lambda}}$ be the subcategory of objects M in $\mathcal{O}'$ such that for any $m \in M$ and any $\xi \in S^W$, then $(\xi - \bar{\lambda}(\xi))^n \cdot m = 0$ for $n \gg 0$.

Then,

$$\mathcal{O}' = \bigoplus_{\bar{\lambda} \in V^*/W} \mathcal{O}_{\bar{\lambda}}$$

4.1.2. *Principal block.* We focus our study on $\mathcal{O} = \mathcal{O}'_0$ (similar descriptions for arbitrary $\bar{\lambda}$ have been partially worked out). We write also $\mathcal{O}_c$ for the category $\mathcal{O}$ of H_c.

Let $\operatorname{Irr}(W)$ be the set of isomorphism classes of irreducible complex representations of W. Given $E \in \operatorname{Irr}(W)$, let $N_E = \sum_{s \in \mathcal{S}} \frac{\operatorname{Tr}(s|E)}{\dim E} c_s = \frac{\operatorname{Tr}(z|E)}{\dim E} + \sum_{s \in \mathcal{S}} c_s$. We define an order on $\operatorname{Irr}(W)$ by $E < F$ if $N_F - N_E \in \mathbf{Z}_{>0}$.

Given $E \in \operatorname{Irr}(W)$, let

$$\Delta(E) = \operatorname{Ind}_{S \rtimes W}^{H} E$$

where E is viewed as an $(S \rtimes W)$-module with V acting as 0.

Then, we have

Theorem 4.1 ([**GGOR**, Theorem 2.19]). *$\mathcal{O}$ is a highest weight category with standard objects the $\Delta(E)$'s.*

About the proof. The approach is similar to the one for affine Lie algebras and makes crucial use of the canonical grading. ☐

In particular, $\Delta(E)$ has a unique simple quotient $L(E)$ and $\{L(E)\}_{E \in \operatorname{Irr}(W)}$ is a complete set of representatives of isomorphism classes of simple objects of $\mathcal{O}$.

It follows also that if no two distinct elements of $\operatorname{Irr}(W)$ are comparable, then $\mathcal{O}$ is semi-simple.

Corollary 4.2. *For generic values of c, then $\mathcal{O}$ is semi-simple and $\Delta(E) = L(E)$ for all $E \in \operatorname{Irr}(W)$.*

The costandard object $\nabla(E)$ is the H-submodule of $\mathrm{Hom}_{S(V^*)\rtimes W}(H,E)$ of elements that are locally nilpotent for S (where E is viewed as an $(S(V^*)\rtimes W)$-module with V^* acting as 0).

Simple objects don't have self-extensions:

Proposition 4.3 ([**BerEtGi2**, Proposition 1.12]). *We have* $\mathrm{Ext}^1_{\mathcal{O}}(L(E),L(E))=0$ *for any* $E\in\mathrm{Irr}(W)$.

4.1.3. *Dualities.* Cf [**GGOR**, §4].

Let $M\in\mathcal{O}$. Let M' be the k-submodule of $S(V^*)$-locally nilpotent elements of $\mathrm{Hom}_{\mathbf{C}}(M,\mathbf{C})$. Consider the anti-involution on H_c:

$$\psi : H_c \xrightarrow{\sim} H_c^{\mathrm{opp}},\quad \xi\ni V\mapsto -F(\xi),\quad V^*\ni x\mapsto -F^{-1}(x),\quad W\ni w\mapsto w^{-1}.$$

Then $M^\vee=\psi_*M'\in\mathcal{O}$ and this defines a duality

$$\mathcal{O}\xrightarrow{\sim}\mathcal{O}^{\mathrm{opp}},\quad M\mapsto M^\vee.$$

We have $L(E)^\vee\simeq L(E)$ and $\Delta(E)^\vee\simeq\nabla(E)$.

The anti-involution

$$H_c\xrightarrow{\sim}H_c^{\mathrm{opp}},\quad \xi\ni V\mapsto -\xi,\quad V^*\ni x\mapsto x,\quad W\ni w\mapsto w^{-1}$$

provides H_c with a structure of $(H_c\otimes H_c)$-bimodule and we obtain a duality

$$R\,\mathrm{Hom}_{H_c}(-,H_c[n]) : D^b(H_c\text{-mod})\xrightarrow{\sim}D^b(H_c\text{-mod})^{\mathrm{opp}}.$$

It restricts to a duality

$$D : D^b(\mathcal{O})\xrightarrow{\sim}D^b(\mathcal{O})^{\mathrm{opp}}.$$

We have $D(\Delta(E))^\vee\simeq\nabla(E\otimes\det)$ and $D(P(E))^\vee\simeq T(E\otimes\det)$ (a tilting module). As a consequence, $\mathcal{O}$ is equivalent to its Ringel dual.

4.1.4. *Dihedral groups.* The structure of the standard modules for $W=I_2(d)$ a dihedral group is given in [**Chm**]. Let us explain the results, in the simpler case $d=2m+1$ is odd. We denote by τ_l the 2-dimensional irreducible representation whose first occurrence in $S(V)$ is in degree l (where $1\le l\le m$). We assume $c>0$ (cf §3.1.4 to deduce the case $c<0$).

- Assume first $c=\frac{r}{d}$ for some $r\in\mathbf{Z}_{>0}$, $d\nmid r$. Let $l\in\{1,\ldots,m\}$ such that $r\equiv\pm l\pmod d$.

We have $L(\rho)=\Delta(\rho)=P(\rho)$ if $\rho\neq\mathbf{C},\det,\tau_l$ (they form simple blocks of category $\mathcal{O}$).

We give now the Loewy series of various modules in $\mathcal{O}$ (socle and radical series coincide):

$$\Delta(\mathbf{C})=P(\mathbf{C})=\begin{array}{c}L(\mathbf{C})\\L(\tau_l)\end{array},\quad \Delta(\det)=L(\det)=T(\det),\quad \Delta(\tau_l)=\begin{array}{c}L(\tau_l)\\L(\det)\end{array}$$

$$P(\det)=T(\tau_l)=\begin{array}{c}L(\det)\\L(\tau_l)\\L(\det)\end{array},\quad P(\tau_l)=T(\mathbf{C})=\begin{array}{c}L(\tau_l)\\L(\det)\oplus L(\mathbf{C})\\L(\tau_l)\end{array}$$

The only simple finite dimensional module is $L(\mathbf{C})$.

- Assume now $c\in\frac{1}{2}+\mathbf{Z}_{\geq 0}$.

We have $L(\rho)=\Delta(\rho)=P(\rho)$ if $\rho\neq\mathbf{C},\det$ (they form simple blocks of category $\mathcal{O}$).

We have

$$\Delta(\mathbf{C})=P(\mathbf{C})=\begin{array}{c}L(\mathbf{C})\\L(\det)\end{array},\quad \Delta(\det)=T(\det)=L(\det),\quad P(\det)=T(\mathbf{C})=\begin{array}{c}L(\det)\\L(\mathbf{C})\\L(\det)\end{array}.$$

- If $c\in\mathbf{Z}_{>0}$ or neither $2c$ nor dc are integers, then $\mathcal{O}$ is semi-simple.

The structure is more complicated when d is even, for special values of the parameter. In particular, finite dimensional modules need not be semi-simple (this occurs as well for W of type D_4 with parameter $c = \frac{1}{2}$ [**BerEtGi2**, Example 6.4]).

4.1.5. In view of the analogy with complex semi-simple Lie algebras, finite-dimensional representations are particularly interesting. A particular class of finite dimensional quotients of $\mathbf{C}[V] = \Delta(\mathbf{C})$ has been studied ("perfect representations"): these are naturally commutative algebras, and they generalize the Verlinde algebras [**Che6**].

Problem 1.
- Find the multiplicities $[\Delta(E) : L(F)]$. They are known when W is dihedral (cf §4.1.4) and when has type A_n and $c \notin \frac{1}{2} + \mathbf{Z}$ (cf Corollary 6.3).
- Describe the category of finite dimensional representations of H. For which values of c does H have non-zero finite dimensional representations ? Cf [**Che6, De, BerEtGi2, Go2, ChmEt, Vas**] for studies of finite dimensional representations. These questions are solved in type A_n, cf Theorem 6.5.
- Is $\mathcal{O}$ Koszul ? If so, is it its own Koszul dual (up to a change of parameters) ?

Analogy 4. The category $\mathcal{O}'$ of finitely generated $U(\mathfrak{g})$-modules that are diagonalizable for $U(\mathfrak{h})$ and locally finite for $U(\mathfrak{n}^+)$ splits into a sum of subcategories corresponding to a fixed central character. The finite dimensional representations are semi-simple and the simple ones correspond to dominant weights. The principal block $\mathcal{O}$ is a highest weight category with standard objects being Verma modules. The parametrizing set is W, with the Bruhat order. The multiplicities of simple objects in standard objects are given by evaluation at 1 of the Kazhdan-Lusztig polynomials for W (Kazhdan-Lusztig conjecture, proven by Beilinson-Bernstein and Brylinski-Kashiwara). The principal block $\mathcal{O}$ is Koszul and it is equivalent to its Koszul dual (Beilinson-Ginzburg-Soergel).

4.2. Dunkl operators and KZ functor.

4.2.1. *Dunkl operators.* Cf [**Che7**], [**EtGi**, §4], [**DuOp**, §2.2], [**GGOR**, §5.2].

Denote by $\mathbf{C}$ the trivial representation of W. Via the canonical isomorphism $\mathbf{C}[V] \xrightarrow{\sim} \Delta(\mathbf{C})$, we obtain an action of H on $\mathbf{C}[V]$. The action of W is the natural action, the action of $\mathbf{C}[V]$ is given by multiplication, and the action of $\xi \in V$ is given by the Dunkl operator (for type A, this is the same as §2)

$$T_\xi = \partial_\xi + \sum_{s \in \mathcal{S}} \frac{\langle \xi, \alpha_s \rangle}{\alpha_s} c_s (s-1).$$

This gives a morphism $\rho : H \to D(V_{reg}) \rtimes W$, where $V_{reg} = V - \bigcup_{s\in\mathcal{S}} \ker \alpha_s$.

A fundamental property is the faithfulness of that representation (Cherednik and [**EtGi**, Proposition 4.5]):

Theorem 4.4. *The morphism ρ is injective and induces an isomorphism $H \otimes_{\mathbf{C}[V]} \mathbf{C}[V_{reg}] \xrightarrow{\sim} D(V_{reg}) \rtimes W$.*

About the proof. One puts a filtration on H with W and V^* in degree 0, and V in degree 1. Then, ρ is compatible with the filtration on $D(V_{reg})$ given by the order of differential operators and the associated graded map is injective. □

Note that $\rho(\mathrm{eu}) = \sum_{b\in\mathcal{B}} b^\vee b \in D(V) \rtimes W$ is the ordinary Euler vector field.

Remark 4.5. Via the canonical isomorphism $D(V_{reg})^W \xrightarrow{\sim} eD(V_{reg})e, f \mapsto ef$, the restriction of ρ to $B_{1,c}$ gives an injective morphism $B_{1,c} \to D(V_{reg})^W$.

4.2.2. *Knizhnik-Zamolodchikov functor.* Cf [**GGOR**, §5.3-5.4].

We are going to associate a vector bundle with a connection on V_{reg} assciated to an object of $\mathcal{O}$. In the case of a standard object $\Delta(E)$, this is essentially the Knizhnik-Zamolodchikov-Cherednik connection [**Che2, Che4, Op**] (cf the affine equation in the trigonometric setting in [**Che4**]). For generic values of the parameter, every object of $\mathcal{O}$ is a sum of $\Delta(E)$'s and this is used in the constructions below via deformation arguments.

Let $M \in \mathcal{O}$ and $M_{reg} = \rho_*(M \otimes_{\mathbf{C}[V]} \mathbf{C}[V_{reg}])$. This corresponds to a W-equivariant vector bundle on V_{reg} with a flat connection. It is shown to have regular singularities, by treating first the case $M = \Delta(E)$.

Applying the de Rham functor $\mathrm{Hom}_{\mathcal{D}(V_{reg})}(\mathcal{O}_{V_{reg}}, -)$ gives a W-equivariant locally constant sheaf on V_{reg}. This corresponds to a locally constant sheaf on V_{reg}/W, hence to a finite dimensional representation $F(M)$ of $B = \pi_1(V_{reg}/W)$ (relative to some base point). Fixing a base point in $(V_{\mathbf{R}})_{reg}$ provides a description of W as a finite Coxeter group, with set of simple reflections $\mathcal{S}_0$. It also provides an identification of B as the corresponding braid group with set of generators $\{\sigma_s\}_{s \in \mathcal{S}_0}$.

Let $\mathcal{H}$ be the Hecke algebra of W with parameters $\{1, -\exp(2i\pi c_s)\}$, *i.e.*, the quotient of $\mathbf{C}[B_W]$ by the relations $(\sigma_s - 1)(\sigma_s + \exp(2i\pi c_s)) = 0$.

Then, the representation $F(M)$ of B factors through a representation $\mathrm{KZ}(M)$ of $\mathcal{H}$. This is proven by first computing the eigenvalues of monodromy when M is a standard module and for generic values of the parameter. A deformation argument shows the result in general.

Let $\mathcal{O}_{tor}$ be the full subcategory of $\mathcal{O}$ of objects M such that $M_{reg} = 0$.
The main properties of KZ are given in the following Theorem [**GGOR**, Theorem 5.14, Theorem 5.16, and Proposition 5.9]:

Theorem 4.6. *The functor* KZ *is exact and it induces an equivalence* $\mathcal{O}/\mathcal{O}_{tor} \xrightarrow{\sim} \mathcal{H}$-mod.

Given $M, N \in \mathcal{O}$*, the canonical map* $\mathrm{Hom}_{\mathcal{O}}(M, N) \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{KZ}(M), \mathrm{KZ}(N))$ *is an isomorphism in the following cases:*

- *when* N *is projective*
- *when* $c_s \notin \frac{1}{2} + \mathbf{Z}$ *for all* $s \in \mathcal{S}$ *and* N *is* Δ*-filtered.*

About the proof. One shows that $\mathcal{O} \to \mathcal{O}/\mathcal{O}_{tor}$ is fully faithful on projective objects by using the duality D. The fully faithfulness of $\mathcal{O}/\mathcal{O}_{tor} \to \mathcal{H}$-mod is a consequence of the Riemann-Hilbert correspondence (Deligne). The essential surjectivity follows from a deformation argument. The statement in the case where N is Δ-filtered is obtained by a computation of residues. ☐

Let P be a progenerator for $\mathcal{O}$. Then, $\mathcal{O} \simeq \mathrm{End}_{\mathcal{H}}(\mathrm{KZ}(P))$-mod. The algebra $\mathrm{End}_{\mathcal{H}}(\mathrm{KZ}(P))$ should be viewed as a "generalized q-Schur algebra" associated to W, cf Theorem 6.2.

Corollary 4.7. *The category* $\mathcal{O}$ *is semi-simple if and only if* $\mathcal{H}$ *is semi-simple.*

Problem 2.
- Provide an explicit construction of a progenerator.
- What is the image of a progenerator P of $\mathcal{O}$? What is $\mathrm{End}_{\mathcal{H}}(\mathrm{KZ}(P))$? This is understood when W has type A_n and $c \notin \frac{1}{2} + \mathbf{Z}$, cf §6.2.1.

Analogy 5. Let W be the Weyl group of G and let $C = \mathbf{C}[\mathfrak{h}]/(\mathbf{C}[\mathfrak{h}]\mathbf{C}[\mathfrak{h}]_+^W)$ be the algebra of coinvariants. Note that there is a canonical isomorphism $C \xrightarrow{\sim} H^*(G/B)$. Let $P = P(w_0)$ be the "antidominant" projective of $\mathcal{O}$. There is an isomorphism $C \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}}(P)$, the functor $\mathrm{Hom}_{\mathcal{O}}(P, -) : \mathcal{O} \to C$-mod is fully faithful when restricted to projectives, and the image of a suitable progenerator is

$$\bigoplus_{w \in W} \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}[\mathfrak{h}]^{s_1}} \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}[\mathfrak{h}]^{s_2}} \cdots \otimes_{\mathbf{C}[\mathfrak{h}]^{s_{r-1}}} \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}[\mathfrak{h}]^{s_r}} C,$$

where $w = s_1 \cdots s_r$ is a reduced decomposition (Soergel).

Remark 4.8. When $s \mapsto c_s$ is constant (equal parameter case), the $\mathcal{H}$-modules $\mathrm{KZ}(\Delta(E))$ are the "standard" modules occurring in Kazhdan-Lusztig theory [**GGOR**, Theorem 6.8] (cf §6.2.1 for type A).

4.3. Primitive ideals and supports.

Cf [**Gi**, §6].

4.3.1. Given M a finitely generated H-module, there is a structure of filtered H-module on M such that $\mathrm{gr}M$ is a finitely generated $\mathrm{gr}H$-module (a "good filtration"). It is in particular a finitely generated $S(V \oplus V^*)$-module (cf §3.1.1). Let $\mathrm{Supp}(M)$ be the support of that module, a W-stable closed subvariety of $V^* \times V$. It is independent of the choice of a good filtration of M.

If $M \in \mathcal{O}$, then $\mathrm{Supp}(M) \subseteq \{0\} \times V$. When $c = 0$, Bernstein's inequality asserts that $\dim \mathrm{Supp}(M) \geq \dim V$. But there are values of c and objects $M \in \mathcal{O}_c$ with $\dim \mathrm{Supp}(M) < \dim V$, cf §7.2.

We have of course $\mathrm{Supp}(\Delta(E)) = \{0\} \times V$, since the restriction of $\Delta(E)$ to P is free.

4.3.2. Recall that an ideal of H is *primitive* if it is the annihilator of a simple H-module.

Theorem 4.9 ([**Gi**, Corollary 6.6]). *Every primitive ideal of H is the annihilator of a simple object of $\mathcal{O}$.*

Let I be an ideal of H. Give I the filtration induced by the canonical filtration on H. Then, $\mathrm{gr}I$ is an ideal of $\mathrm{gr}H = \mathbf{C}[V^* \times V] \rtimes W$, hence defines a W-invariant closed subvariety of $V^* \times V$, the *associated variety* of I.

A *parabolic subgroup* of W is the pointwise stabilizer in W of a subspace of V. We denote by $\mathrm{Par}(W)$ the set of parabolic subgroups of W.

Theorem 4.10 ([**Gi**, Proposition 6.4]). *The associated variety of a primitive ideal of H is of the form $W \cdot (V^* \times V)^{W'}$ for some $W' \in \mathrm{Par}(W)$. In particular, its image in $(V^* \times V)/W$ is irreducible.*

4.3.3.

Theorem 4.11 ([**Gi**, Theorem 6.8]). *Given M simple in $\mathcal{O}$, there is $W' \in \mathrm{Par}(W)$ such that $\mathrm{Supp}(M) = \{0\} \times (W \cdot V^{W'})$.*

Problem 3.
- Determine $\mathrm{Supp}\, L(E)$. This generalizes the problem about finite dimensional $L(E)$'s (Problem 1), they correspond to the case $\mathrm{Supp}\, L(E) = 0$.
- Study the order on $\mathrm{Irr}(W)$ defined by $E \prec E'$ if $\mathrm{Supp}\, L(E) \subset \mathrm{Supp}\, L(E')$.

Analogy 6. The associated variety of a primitive ideal of $U(\mathfrak{g})$ is the closure of a nilpotent class, hence it is irreducible (Borho-Brylinski, Joseph, Kashiwara-Tanisaki).

Every primitive ideal of $U(\mathfrak{g})$ is the annihilator of some simple object of $\mathcal{O}'$ (Duflo).

The annihilator of $L(w)$ is contained in the annihilator of $L(w')$ if and only if w is smaller than w' for the "left cell order" (Joseph, Vogan).

4.4. Harish-Chandra bimodules.

4.4.1. The definitions and results of this section 4.4.1 follow [**BerEtGi3**, §3 and §8].

Let $c, c' \in \mathbf{C}^{\bar{\mathcal{S}}}$.

Definition 4.12. *A $(H_c, H_{c'})$-bimodule is a* Harish-Chandra bimodule *if it is finitely generated and the action of $a \otimes 1 - 1 \otimes a$ is locally nilpotent for every $a \in S(V)^W \cup S(V^*)^W$.*

Such a bimodule is finitely generated as a left H_c-module, as a right $H_{c'}$-module, as a $(S(V)^W, S(V^*)^W)$-bimodule and as a $(S(V^*)^W, S(V)^W)$-bimodule.

We denote by $\mathcal{HC}_{c,c'}$ the category of Harish-Chandra $(H_c, H_{c'})$-bimodules. The inclusion functor $\mathcal{HC}_{c,c'} \to (H_c \otimes H_{c'}^{\mathrm{opp}})$-mod has a right adjoint $M \mapsto M_{fin}$.

Given $M \in \mathcal{HC}_{c,c'}$ and $N \in \mathcal{HC}_{c',c''}$, then $M \otimes_{H_{c'}} N \in \mathcal{HC}_{c,c''}$.

Theorem 4.13. *Assume $c, c' \in \mathbf{Z}_{\geq 0}^{\bar{\mathcal{S}}}$. There is a parametrization $\{V_{c,c'}(E)\}_{E \in \mathrm{Irr}(W)}$ of the set of isomorphism classes of simple objects of $\mathcal{HC}_{c,c'}$ such that given $E_1, E_2 \in \mathrm{Irr}(W)$, then*

$$\mathrm{Hom}_{\mathbf{C}}(\Delta_{c'}(E_1), \Delta_c(E_2))_{fin} \simeq \bigoplus_{E \in \mathrm{Irr}(W)} \mathrm{Hom}_{\mathbf{C}[W]}(E \otimes E_1, E_2) \otimes_{\mathbf{C}} V_{c,c'}(E).$$

Furthermore, $E \mapsto V_{c,c}(E)$ extends to an equivalence of monoidal categories

$$\mathbf{C}[W]\text{-mod} \xrightarrow{\sim} \mathcal{HC}_{c,c}.$$

Problem 4. Describe the structure of the 2-category with set of objects $\bar{\mathcal{S}}$, 1-arrows the objects of $\mathcal{HC}_{c,c'}$ and 2-arrows the morphisms of $\mathcal{HC}_{c,c'}$.

4.4.2.

Theorem 4.14 ([**BerEtGi3**, Theorem 3.1]). *Assume $\mathcal{H}$ is semi-simple. Then, H_c is a simple algebra and H_ce gives a Morita equivalence between H_c and B_c.*

About the proof. The key point is Theorem 4.9. It says in particular that H_ce gives a Morita equivalence if and only if e kills no simple object of category $\mathcal{O}$. ☐

Let $\varepsilon : W \to \{\pm 1\}$ be a one-dimensional representation of W. Let $e_\varepsilon = \frac{1}{|W|}\sum_{w\in W}\varepsilon(w)w$. Define $1_\varepsilon : \mathcal{S} \to \mathbf{C},\ s \mapsto \begin{cases} 1 & \text{if } \varepsilon(s) = -1 \\ 0 & \text{otherwise.}\end{cases}$

Proposition 4.15 ([**BerEtGi3**, Proposition 4.11]). *Assume $\mathcal{H}$ is semi-simple. Then, the algebras $e_\varepsilon H_c e_\varepsilon$ and $eH_{c-1_\varepsilon}e$ are isomorphic.*

Theorem 4.16 ([**BerEtGi3**, Theorem 8.1]). *Assume $\mathcal{H}$ is semi-simple. Let $m \in \mathbf{Z}^{\bar{\mathcal{S}}}$. Then, the algebras H_c and H_{c-m} are Morita equivalent.*

Problem 5 ([**BerEtGi3**, Conjecture 8.12]). Assume $\mathcal{H}$ is semi-simple and let $c' \in \mathbf{C}^{\bar{\mathcal{S}}}$. If H_c and $H_{c'}$ are Morita equivalent, show that there is $\zeta : W \to \{\pm 1\}$ a character such that $c\zeta - c' \in \mathbf{Z}^{\bar{\mathcal{S}}}$. Cf Theorem 6.8 for a partial answer in type A.

Analogy 7. Two blocks of category $\mathcal{O}'$ associated to regular weights are equivalent via a translation functor.

5. Representation theory at $t = 0$

5.1. General representations.

5.1.1. *Limit Dunkl operators.* Following [**EtGi**, §4], the construction of §4.2.1 can be done for the algebra $H_{t,c}$, $t \neq 0$, and it then possible to pass to the limit $t = 0$. One obtains an injective algebra morphism

$$H_{0,c} \hookrightarrow \mathbf{C}[V^* \times V_{reg}] \rtimes W,\quad x \mapsto x,\quad \xi \mapsto \xi + \sum_{s\in\mathcal{S}} c_s \frac{\langle \xi, \alpha_s\rangle}{\alpha_s} s,\quad w \mapsto w.$$

5.1.2. Since $H_{0,c}$ is a finitely generated module over its centre $Z(H_{0,c})$, it follows that all simple $H_{0,c}$-modules are finite dimensional. Furthermore, the category of finite dimensional $H_{0,c}$-modules decomposes into a sum of subcategories according to the central character (a point of $\mathcal{CM}_c$).

The smoothness of the Calogero-Moser space is related to representation theory of $H_{0,c}$:

Theorem 5.1 ([**BrGo**, Theorem 7.8], [**EtGi**, Theorems 1.7 and 3.7, and Proposition 3.8], [**GoSm**, Lemma 2.8]). *Let $m \in \mathcal{CM}_c$. The following assertions are equivalent*

- *m is a smooth point of $\mathcal{CM}_c$*
- *the Poisson bracket of $\mathcal{CM}_c$ is non-degenerate at m*
- *there is a unique simple $H_{0,c}$-module with central character m*
- *the simple $H_{0,c}$-modules with central character m have dimension $\geq |W|$*
- *the simple $H_{0,c}$-modules with central character m are isomorphic to the regular representation of W, as $\mathbf{C}[W]$-modules.*

In particular, if $\mathcal{CM}_c$ is smooth, then its points parametrize the (isomorphism classes of) simple $H_{0,c}$-modules.

5.1.3. There is a stratification of $\mathcal{CM}_c$ by *symplectic leaves* [**BrGo**, §3]. Given I a Poisson prime ideal of $B_{0,c}$, the associated symplectic leaf is the set of points $m \in \mathcal{CM}_c$ such that I is a maximal Poisson ideal contained in m. There are only finitely many symplectic leaves [**BrGo**, Theorem 7.8].

The representation theory of $H_{0,c}$ doesn't change inside a symplectic leaf:

THEOREM 5.2 ([**BrGo**, Theorem 4.2]). *Let $m, m' \in \mathcal{CM}_c$ be two points in the same symplectic leaf. Then, $H_{0,c}/H_{0,c}m \simeq H_{0,c}/H_{0,c}m'$.*

One has an irreducibility statement for associated varieties of Poisson ideals:

THEOREM 5.3 ([**Ma**, Corollary 3]). *The associated variety in $(V^* \times V)/W$ of a Poisson prime ideal of $B_{0,c}$ is irreducible.*

5.2. 0-fiber.

5.2.1. Cf [**Go1**].

Let I be the ideal of $H_{0,c}$ generated by $\mathbf{C}[V]^W_+$ and $\mathbf{C}[V^*]^W_+$ and let $\bar{H}_{0,c} = H_{0,c}/I$. The $\bar{H}_{0,c}$-modules are $H_{0,c}$-modules whose central character is in $\Upsilon^{-1}(0)$ and every simple $H_{0,c}$-module with such a central character is a $\bar{H}_{0,c}$-module. The blocks of $\bar{H}_{0,c}$ are given by the central character, *i.e.*, are in bijection with $\Upsilon^{-1}(0)$:

$$\bar{H}_{0,c} = \bigoplus_{m \in \Upsilon^{-1}(0)} \bar{H}_{0,c} b_m,$$

where b_m is the primitive central idempotent of $\bar{H}_{0,c}$ corresponding to m. Let $Z_m = \Gamma(\Upsilon^*(0)_m)$, where $\Upsilon^*(0)$ is the scheme theoretic fiber.

We have a vector space decomposition $\bar{H}_{0,c} = C \otimes \mathbf{C}[W] \otimes C'$, where $C = \mathbf{C}[V^*]/(\mathbf{C}[V^*]\mathbf{C}[V^*]^W_+)$ and $C' = \mathbf{C}[V]/(\mathbf{C}[V]\mathbf{C}[V]^W_+)$ are the coinvariant algebras.

5.2.2. The *Baby-Verma module* associated to $E \in \mathrm{Irr}(W)$ is $M(E) = \mathrm{Ind}_{C \rtimes W}^{\bar{H}_{0,c}} E$. It has a unique simple quotient $L(E)$ and $\{L(E)\}_{E \in \mathrm{Irr}(W)}$ is a complete set of representatives of isomorphism classes of simple $\bar{H}_{0,c}$-modules. Define a map $\mathrm{Irr}(W) \to \Upsilon^{-1}(0)$, $E \mapsto m_E$, by the property that $L(E)$ is in the block $\bar{H}_{0,c} b_{m_E}$.

The blocks corresponding to smooth points can be described precisely:

THEOREM 5.4 ([**Go1**, Corollary 5.8]). *Let $m \in \Upsilon^{-1}(0)$ be a smooth point of $\mathcal{CM}_c$. Then, $\bar{H}_{0,c} b_m \simeq \mathrm{Mat}_{|W|}(Z_m)$.*

If all points in $\Upsilon^{-1}(0)$ are smooth, then the canonical map $\mathrm{Irr}(W) \to \Upsilon^{-1}(0)$ is bijective and $\dim Z_{m_E} = (\dim E)^2$.

PROBLEM 6.
- Is $\bar{H}_{0,c}$ a symmetric algebra ?
- Find the (graded) multiplicities $[M(E) : L(F)]$. In particular, what is the block distribution of the $M(E)$'s ?

REMARK 5.5. It is known ([**EtGi**, Proposition 16.4], [**Go1**, Proposition 7.3]) that $\mathcal{CM}_c$ is singular for all values of c, when W has type different from A_n, B_n and D_{2n+1}. It is conjectured that it will always be singular in type D_{2n+1} ($n \geq 2$).

Let us give an example where $E \mapsto m_E$ is not bijective [**Go1**, §7.4]. Let $W = G_2$ and fix a generic value of c. Given $E \neq F \in \mathrm{Irr}(W)$, then $m_E = m_F$ if and only if E and F are the two distinct 2-dimensional representations.

6. Type A

6.1. Structure.

6.1.1. In this section, let $W = \mathfrak{S}_n$ be the symmetric group on $\{1, \ldots, n\}$ in its permutation representation on $V = \mathbf{C}^n$. We consider the canonical bases $(\xi_1, \ldots, \xi_n)$ of V and $(x_1, \ldots, x_n)$ of V^*. Then, $\mathbf{H}$ is the $\mathbf{C}[\mathbf{t}, \mathbf{c}]$-algebra with generators $\mathfrak{S}_n$, $x_1, \ldots, x_n$ and $\xi_1, \ldots, \xi_n$ and relations
$[\xi_i, \xi_j] = [x_i, x_j] = 0$ for all i, j
$wx_iw^{-1} = x_{w(i)}$, $w\xi_iw^{-1} = \xi_{w(i)}$,
$[\xi_i, x_j] = \mathbf{c} \cdot (ij)$ if $i \neq j$ and $[\xi_i, x_i] = \mathbf{t} - \mathbf{c} \sum_{k \neq i}(ik)$.

One can also consider the action of W on the hyperplane $V' = \ker(x_1 + \cdots + x_n)$. The rational Cherednik algebra of (W, V') is canonically isomorphic to the subalgebra $\mathbf{H}'$ of $\mathbf{H}$ generated by $\xi_i - \xi_j$, $x_i - x_j$ and W, for $1 \leq i, j \leq n$ and we have a decomposition $\mathbf{H} = \mathbf{H}' \otimes \mathbf{H}^1$, where $\mathbf{H}^1$ is generated by $\xi_1 + \cdots + \xi_n$ and $x_1 + \cdots + x_n$, and is isomorphic to the first Weyl algebra.

The algebra H'_c is interesting since it carries finite dimensional non-zero representations for certain values of c, while H_c is the one that relates most directly to Hilbert schemes of points on the plane. Note that the categories $\mathcal{O}$ for H_c and H'_c are canonically equivalent.

6.1.2. The variety $\mathcal{CM}_1$ is isomorphic to the "usual" Calogero-Moser space (a smooth symplectic variety)

$$\{(M, M') \in \operatorname{Mat}_n(\mathbf{C}) \times \operatorname{Mat}_n(\mathbf{C}) | \operatorname{rank}([M, M'] + \operatorname{Id}) = 1\} / \operatorname{GL}_n(\mathbf{C}),$$

where $\operatorname{GL}_n(\mathbf{C})$ acts diagonally by conjugation [**EtGi**, Theorem 11.16].

At the level of points, this isomorphism is constructed as follows [**EtGi**, Theorem 11.16]: let L be a simple representation of $H_{0,1}$. Fix a basis of the n-dimensional space $L^{\mathfrak{S}_{n-1}}$. The actions of ξ_n and x_n on that space give matrices M and M' such that $\operatorname{rank}([M, M'] + \operatorname{Id}) = 1$.

The morphism Υ sends (M, M') to the pair of roots of the characteristic polynomials of M and M'.

Remark 6.1 (Etingof). Let $w \in W - \{1\}$. Then, there is $i \in \{1, \ldots, n\}$ such that $w(i) = j \neq i$. We have $[\xi_i, x_j(ij)w] = [\xi_i, x_j](ij)w = w$, hence $w \in [H_{0,1}, H_{0,1}]$. It follows that the restriction to W of a representation of $H_{0,1}$ is a multiple of the regular representation. This proves the smoothness of $\mathcal{CM}_1$, via Theorem 5.1.

6.2. Category $\mathcal{O}$.

6.2.1. Let $q = \exp(2i\pi c)$. The Hecke algebra $\mathcal{H}$ of $\mathfrak{S}_n$ with parameters $(1, -q)$ is the $\mathbf{C}$-algebra with generators $T_1, \ldots, T_{n-1}$ and relations

$$T_iT_j = T_jT_i \text{ if } |i - j| > 1,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \text{ and } (T_i - 1)(T_i + q) = 0.$$

We use the standard parametrization of $\operatorname{Irr}(W)$ by partitions of n.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ be a partition of n. Let $\mathcal{H}(\lambda)$ be the subalgebra of $\mathcal{H}$ generated by $T_1, \ldots, T_{\lambda_1 - 1}, T_{\lambda_1 + 1}, \ldots, T_{\lambda_1 + \lambda_2 - 1}, \ldots$. It is isomorphic to the tensor product of the Hecke algebras of $\mathfrak{S}_{\lambda_1}, \ldots, \mathfrak{S}_{\lambda_r}$. Given λ a partition of n, let $d(\lambda)$ be the number of r-uples $(\beta_1, \ldots, \beta_r)$ whose associated multiset is that of λ. Let $M(\lambda) = \operatorname{Ind}_{\mathcal{H}(\lambda)}^{\mathcal{H}} \mathbf{C}$, where $\mathbf{C}$ is the one-dimensional representations of $\mathcal{H}$ where the T_i's act as 1 and let $M = \bigoplus_\lambda M(\lambda)^{d_\lambda}$.

The *q-Schur algebra of* $\mathfrak{S}_n$ is $S(n) = \operatorname{End}_{\mathcal{H}}(M)$. Note that $S(n)$-mod is a highest category with parametrizing set the set of partitions of n.

The q-Schur algebra occurs also as a quotient of the quantum general linear group $U_q(\mathfrak{gl}_m)$ for $m \geq n$ (via its action on quantum tensor space $(\mathbf{C}^m)^{\otimes_q n}$) and when q is a prime power, as a quotient of the group algebra of the finite group $\operatorname{GL}_n(\mathbf{F}_q)$.

The category $\mathcal{O}$ is described as follows, as conjectured in [**GGOR**, Remark 5.17]:

Theorem 6.2 ([**Rou**]). *Assume $c \notin \frac{1}{2} + \mathbf{Z}$. Then, there is an equivalence $\mathcal{O} \xrightarrow{\sim} S(n)$-mod making the following diagram commutative*

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{KZ}"'] & & S(n)\text{-mod} \arrow[dl, "M \otimes_{S(n)} -"] \\
& \mathcal{H}\text{-mod} &
\end{tikzcd}$$

and sending $\Delta(\lambda)$ *to the standard object of* $S(n)$-mod *associated to* λ *if* $c \leq 0$ *and to the transposed partition of* λ *if* $c > 0$.

About the proof. The proof proceeds by deformation: the parameter ring becomes a discrete valuation ring and at the generic point the categories are semi-simple. Then, one shows that the image of the Δ-filtered objects under the Schur and KZ-functors is a full subcategory closed under extensions. □

In particular, there is a progenerator P of $\mathcal{O}$ such that $\mathrm{KZ}(P) = M$. Furthermore, the modules $\mathrm{KZ}(\Delta(\lambda))$ are the q-Specht modules.

Assume $c \in \mathbf{Q}$ and let d be the order of c in $\mathbf{Q}/\mathbf{Z}$.

Let Sym be the space of symmetric functions. Given λ a partition of n, let s_λ be the corresponding Schur function.

The Fock space Sym has a natural action of the affine Lie algebra $\hat{\mathfrak{sl}}_d$. There is a lower canonical basis $\{G_\lambda^-\}_{\lambda \text{ a partition}}$ of Sym [**LeThi**]. By [**VarVas2**], the multiplicity $[\Delta_{S(n)}(\lambda) : L_{S(n)}(\mu)]$ is the coefficient of G_μ^- in a decomposition of s_λ in the lower canonical basis (a generalisation of the Lascoux-Leclerc-Thibon conjecture on Hecke algebras, proven by Ariki). So, we deduce the corresponding result for category $\mathcal{O}$:

Corollary 6.3. *Assume* $c \notin \frac{1}{2} + \mathbf{Z}$. *Then,* $[\Delta(\lambda) : L(\mu)]$ *is the coefficient of* G_μ^- *in a decomposition of* s_λ *in the lower canonical basis.*

Remark 6.4. There is a counterpart of Theorem 6.2 in the trigonometric case [**VarVas2**], which builds on an explicit computation of monodromy (this can't be done in the rational case). It might be possible to deduce Theorem 6.2 from the trigonometric case by using [**Su**].

6.2.2. *Finite dimensional representations.* They are completely understood ([**BerEtGi2**, Theorem 1.2], cf also [**Che6**, §7.1]):

Theorem 6.5. *The algebra* H'_c *has non-zero finite dimensional representations if and only if* $c = \pm\frac{r}{n}$ *for some* $r \in \mathbf{Z}_{>0}$ *with* $(r, n) = 1$. *When* c *takes such a value, all finite dimensional representations are semi-simple and the only irreducible representation is* $L(\mathbf{C})$ *when* $c > 0$ *and* $L(\det)$ *when* $c < 0$.

6.3. Shift functors.

6.3.1. We have $\rho(\delta^{-1}e_{\det}H_{c+1}e_{\det}\delta) = \rho(eH_ce)$ [**BerEtGi2**, Proposition 4.1]. So, left and right multiplication make $Q_c^{c+1} = \rho(eH_{c+1}e_{\det}\delta)$ into a (B_{c+1}, B_c)-bimodule. Let $S_c = Q_c^{c+1} \otimes_{B_c} - : B_c\text{-mod} \to B_{c+1}\text{-mod}$ ("Heckmann-Opdam shift functor").

The following result generalizes [**BerEtGi2**, Proposition 4.3].

Theorem 6.6 ([**GoSt1**, Theorem 3.3 and Proposition 3.16]). *If* $c \in \mathbf{R}_{\geq 0}$ *and* $c \notin \frac{1}{2} + \mathbf{Z}$, *then*

- $eH_c \otimes_{H_c} - : H_c\text{-mod} \to B_c\text{-mod}$ *is an equivalence*
- $S_c : B_c\text{-mod} \to B_{c+1}\text{-mod}$ *is an equivalence*
- $M \mapsto H_{c+1}e_{\det}\delta \otimes_{B_c} eM : H_c\text{-mod} \to H_{c+1}\text{-mod}$ *is an equivalence. It restricts to an equivalence* $\mathcal{O}_c \xrightarrow{\sim} \mathcal{O}_{c+1}$ *sending* $\Delta_c(E)$ *to* $\Delta_{c+1}(E)$.

About the proof. The key point is to show that $H_{c+1}e_{\det}H_{c+1} = H_{c+1}$ and $H_ceH_c = H_c$ for $c \geq 0$. Let us consider the first equality, the second one has a similar proof. If the equality fails, then $e_{\det}$ will kill some simple object of $\mathcal{O}$ by Theorem 4.9. One uses the canonical $\mathbf{C}$-grading on $\Delta(\lambda)$, λ a partition. One shows that the lowest weight where the representation det of W appears in $\Delta(\lambda)$ is strictly larger than the lowest weight where it appears in $\Delta(\mu)$ whenever $\lambda < \mu$ (this is where the assumption $c \geq 0$ enters). As a consequence, det occurs in $L(\mu)$, for any μ, hence $e_{\det}L(\mu) \neq 0$ and we deduce the first equality. Note that the assumption $c \notin \frac{1}{2} + \mathbf{Z}$ comes from the use of Theorem 4.6. To check that $\Delta_c(E)$ goes to $\Delta_{c+1}(E)$, one proves this after localizing to V_{reg} and then show that the equivalence $\mathcal{O}_c \xrightarrow{\sim} \mathcal{O}_{c+1}$ must preserve the highest weight structure. □

Remark 6.7. Note that, by Theorem 4.14 and §3.1.4, we deduce that H_c and B_c are Morita equivalent for every $c \in \mathbf{C}$ satisfying the conditions

$$c \notin \frac{1}{2} + \mathbf{Z} \text{ and } c \notin \{-\frac{m}{d} | m, d \in \mathbf{Z},\ 2 \leq d \leq n,\ 0 < m < d\}.$$

6.3.2. There is a Morita equivalence classification of the algebras H_c, when c is not algebraic.

Theorem 6.8 ([**BerEtGi1**, Theorem 2]). *Let $c \notin \bar{\mathbf{Q}}$ and $c' \in \mathbf{C}$. The algebras H_c and $H_{c'}$ are*

- *isomorphic if and only if $c' = \pm c$*
- *Morita equivalent if and only if $c \pm c' \in \mathbf{Z}$.*

About the proof. The criterion is obtained by computing the traces $K_0(H_c) \to HH_0(H_c)$. □

6.4. Hilbert schemes. The existence of a link between Hilbert schemes of points on $\mathbf{C}^2$ and rational Cherednik algebras of type A was pointed out in [**EtGi**] and [**BerEtGi2**, §7.2]. We describe here some of the constructions and results of [**GoSt1, GoSt2**].

6.4.1. *Quantization.* Cf [**GoSt1**, §4–6].

Let $\mathrm{Hilb}^n \mathbf{C}^2$ be the Hilbert scheme of n points in $\mathbf{C}^2$. Let X_n be the reduced scheme of $\mathrm{Hilb}^n \mathbf{C}^2 \times_{S^n \mathbf{C}^2} \mathbf{C}^{2n}$ (the isospectral Hilbert scheme). Following Haiman, we have a diagram

(2)
$$\begin{tikzcd}
X_n \arrow[r, "f"] \arrow[d, "\text{flat}"', "p"] & \mathbf{C}^{2n} = V^* \times V \arrow[d] \\
\mathrm{Hilb}^n \mathbf{C}^2 \arrow[r, "\tau", "\text{resolution}"'] & S^n \mathbf{C}^2 = (V^* \times V)/W
\end{tikzcd}$$

Denote by $\mathcal{Z}_n = \tau^{-1}(0)$ the punctual Hilbert scheme.

Fix $c \in \mathbf{R}_{\geq 0}$, $c \notin \frac{1}{2} + \mathbf{Z}$.

Given $i > j \in \mathbf{Z}_{\geq 0}$, let $\mathcal{B}^{jj} = B_{c+j}$ and $\mathcal{B}^{ij} = Q^{c+i}_{c+i-1} \otimes_{B_{c+i-1}} Q^{c+i-1}_{c+i-2} \otimes \cdots \otimes Q^{c+j+1}_{c+j}$. Let $\mathcal{B} = \bigoplus_{i,j \geq 0} \mathcal{B}^{ij}$. This is a non-unital algebra. We denote by 1_i the unit of $\mathcal{B}^{ii}$. We denote by $\mathcal{B}$-mod the category of finitely generated $\mathcal{B}$-modules M such that $M = \bigoplus_{i \geq 0} 1_i M$. We denote by $\mathcal{B}$-qmod the abelian category quotient of $\mathcal{B}$-mod by the Serre subcategory of objects M such that $1_i M = 0$ for $i \gg 0$.

The filtration by the order of differential operators on $D(V_{reg}) \rtimes W$ induces a filtration on $\mathcal{B}$ and we denote by $\operatorname{ogr} \mathcal{B} = \bigoplus_{i \geq j \geq 0} \operatorname{ogr} \mathcal{B}^{ij}$ the associated graded (non-unital) algebra. We define a category $(\operatorname{ogr} \mathcal{B})$-qmod as above.

Theorem 6.9 ([**GoSt1**, Theorem 6.4]). *There are equivalences*

$$B_c\text{-mod} \xrightarrow{\sim} \mathcal{B}\text{-qmod}$$

$$\mathrm{Hilb}^n \mathbf{C}^2\text{-coh} \xrightarrow{\sim} (\operatorname{ogr} \mathcal{B})\text{-qmod}.$$

About the proof. The first equivalence is an immediate consequence of the fact that the $\mathcal{B}^{ij}$'s induce Morita equivalences (Theorem 6.6).

Consider $A^1 = \mathbf{C}[\mathbf{C}^{2n}]^{\det}$, the det-isotypic part for the action of W on $\mathbf{C}[\mathbf{C}^{2n}]$ and let A'^1 be the ideal of $\mathbf{C}[\mathbf{C}^{2n}]$ generated by A^1. Let $A^d = (A^1)^d$ and $A'^d = (A'^1)^d$ (inside $\mathbf{C}[\mathbf{C}^{2n}]$) for $d \geq 1$ and let $A^0 = \mathbf{C}[\mathbf{C}^{2n}]^{\mathfrak{S}_n}$ and $A'^0 = \mathbf{C}[\mathbf{C}^{2n}]$. Let $A = \bigoplus_{d \geq 0} A^d$ and $A' = \bigoplus_{d \geq 0} A'^d$.

Then, there are isomorphisms $X_n \xrightarrow{\sim} \operatorname{Proj} A'$ and $\operatorname{Hilb}^n \mathbf{C}^2 \xrightarrow{\sim} \operatorname{Proj} A$ so that the diagram (2) above becomes the following diagram, with canonical maps (Haiman)

$$\begin{tikzcd}
\operatorname{Proj} \bigoplus_{d\geq 0} A'^d \arrow[r] \arrow[d] & \operatorname{Spec} A'^0 \arrow[d] \\
\operatorname{Proj} \bigoplus_{d\geq 0} A_d \arrow[r] & \operatorname{Spec} A_0
\end{tikzcd}$$

Now, the Theorem follows from the equalities $\operatorname{ogr} \mathcal{B}^{ij} = eA^{i-j}\delta^{i-j}e$ between subspaces of $\mathbf{C}[\mathbf{C}^{2n}]^{\mathfrak{S}_n}$, whose delicate proof involves understanding the graded structure of these two subspaces. ☐

6.4.2. *Localization.* Cf [**GoSt2**].

Consider the order filtration on H_c : $\operatorname{ord}^0 H_c = \mathbf{C}[V] \rtimes W$, $\operatorname{ord}^1 H_c = V \cdot \operatorname{ord}^0 H_c + \operatorname{ord}^0 H_c$ and $\operatorname{ord}^i H_c = (\operatorname{ord}^1 H_c)^i$ for $i \geq 2$. This induces a filtration on B_c.

Theorem 6.9 gives a functor Φ_c from the category B_c-filt of B_c-modules with a good filtration (for the order filtration) to the category $\operatorname{Hilb}^n \mathbf{C}^2$-coh of coherent sheaves over $\operatorname{Hilb}^n \mathbf{C}^2$.

We have

$$\Phi_c(B_c) \simeq \mathcal{O}_{\operatorname{Hilb}^n \mathbf{C}^2}$$

The image of eH_c with the order filtration is the Procesi bundle [**GoSt2**, Theorem 4.5]:

$$\Phi_c(eH_c) \simeq p_*\mathcal{O}_{X_n}.$$

We have [**GoSt2**, Proposition 5.4] (this relates to [**BerEtGi2**, Conjectures 7.2 and 7.3]):

$$\Phi_{1/n}(L_{1/n}(\mathbf{C})) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Z}_n}.$$

Given $M \in B_c$-filt, there is an induced tensor product filtration on $S_c(M) = Q_c^{c+1} \otimes_{B_c} M$. Let $\mathcal{L}$ be the determinant of the universal rank n vector bundle on $\operatorname{Hilb}^n \mathbf{C}^2$ (an ample line bundle). The geometric importance of S_c is provided by the next result, which explains the constructions of Gordon-Stafford.

Theorem 6.10 ([**GoSt2**, Lemma 4.4]). *There is an isomorphism of functors* B_c-filt $\to \operatorname{Hilb}^n \mathbf{C}^2$-coh*:*

$$\Phi_{c+1} \circ S_c(-) \xrightarrow{\sim} \mathcal{L} \otimes \Phi_c(-).$$

Problem 7 ([**GoSt2**, Question 1]). Let M be an H_c-module with a good filtration. Then, $\operatorname{gr} M$ is a finitely generated $(\mathbf{C}[V^* \times V] \rtimes W)$-module. Let $\tilde{\Phi}(M) \in D^b(\operatorname{Hilb}^n \mathbf{C}^2\text{-coh})$ be its image under the equivalence of derived categories (Bridgeland-King-Reid, Haiman):

$$(p_*Lf^*(-))^W : D^b((\mathbf{C}[V^* \times V] \rtimes W)\text{-mod}) \xrightarrow{\sim} D^b(\operatorname{Hilb}^n \mathbf{C}^2\text{-coh}).$$

Is there an isomorphism $\tilde{\Phi}(M) \xrightarrow{\sim} \Phi(eM)$? Gordon and Stafford construct a morphism which is surjective on $\mathcal{H}^0$. A related question is to understand which $(\mathbf{C}[V^* \times V] \rtimes W)$-modules can be quantized to H_c-modules for some value of $c \in \mathbf{R}_{\geq 0}$.

Remark 6.11. Gordon and Stafford actually work with H'_c and they consider the W-Hilbert scheme $\operatorname{Hilb}(n)$ of $(V')^* \times V'$. There is an isomorphism $\operatorname{Hilb}^n \mathbf{C}^2 \xrightarrow{\sim} \operatorname{Hilb}(n) \times \mathbf{C}^2$, so the geometric properties of $\operatorname{Hilb}(n)$ and $\operatorname{Hilb}^n \mathbf{C}^2$ are easily related [**GoSt1**, §4.9].

6.4.3. *Characteristic cycles.* Cf [**GoSt2**, §6].

Let $Z = Z(n) = \tau^{-1}(\{0\} \times V/W)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ be a partition of n and $S^\lambda \mathbf{C}^2$ be the subvariety of $S^n \mathbf{C}^2$ of 0-cycles of $\mathbf{C}^2$ of the form $\sum_i \lambda_i x_i$, where $x_1, \ldots, x_r \in \mathbf{C}^2$ are distinct.

Let Z_λ be the closure of $Z \cap \tau^{-1}(S^\lambda \mathbf{C}^2)$. This is a Lagrangian subvariety of $\operatorname{Hilb}^n \mathbf{C}^2$ and the Z_λ's, where λ runs over the partitions of n, are the irreducible components of Z (Grojnowski, Nakajima).

Given λ a partition of n, let $m_\lambda \in \mathrm{Sym}$ be the corresponding monomial symmetric function. There is an isomorphism (Grojnowski, Nakajima)

$$\xi : \bigoplus_{n \geq 0} H_n(Z(n)) \xrightarrow{\sim} \mathrm{Sym}, \quad [Z_\lambda] \mapsto m_\lambda$$

where H_n is the Borel-Moore homology with complex coefficients.

Recall that the cycle support of a coherent sheaf $\mathcal{F}$ is the cycle $\sum_i n_i [Z_i]$, where Z_i runs over irreducible components of the support of $\mathcal{F}$ and n_i is the dimension of $\mathcal{F}$ at the generic point of Z_i.

Let $M \in \mathcal{O}_c$. Fix a good filtration of eM and consider the part of the cycle support of $\Phi(eM)$ involving only subvarieties of dimension n. This is independent of the choice of the good filtration and this gives an isomorphism [**GoSt2**, Corollary 6.10]

$$\gamma : K(\mathcal{O}_c) \otimes_{\mathbf{Z}} \mathbf{C} \xrightarrow{\sim} H_n(Z)$$

The following Theorem describes the characteristic cycle of $\Delta(\mu)$.

Theorem 6.12 ([**GoSt2**, Theorem 6.7]). *Let μ be a partition of n. The support of $\Phi(e\Delta(\mu))$ is a union of Z_λ's. We have*

$$\xi\gamma([\Delta(\mu)]) = s_\mu.$$

Problem 8 ([**GoSt2**, Question 4.9 and §6.8]). From Corollary 6.3, one deduces that $\xi\gamma([L(\mu)])$ is the lower canonical basis element of the Fock space corresponding to μ. Are the irreducible components of the support of $\Phi(eL(\mu))$ all of dimension n ? If so, this would completely describe the characteristic cycle of $L(\mu)$.

Problem 9 ([**GoSt2**, Problem 7.7]). Gordon and Stafford show [**GoSt2**, Lemma 7.7] that the top Borel-Moore homology of $\operatorname{Hilb}^n \mathbf{C}^2 \times_{S^n \mathbf{C}^2} \operatorname{Hilb}^n \mathbf{C}^2$ with the convolution product is isomorphic to the representation ring of W. The determination of the $(\mathbf{C}^\times)^2$-equivariant K-theory ring under convolution remains to be done.

Remark 6.13. There is a different geometric approach started in [**GanGi**], which also leads to the construction of characteristic cycles on the Hilbert scheme.

6.4.4. Let us summarize

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, dotted, leftrightarrow, "\sim"] \arrow[d, dotted, "\text{quantization}"'] & & B_c \arrow[d, dotted, "\text{quantization}"] \\
\operatorname{Hilb}^n \mathbf{C}^2 \arrow[rr, "\text{resolution}"'] & & (V^* \times V)/W
\end{tikzcd}$$

The algebra H_c is a "quantization" of the orbifold $[(V^* \times V)/W]$. Furthermore, X_n, viewed as a scheme over $\operatorname{Hilb}^n \mathbf{C}^2$ and over $[(V^* \times V)/W]$, becomes after "quantization" the (B_c, H_c)-bimodule eH_c.

The transform with kernel $\mathcal{O}_{X_n}$ gives an equivalence of triangulated categories ("McKay correspondence") $D^b(\operatorname{Hilb}^n \mathbf{C}^2\text{-coh}) \xrightarrow{\sim} D^b([(V^* \times V)/W]\text{-coh})$. This is simpler in the non-commutative case, where eH_c gives an equivalence of abelian categories $B_c\text{-mod} \xrightarrow{\sim} U_c\text{-mod}$ (for suitable c's).

Analogy 8. Let $\mathcal{B} = G/B$ be the flag variety of G. Given $w \in W$, we put $\mathcal{B}_w = BwB/B$. We have the Springer resolution of singularities $T^*\mathcal{B} \to \mathcal{N}$. We have a canonical isomorphism $\bar{U}_0(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{D}_\mathcal{B})$ and we have $\Gamma(\mathcal{B},\mathcal{D}_{\mathcal{B}})$-mod $\simeq \mathcal{D}_{\mathcal{B}}$-coh (Beilinson-Bernstein). We have $\mathrm{gr}\mathcal{D}_{\mathcal{B}} \xrightarrow{\sim} \mathcal{O}_{T^*\mathcal{B}}$.

$$\begin{tikzcd}
\mathcal{D}_{\mathcal{B}} \arrow[r, leftrightarrow, dotted, "\sim"] \arrow[d, dotted, "\text{quantization}"'] & \bar{U}_0(\mathfrak{g}) \arrow[d, dotted, "\text{quantization}"] \\
T^*\mathcal{B} \arrow[r, "\text{resolution}"'] & \mathcal{N}
\end{tikzcd}$$

Given an object M of $\mathcal{O}'$, we can consider the characteristic cycle of the corresponding $\mathcal{D}_{\mathcal{B}}$-module, an element of $\bigoplus_{w\in W}\mathbf{Z}[\overline{T^*_{\mathcal{B}_w}\mathcal{B}}]$.

There is a canonical isomorphism of algebras between the top homology of the Steinberg variety $Z = T^*\mathcal{B}\times_{\mathcal{N}} T^*\mathcal{B}$ and $\mathbf{C}W$ (Kazhdan-Lusztig). The images of the components of Z give a basis $(b_w)_{w\in W}$ of $\mathbf{C}W$. Define $(\beta_{w,w'})_{w,w'}$ by $w=\sum_{w'}\beta_{w,w'}b_{w'}$. Then, the characteristic cycle of $\Delta(w)$ is $\sum_{w'}\beta_{w,w'}[\overline{T^*_{\mathcal{B}_{w'}}\mathcal{B}}]$ (Kashiwara-Tanisaki).

The $(G\times\mathbf{C}^\times)$-equivariant K-theory of Z is isomorphic to the affine Hecke algebra of type W (Kazhdan-Lusztig, Chriss-Ginzburg).

Problem 10. Can one deform the category of W-equivariant mixed Hodge modules on V ?

7. Type A_1

7.1. Presentation. Take $W = A_1 = \langle s\rangle$ acting on $V=\mathbf{C}\xi$ and put $V^*=\mathbf{C}x$ where $\langle\xi,x\rangle = 1$. Then, $\mathbf{H}$ is the $\mathbf{C}[\mathbf{t},\mathbf{c}]$-algebra with generators s,x,ξ and relations

$$s^2=1,\ sxs=-s,\ s\xi s=-\xi,\ [\xi,x]=\mathbf{t}-2\mathbf{c}s.$$

We have $e=\frac{1}{2}(1+s)$.

Fix $t,c\in\mathbf{C}$. Recall that $H_{t,c}\simeq H_{1,t^{-1}c}$ if $t\neq 0$ and $H_{0,c}\simeq H_{0,1}$ if $c\neq 0$. So, there are three types of algebras in the family : $H_{1,c}$, $H_{0,1}$ and $H_{0,0}$.

7.2. Category $\mathcal{O}$ and KZ.

7.2.1. We take $t=1$.

We identify $\Delta(\mathbf{C})$ with $\mathbf{C}[x]$. The action of the generators is given by

$$\begin{aligned}
x :& x^i \mapsto x^{i+1}\\
s :& x^i \mapsto (-1)^i x^i\\
\xi :& x^i \mapsto \begin{cases} ix^{i-1} \text{ if } i \text{ is even}\\ (i-2c)x^{i-1} \text{ if } i \text{ is odd}\end{cases}
\end{aligned}$$

In particular, $\Delta(\mathbf{C})=L(\mathbf{C})$ if and only if $c\not\in\frac{1}{2}+\mathbf{Z}_{\geq 0}$. If $c=\frac{1}{2}+n$ with $n\geq 0$, then $x^{2n+1}\mathbf{C}[x]$ is the radical of $\Delta(\mathbf{C})$ and $L(\mathbf{C})=\mathbf{C}[x]/(x^{2n+1})$.

The Dunkl operator is $T_\xi=\frac{d}{dx}+\frac{c}{x}(s-1)$. The connection on the trivial line bundle over V_{reg} is $\frac{d}{dx}$. The solutions are constant functions, the monodromy operator is trivial.

7.2.2. We identify $\Delta(\det)$ with $\mathbf{C}[x]$. The action of the generators is given by

$$\begin{aligned}
x :& x^i \mapsto x^{i+1}\\
s :& x^i \mapsto (-1)^{i+1} x^i\\
\xi :& x^i \mapsto \begin{cases} ix^{i-1} \text{ if } i \text{ is even}\\ (i+2c)x^{i-1} \text{ if } i \text{ is odd}\end{cases}
\end{aligned}$$

In particular, $\Delta(\det)=L(\det)$ if and only if $c\not\in-(\frac{1}{2}+\mathbf{Z}_{\geq 0})$. If $c=-(\frac{1}{2}+n)$ with $n\geq 0$, then $x^{2n+1}\mathbf{C}[x]$ is the radical of $\Delta(\det)$ and $L(\det)=\mathbf{C}[x]/(x^{2n+1})$.

The connection on the trivial line bundle over V is $\frac{d}{dx}+\frac{2c}{x}$. In a neighborhood of $x=1$, we have the solution $f=x^{-2c}$ with $f(1)=1$. Analytical continuation in a tubular neighborhood of the path $t\in[0,1]\mapsto\exp(i\pi t)$ gives a function with value $\exp(-2i\pi c)$ at -1. The action of the monodromy operator is $-\exp(-2i\pi c)$. The KZ-construction involves the de Rham functor while here we considered the solution functors (=horizontal sections). We pass from one to the other by dualizing the representation of the fundamental group. So, the generator σ_s of the braid group acts now by $-\exp(2i\pi c)$ on $\mathrm{KZ}(\Delta(\det))$.

7.2.3. When $c\notin\frac{1}{2}+\mathbf{Z}$, then $\mathcal{O}$ is semi-simple.

When $c\in-(\frac{1}{2}+\mathbf{Z}_{\geq 0})$, then we have an exact sequence $0\to\Delta(\mathbf{C})\to\Delta(\det)\to L(\det)\to 0$. We have $P(\det)=\Delta(\det)$ and we have an exact sequence $0\to\Delta(\det)\to P(\mathbf{C})\to\Delta(\mathbf{C})\to 0$.

When $c\in\frac{1}{2}+\mathbf{Z}_{\geq 0}$, then we have an exact sequence $0\to\Delta(\det)\to\Delta(\mathbf{C})\to L(\mathbf{C})\to 0$. We have $P(\mathbf{C})=\Delta(\mathbf{C})$ and we have an exact sequence $0\to\Delta(\mathbf{C})\to P(\det)\to\Delta(\det)\to 0$.

So, when $c\in\frac{1}{2}+\mathbf{Z}$, then $\mathcal{O}$ is equivalent to the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_2(\mathbf{C})$ (this is no miracle, cf §7.3.3).

7.2.4. We have $ex+xe=2x$, $e\xi+\xi e=2\xi$, so $\bigoplus_{i,j}\mathbf{C}x^i\xi^j\subset HeH$. Since $[\xi,x]=t-2cs$, we have $t-2cs\in HeH$. So, $2c+t\in HeH$. It follows that $HeH=H$ if $2c+t\neq 0$.

Assume now $2c=-t$. We put a structure of H-module on $L=\mathbf{C}$ by letting x and ξ act by 0 and s by -1. Then, e acts by 0, so HeH annihilates L, hence $HeH\neq H$. So, we have $HeH=H$ if and only if $2c\neq -t$. Note that when $c=-\frac{1}{2}$ and $t=1$, via the identification $\Delta(\det)=\mathbf{C}[x]$, then $L\simeq L(\det)=\mathbf{C}[x]/x\mathbf{C}[x]$. Note also that the algebra $B_{1,-\frac{1}{2}}$ is simple.

Finally, the following assertions are equivalent:

- $H_{t,c}eH_{t,c}=H_{t,c}$
- $H_{t,c}$ and $B_{t,c}$ are Morita equivalent
- $c\neq-\frac{1}{2}t$.

7.3. Spherical subalgebra.

7.3.1. One has $ex^i\xi^je=\begin{cases}ex^i\xi^j & \text{if } i+j \text{ is even}\\ 0 & \text{otherwise}\end{cases}$ and $ex^i\xi^je=\pm ex^is\xi^je$. So, $\mathbf{B}$ has a basis $(ex^i\xi^j)_{i,j\geq 0,i+j\text{ even}}$. It is generated as a $\mathbf{C}[\mathbf{t},\mathbf{c}]$-algebra by $u=\frac{1}{2}ex^2e$, $v=-\frac{1}{2}e\xi^2e$ and $w=ex\xi e$. We obtain an isomorphism between $\mathbf{B}$ and the $\mathbf{C}[\mathbf{t},\mathbf{c}]$-algebra generated by u, v and w with the relations $w(w-\mathbf{t}-2\mathbf{c})=-4uv$ and $[u,v]=\mathbf{t}w+\mathbf{t}(\frac{1}{2}\mathbf{t}-\mathbf{c})$.

7.3.2. We have $\mathbf{C}[u,v,w]/(w(w-2c)+4uv)\xrightarrow{\sim}B_{0,c}$. The variety $\operatorname{Spec}B_{0,c}$ is smooth if and only if $c\neq 0$.

7.3.3. For $t=1$, we have $[u,v]=w+\frac{1}{2}-c$, $[v,w]=2v$ and $[u,w]=-2u$. Let e_+, e_- and h be the standard generators of $\mathfrak{sl}_2(\mathbf{C})$ and let $C=e_+e_-+e_-e_++\frac{1}{2}h^2$ be the Casimir element. We have an isomorphism [**EtGi**, Proposition 8.2]

$$U(\mathfrak{sl}_2)/\langle C-\frac{1}{2}(c-\frac{1}{2})(c+\frac{3}{2})\rangle\xrightarrow{\sim}B_{1,c},\quad e_+\mapsto u,\quad e_-\mapsto v.$$

We have $B_{1,c}$ Morita equivalent to $B_{1,c+1}$ if and only if $c\neq-\frac{3}{2},-\frac{1}{2}$.

7.3.4. The representation theory of $U(\mathfrak{sl}_2)$ is related to $T^*\mathbf{P}^1$, since the flag variety of $\mathfrak{sl}_2$ is a projective line. Note that $\operatorname{Hilb}^W(V^*\times V)\simeq T^*\mathbf{P}^1$, the minimal resolution of singularities of the cone $\operatorname{Spec}B_{0,0}$.

7.4. Double affine Hecke algebra. We finish with some words on the daha and its relation with the rational daha.

7.4.1. The double affine Hecke algebra $\mathbf{H}^{ell}$ is the $\mathbf{C}[\tau^{\pm 1},q^{\pm 1}]$-algebra with generators $X^{\pm 1},Y^{\pm 1},T$ and relations

$$(T-\tau)(T+\tau^{-1})=0,\ TXT=X^{-1},\ TY^{-1}T=Y\text{ and }Y^{-1}X^{-1}YXT^2=q. \tag{3}$$

There is a triangular decomposition $\mathbf{H}^{ell}=\mathbf{C}[X^{\pm 1}]\otimes\mathcal{H}\otimes\mathbf{C}[Y^{\pm 1}]$ [**Che7**, §1.4.2]. Let us consider the $\mathbf{H}^{ell}$-module $\operatorname{Ind}_{\mathbf{C}[Y^{\pm 1}]\otimes\mathcal{H}}^{\mathbf{H}^{ell}}\mathbf{C}$, where Y and T act on $\mathbf{C}$ by multiplication by τ [**Che7**, Proof of Lemma 1.4.5].

We identify this module with $\mathbf{C}[X^{\pm 1}]$. This gives a faithful representation of $\mathbf{H}^{ell}$. The action of $\mathbf{H}^{ell}$ is given by

$$\begin{aligned} X : X^i &\mapsto X^{i+1} \\ T : X^i &\mapsto X^{-i}T + (\tau^{-1} - \tau)(X^{i-2} + X^{i-4} + \cdots + X^{-i}) \\ YT^{-1} : X^i &\mapsto q^{-i}X^{-i} \end{aligned}$$

So, T acts by $\tau s + \frac{\tau - \tau^{-1}}{X^2 - 1}(s-1)$ and Y acts by spT, where $p(f)(X) = f(q^{-1}X)$.

7.4.2. We follow [**EtGi**, Proposition 4.10].

Fix $c \in \mathbf{C}$. We consider the ring $\mathbf{C}[[h]]$ with its h-adic topology. Let $\hat{H}^{ell}$ be the $\mathbf{C}[[h]]$-algebra topologically generated by three elements s, x and y with relations (3) for $X = e^{hx}$, $Y = e^{hy}$, $T = se^{h^2 cs}$, $q = e^{h^2}$ and $\tau = e^{h^2 c}$.

The first relation, taken at order 0, gives $(s-1)(s+1) = 0$. The second and third relations, taken at order 1, give $sxs = -x$ and $sys = -y$. Finally, the last relation, taken at order 2, gives $yx - xy = 1 - 2cs$. This gives rise to an isomorphism of $\mathbf{C}$-algebras $H \xrightarrow{\sim} \hat{H}^{ell} \otimes_{\mathbf{C}[[h]]} \mathbf{C}[[h]]/(h)$. Note finally that $\hat{H}^{ell}$ is actually a trivial deformation of H, *i.e.*, there is an isomorphism of topological $\mathbf{C}[[h]]$-algebras $H \hat{\otimes} \mathbf{C}[[h]] \xrightarrow{\sim} \hat{H}^{ell}$ [**Che6**, p.65].

8. Generalizations

8.1. Complex and symplectic reflection groups.

8.1.1. The definition of the rational Cherednik algebra generalizes to the case where W is a complex reflection group on V and more generally, when W is a symplectic reflection group on a space L (which is $V \oplus V^*$ in the complex reflection case). Theorem 3.1 remains valid in that setting. The construction and the main results on category $\mathcal{O}$ generalize to complex reflection groups.

Such deformations have been introduced and studied before in [**CrBoHo**] in the case of a symplectic reflection group acting on a symplectic space of dimension 2.

These constructions of symplectic reflection algebras are special cases of a more general construction [**Dr**]. An even more general construction is given in [**EtGanGi**], where finite groups are replaced by reductive groups.

Another direction of generalization is globalization, where V is replaced by an algebraic variety acted on by a finite group [**Et**].

8.1.2. Many results of §6 should generalize to the complex reflection groups $B_n(d) \simeq \mathbf{Z}/d \wr \mathfrak{S}_n$ (cf [**Mu**] for a generalization of some of the constructions of §6.4). Some geometric aspects (should) generalize even to $\Gamma \wr \mathfrak{S}_n$, where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$. For example, the Hilbert scheme to consider is $\mathrm{Hilb}^n(\mathrm{Hilb}^\Gamma \mathbf{C}^2)$. We refer to [**Wa**] for a survey on Hilbert schemes and wreath products. The description of multiplicities for $B_n(d)$ should generalize Corollary 6.3 using suitable canonical bases of higher level Fock spaces [**Rou**].

Some finite dimensional representations have been constructed in the case $\Gamma \wr \mathfrak{S}_n$. Those where the x_i's and ξ_i's act by zero have been classified in [**Mo1**]. More general finite dimensional simple representations have be shown to exist by cohomological methods [**EtMo, Mo2**]. These representations come from ones where the x_i's and the ξ_i's act by zero via reflection functors and every simple finite dimensional representation for parameters "close to 0" is of this form [**Gan**].

8.1.3. Assume $W = \Gamma \wr \mathfrak{S}_n$ acting on $\mathbf{C}^{2n}$, for some finite subgroup Γ of $\mathrm{SL}_2(\mathbf{C})$,

There is a crepant (=symplectic) resolution $X_c \to \mathcal{CM}_c$ and an equivalence [**GoSm**, Theorem 1.2]

$$D^b(H_{0,c}\text{-mod}) \xrightarrow{\sim} D^b(X_c\text{-coh})$$

The variety X_c is constructed as a moduli space of representations of H_c and the kernel of the equivalence is the universal bundle. The case $c = 0$ is the McKay correspondence.

The algebra $H_{0,c}$ is actually a non-commutative crepant resolution of $\mathcal{CM}_c$, in the sense of Van den Bergh [**GoSm**, Lemma 3.10].

Note that $\mathcal{CM}_c$ is smooth for generic values of c [**EtGi**, Proposition 11.11] and it has then a description generalizing that of §6.1.2 [**EtGi**, Theorem 11.16].

8.2. Characteristic $p > 0$. The rational Cherednik algebra can be defined over $\mathbf{Z}$, and in particular over an algebraic closure $\bar{\mathbf{F}}_p$ of the field with p elements. The representation theory in characteristic $p > 0$ is quite different from that in characteristic 0, due to the presence of a big center: we have $\mathrm{gr}Z(H_c) \xrightarrow{\sim} (\mathbf{C}[V^* \times V]^p)^W$ when $p > n$ (Etingof, cf [**BezFiGi**, Theorem 10.1.1]). In particular, H_c is a finite dimensional module over its center, hence all its simple representations are finite-dimensional.

There is a localization result in characteristic p.

THEOREM 8.1 ([**BezFiGi**, Theorem 7.3.2]). *Assume $c \leq 0$ and $c \notin \frac{1}{2} + \mathbf{Z}$. Assume p is large enough. Then, there is a sheaf $\mathcal{F}_c$ of Azumaya algebras over the Frobenius twist $\mathrm{Hilb}^{(1)}$ of $\mathrm{Hilb}^n \mathbf{A}^2$ and an equivalence*

$$D^b(\mathcal{F}_c\text{-mod}) \xrightarrow{\sim} D^b(H_c\text{-mod}).$$

This comes from an isomorphism $H^0(\mathrm{Hilb}^{(1)}, \mathcal{F}_c) \xrightarrow{\sim} H_c$ and from the vanishing $H^{>0}(\mathrm{Hilb}^{(1)}, \mathcal{F}_c) = 0$.

Cf [**La**] for the determination of irreducible representations of Cherednik algebras associated to the rank 1 groups $W = \mathbf{Z}/d$, over a field of characteristic $p \not| d$, making explicit results of [**CrBoHo**].

9. Table of analogies

$\mathbf{H}$ or $\mathbf{B}$	$U(\mathfrak{g})$
$H_{t,c} = S(V) \otimes \mathbf{C}[W] \otimes S(V^*)$	$U(\mathfrak{g}) = U(\mathfrak{n}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^-)$
$\mathbf{C}^{\bar{\mathcal{S}}}$	$\mathfrak{h}^*/W$
$H_{1,c}$ or $B_{1,c}$	$U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\lambda$
$(V^* \times V)/W$	$\mathcal{N}$
parabolic subgroups of W	nilpotent classes
$\mathrm{Irr}(W)$	W
$\mathcal{H}$	$S(\mathfrak{h}^*)/S(\mathfrak{h}^*)S(\mathfrak{h}^*)_+^W \simeq H^*(G/B)$
?	dominant weights
$\mathrm{Hilb}^n \mathbf{C}^2$ (type A_{n-1})	T^*G/B

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