Centers and Simple Modules for Iwahori-Hecke Algebras

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1. Introduction

The work of Dipper and James on Iwahori-Hecke algebras associated with the finite Weyl groups of type A_n has shown that these algebras behave in many ways like group algebras of finite groups. Moreover, there are "generic" features in the modular representation theory of these algebras which, at present, can only be verified in examples by explicit computations. This paper arose from an attempt to provide a conceptual explanation of these phenomena, in the general framework of the representation theory of (symmetric) algebras. We will study relations between the center of such algebras and properties of decomposition maps, and we will use this to obtain a general result about the "genericity" of the number of simple modules of Iwahori-Hecke algebras.

Usually, the formalism of decomposition maps is developed for algebras over a complete discrete valuation ring. However, in our applications to Iwahori-Hecke algebras, we have to make sure that this also works over the ring of Laurent polynomials in one indeterminate over the integers. Roughly speaking, this will be achieved by using the theory of Henselian rings (see [Ray]). In Section 2, we describe such a general setting for decomposition maps of algebras over integrally closed ground rings (see Proposition 2.11). Furthermore, we extend the standard results on the "Brauer-Cartan triangle" to the case of orders in non-semisimple and non-split algebras, by using enlargements of the usual Grothendieck groups. As a formal consequence of the definition, we get a factorization property of decomposition maps (see Proposition 2.12). Previously, this factorization was only established using strong additional assumptions on the realizability of representations (cf. [Ge1], (2.4), (5.3)).

Let H be an algebra over a local integrally closed domain $\mathcal{O}$ with residue field k. Then we have a canonical map from central functions on

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H to central functions on kH (induced by reduction modulo the maximal ideal of $\mathcal{O}$). In Proposition 3.1, generalizing a theorem of Hattori, we show that the surjectivity of this map implies that the decomposition map has finite cokernel and that the Cartan matrix of kH has non-zero determinant. For a symmetric algebra, this surjectivity is equivalent to the surjectivity of the reduction map $Z(H) \to Z(kH)$. In Theorem 5.2 we prove that this surjectivity holds for Iwahori-Hecke algebras, by constructing a basis of the center from [Ge-Pf]. This part is inspired by the work of [Di-Ja], where the type A_n was considered. We believe that this stability of the center under reduction is an important similarity between group algebras and Iwahori-Hecke algebras.

In another direction we show that, under suitable hypothesis, the number of simple modules of the algebra kH is "generic", in the following sense. Assume that the ground ring $\mathcal{O}$ has Krull dimension 2. Fix a height 1 prime ideal $\mathfrak{p}$ and let $k_\mathfrak{p}$ be the quotient field of $\mathcal{O}/\mathfrak{p}$. Then Theorem 3.3 gives a condition on $\mathfrak{p}$ which implies that the number of simple modules of $k_\mathfrak{p}H$ equals the number of simple modules of kH. In our applications to Iwahori-Hecke algebras, $\mathcal{O}$ will be the localization of the ring of Laurent polynomials over $\mathbb{Z}$ in one indeterminate. The choice of height 1 and height 2 prime ideals yield algebras kH and $k_\mathfrak{p}H$, where k is a finite field of characteristic ℓ and $k_\mathfrak{p}$ is a cyclotomic field of characteristic 0 (see [Ge1] for more details). In Theorem 5.4 we check that the above hypotheses are satisfied whenever the prime ℓ is not too small (e.g., does not divide the order of the underlying finite Weyl group). Hence the number of simple modules of kH is determined by the algebra $k_\mathfrak{p}H$, i.e., it is "generic". This is one step in an attempt to prove the more general conjecture of [Ge1], (5.6), that even the decomposition maps themselves are "generic".

2. Decomposition maps

It is the purpose of this section to develop the basic theory of decomposition maps for algebras over integrally closed rings. Much of what follows is inspired by [Bra-Ne] and [Se].

2.1 Grothendieck groups and bilinear forms

Let $\mathcal{O}$ be a commutative local ring and H an $\mathcal{O}$-algebra, finitely generated and free as an $\mathcal{O}$-module. We denote by $K_0(H)$ the Grothendieck group of the category of finitely generated projective left H-modules and by $R_0(H)$ the Grothendieck group of the category of finitely generated H-modules which are free as $\mathcal{O}$-modules (such modules are called

H*-lattices*). The imbedding of the first category into the second one induces a map ("Cartan map") $c_{\mathcal{O}} : K_0(H) \to R_0(H)$. We denote by $R_0^+(H)$ the subset of $R_0(H)$ given by the classes of the H-lattices. Note that $R_0^+(H)$ generates $R_0(H)$.

In what follows, all modules are supposed to be finitely generated.

There is a bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{O}} : K_0(H) \times R_0(H) \to \mathbb{Z}$ defined by

$$\langle [P], [V] \rangle_{\mathcal{O}} = \mathrm{rank}_{\mathcal{O}} \mathrm{Hom}_H(P, V)$$

for P a projective H-module and V an H-lattice (where $[P]$ and $[V]$ denote the classes of P and V in $K_0(H)$ and $R_0(H)$ respectively). The fact that $\mathrm{Hom}_H(P, V)$ is free over $\mathcal{O}$ follows from the existence of an integer n such that $P|H^n$, since then $\mathrm{Hom}_H(P, V)$ is a direct summand of $\mathrm{Hom}_H(H^n, V) \simeq V^n$ as an $\mathcal{O}$-module. Let us now prove that this form is well defined. If $[P] = [P_1] + [P_2]$, then $\mathrm{Hom}_H(P, V) \simeq \mathrm{Hom}_H(P_1, V) \oplus \mathrm{Hom}_H(P_2, V)$. If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of H-lattices, then $0 \to \mathrm{Hom}_H(P, V_1) \to \mathrm{Hom}_H(P, V) \to \mathrm{Hom}_H(P, V_2) \to 0$ is exact because P is projective. These two facts show that $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ is indeed well defined.

Let us denote by $\mathrm{CF}(H) = \mathrm{Hom}_{\mathcal{O}}(H/[H, H], \mathcal{O})$ the module of *class functions* (where $[H, H]$ denotes the $\mathcal{O}$-submodule of H generated by the commutators $hh' - h'h$, $h, h' \in H$). We introduce now a bilinear form $(\cdot, \cdot)_{\mathcal{O}} : K_0(H) \times \mathrm{CF}(H) \to \mathcal{O}$ as follows:

Let P be a projective H-module. There exists an integer n such that P is a direct summand of H^n. Let e be the corresponding idempotent in $\mathrm{End}_H(H^n)$. The latter space can be canonically identified with the space $M_n(H)$ of $n \times n$-matrices over H. Let $\mathrm{Tr}(\mathrm{e}) \in H$ be the trace of e. It is straightforward to check that the image of $\mathrm{Tr}(e)$ in $H/[H, H]$ depends only on the class $[P]$ of P in $K_0(H)$ and that the corresponding map $K_0(H) \to H/[H, H]$ is additive. If $f \in \mathrm{CF}(H)$ then we define $([P], f)_{\mathcal{O}} = f(\mathrm{Tr}(e))$.

Assume that f is the character $\mathrm{ch}([V])$ of an H-lattice V (where we denote by $\mathrm{ch} : R_0(H) \to \mathrm{CF}(H)$ the character map). One has $\mathrm{Hom}_H(eH^n, V) \simeq \mathrm{e}V^n$ as $\mathcal{O}$-modules, hence $\langle [P], [V] \rangle = \mathrm{rank}_{\mathcal{O}} \mathrm{e}V^n$. But $([P], f)_{\mathcal{O}} = f(\mathrm{Tr}(\mathrm{e})) = 1_{\mathcal{O}} \cdot \mathrm{rank}_{\mathcal{O}} \mathrm{e}V^n$, hence $([P], f) = \langle [P], [V] \rangle \cdot 1_{\mathcal{O}}$. This proves the following :

Lemma 2.1 *One has* $(x, \mathrm{ch}(y)) = \langle x, y \rangle \cdot 1_{\mathcal{O}}$ *for* $x \in K_0(H)$ *and* $y \in R_0(H)$.

We define a semi-group morphism $\rho_{\mathcal{O}}$ from $R_0^+(H)$ to the set $\mathrm{Maps}(H, \mathcal{O}[X])$ of maps $H \to \mathcal{O}[X]$ (with operation given by pointwise multiplication of maps) as follows:

Given an H-lattice M and $x \in H$, we let $\rho_{\mathcal{O}}([M])(x)$ be the characteristic polynomial of x acting on the free $\mathcal{O}$-module M.

Let B be a commutative local $\mathcal{O}$-algebra given by $t_B : \mathcal{O} \to B$. If M is an $\mathcal{O}$-module, we denote by BM the B-module $B \otimes_{\mathcal{O}} M$. Without specification, tensor products are taken over $\mathcal{O}$, *i.e.*, $B \otimes M$ means $B \otimes_{\mathcal{O}} M$. There are canonical maps $t_B^{K_0} : K_0(H) \to K_0(BH)$, $t_B^{R_0} : R_0(H) \to R_0(BH)$, $t_B^{\mathrm{CF}} : \mathrm{CF}(H) \to \mathrm{CF}(BH)$ and $t_B^M : \mathrm{Maps}(H, \mathcal{O}[X]) \to \mathrm{Maps}(BH, B[X])$ induced by extension of scalars. The following lemma gives the compatiblities with extension of scalars:

Lemma 2.2 *Let $x \in K_0(H)$, $y \in R_0(H)$ and $f \in \mathrm{CF}(H)$. Then*

$$t_B((x,f)_{\mathcal{O}}) = (t_B(x), t_B(f))_B, \qquad \langle x, y\rangle_{\mathcal{O}} = \langle t_B(x), t_B(y)\rangle_B, \quad \textit{and}$$

$$t_B^{R_0} \circ c_{\mathcal{O}} = c_B \circ t_B^{K_0}, \qquad t_B^{CF} \circ \mathrm{ch} = \mathrm{ch} \circ t_B^{R_0}, \qquad \rho_B \circ t_B^{R_0} = t_B^M \circ \rho_{\mathcal{O}}.$$

Proof. Only the second assertion does not follow directly from the definitions. If $x = [P]$ and $y = [V]$, then $\langle t_B(x), t_B(y)\rangle_B = \mathrm{rank}_B(eV^n) \otimes_{\mathcal{O}} B$, where n is such that $P|H^n$ and e is the corresponding idempotent of $\mathrm{End}_H(H^n)$. Since eV^n is a free $\mathcal{O}$-module, one has $\mathrm{rank}_B(eV^n) \otimes_{\mathcal{O}} B = \mathrm{rank}_{\mathcal{O}} eV^n$, hence $\langle t_B(x), t_B(y)\rangle_B = \langle x, y\rangle_{\mathcal{O}}$. □

Lemma 2.3 *Assume B is flat over $\mathcal{O}$. Then, the map $1_B \otimes t_B^{CF} : B \otimes \mathrm{CF}(H) \to \mathrm{CF}(BH)$ is an isomorphism.*

Proof. From the exact sequence $0 \to [H,H] \to H \to H/[H,H] \to 0$, one gets the exact sequence $0 \to [BH, BH] \to BH \to B \otimes (H/[H,H]) \to 0$. Hence, $B \otimes (H/[H,H]) \simeq BH/[BH, BH]$ and finally $\mathrm{Hom}_B(BH/[BH,BH], B) \simeq B \otimes \mathrm{Hom}_{\mathcal{O}}(H/[H,H], \mathcal{O})$. □

2.2 Algebras over a field

Let us first recall without proof some classical results about simple algebras (cf [Bki1]). Assume that $\mathcal{O} = K$ is a field. We have the following commutative diagram:

$$\begin{array}{ccccc} H-\mathrm{mod} & \longrightarrow & R_0^+(H) & \xrightarrow{\rho} & \mathrm{Maps}(H, K[X]) \\ \uparrow & & \uparrow\simeq & & \uparrow \\ (H/J(H))-\mathrm{mod} & \longrightarrow & R_0^+(H/J(H)) & \xrightarrow{\rho} & \mathrm{Maps}(H/J(H), K[X]) \end{array}$$

where $J(H)$ denotes the radical of H. Hence, in order to study $R_0^+(H)$ and its image in $\mathrm{Maps}(H, K[X])$, we can assume that $J(H) = 0$.

Now, the algebra H is semisimple, *i.e.*, is isomorphic to a finite direct product of simple algebras. So, let us assume that H is simple. Let V

be a simple H-module, $D = \mathrm{End}_H(V)$ and $n = \dim_D(V)$. Then, H is isomorphic to the ring $M_n(D^\circ)$ of $(n \times n)$-matrices over the skewfield D° opposite to D.

Let m be the integer such that $[D : Z(D)] = m^2$. Let $\mathrm{Trd} : H \to Z(D)$ be the reduced trace of the central simple $Z(D)$-algebra H. It has the property that if L is a neutralizing field for H, *i.e.*, such that $Z(D) \subset L \subset D$ and $H \otimes_{Z(D)} L \simeq M_{mn}(L)$, then the usual trace $M_{mn}(L) \to L$ is given by $\mathrm{Trd} \otimes 1_L$. Denote by $\mathrm{ch}_K(V)$ and $\mathrm{ch}_{Z(D)}(V)$ the character of V respectively viewed as a module over the K-algebra H and as a module over the $Z(D)$-algebra H. We have

$$\mathrm{ch}_K(V) = \mathrm{Tr}_{Z(D)/K}\mathrm{ch}_{Z(D)}(V) \quad \text{and} \quad \mathrm{ch}_{Z(D)}(V) = m\,\mathrm{Trd}$$

where $\mathrm{Tr}_{Z(D)/K} : Z(D) \to K$ denotes the trace map of the K-algebra $Z(D)$, *i.e.*, the character of the module $Z(D)$ for the K-algebra $Z(D)$.

Similarly, we have

$$\rho_K(V) = \mathrm{N}_{Z(D)[X]/K[X]}(\rho_{Z(D)}(V)) \quad \text{and} \quad \rho_{Z(D)}(V) = \mathrm{Prd}^m$$

where $\mathrm{N}_{Z(D)[X]/K[X]} : Z(D)[X] \to K[X]$ is the norm map of the $K[X]$-algebra $Z(D)[X]$ and where $\mathrm{Prd} : H \to Z(D)[X]$ is the reduced characteristic polynomial map and $\rho_K(V)$ and $\rho_{Z(D)}(V)$ are the characteristic polynomial maps of V respectively viewed as a module over the K-algebra H and as a module over the $Z(D)$-algebra H.

Lemma 2.4 *The following statements are equivalent for H a simple K-algebra with simple module V and $D = \mathrm{End}_H(V)$:*

(1) the extension $Z(D)$ of K is separable,

(2) $\mathrm{Tr}_{Z(D)/K} \neq 0$,

(3) $\mathrm{Tr}_{Z(D)/K}\mathrm{Trd} \neq 0$,

(4) the algebra $H \otimes_K Z(D)$ is semisimple,

(5) the algebra $D \otimes_K Z(D)$ is semisimple,

(6) the $H \otimes_K Z(D)$-module $V \otimes_K Z(D)$ is semisimple.

If $Z(D)$ is a separable extension of K, then the algebra $H \otimes_K Z(D)$ is isomorphic to a direct product of $[Z(D) : K]$ central simple $Z(D)$-algebras and the module $V \otimes_K Z(D)$ is isomorphic to the direct sum of $[Z(D) : K]$ non-isomorphic simple modules.

Let us go back to the case where H is any finite dimensional algebra over K. From now, we assume that K is perfect (one could as well work with the weaker assumption that given any simple H-module V, then $Z(\mathrm{End}_H(V))$ is a separable extension of K).

A basis for $R_0(H)$ (resp. $K_0(H)$) is given by the images of a complete set of representatives of isomorphism classes of simple (resp. projective indecomposable) modules. Hence, one has an isomorphism $R_0(H) \to K_0(H)$ given by sending the class of a simple module to the class of one of its projective covers.

An irreducible character of H is defined as the character of a simple H-module and we denote by $\mathrm{Irr}(H)$ the set of irreducible characters of H.

If K' is a field extension of K, then the canonical maps $t_{K'}^{K_0} : K_0(H) \to K_0(K'H)$ and $t_{K'}^{R_0} : R_0(H) \to R_0(K'H)$ are injective. There exists a finite Galois extension K' of K such that $K'H$ is split, *i.e.*, such that for every simple $K'H$-module V, the canonical map $K' \to \mathrm{End}_{K'H}(V)$ given by multiplication is an isomorphism; we call such a field K' a *neutralizing* field for H. Let V be a simple H-module. Then, there are non-isomorphic simple $K'H$-modules $V_1, \ldots, V_s$ and an integer m_V (the *Schur index* of V) such that

$$K'V \simeq (V_1 \oplus \cdots \oplus V_s)^{m_V}.$$

Note that we have $[K_V : Z(K_V)] = m_V^2$ where $K_V = \mathrm{End}_H(V)$. Let V' be another simple H-module, with $K'V' \simeq (V_1' \oplus \cdots \oplus V_{s'}')^{m_{V'}}$ where the modules V_i' are simple and $V_i' \not\simeq V_j'$ for $i \neq j$. Then, if $V \not\simeq V'$, we have $V_i \not\simeq V_j'$, for all i, j. Let P_V be a projective cover of V. Then, $\mathrm{Hom}_{K'H}(K'P_V, K'V') \simeq K' \otimes \mathrm{Hom}_H(P_V, V') \simeq K' \otimes \mathrm{Hom}_H(V, V') \simeq \mathrm{Hom}_{K'H}(K'V, K'V')$. Hence, $K'P_V$ is a projective cover of $K'V$, *i.e.*, denoting by P_i a projective cover of V_i, we have

$$K'P_V \simeq (P_1 \oplus \cdots \oplus P_s)^{m_V}.$$

We then define $\bar{R}_0(H)$ as the subgroup of $R_0(K'H)$ with basis $\{\frac{1}{m_V}[V]\}$ where V runs over the simple H-modules (cf [Se, §12.1]). Similarly, we define $\bar{K}_0(H)$ as the subgroup of $K_0(K'H)$ with basis $\{\frac{1}{m_V}[P_V]\}$ where V runs over the simple H-modules. Note that the group $R_0(H)$ (resp. $K_0(H)$) is a subgroup of finite index of$\bar{R}_0(H)$ (resp. $\bar{K}_0(H)$). In particular, $\mathrm{rank}\, R_0(H) = \mathrm{rank} \bar{R}_0(H)$ and $\mathrm{rank}\, K_0(H) = \mathrm{rank} \bar{K}_0(H)$.

It is clear that $c_K, \langle \cdot, \cdot \rangle_K$ and $(\cdot, \cdot)_K$ extend to maps $\bar{K}_0(H) \to \bar{R}_0(H)$, $\bar{K}_0(H) \times \bar{R}_0(H) \to \mathbb{Z}$ and $\bar{K}_0(H) \times \mathrm{CF}(H) \to K$ compatible with the extension to K'. Furthermore, we define $\bar{R}_0^+(H)$ as $\bar{R}_0(H) \cap R_0^+(K'H)$.

Proposition 2.5 *Recall that K is assumed to be perfect. Then the map $\rho_K : \bar{R}_0^+(H) \to \mathrm{Maps}(H, K[X])$ is an injection.*

Proof. (cf [Bra-Ne, Lemma 2]) It is enough to prove the lemma in the case KH split, which we assume now. Let M and N be two H-modules such that $\rho([M]) = \rho([N])$. By replacing M and N with their associated semisimple modules, one can assume that M and N are semisimple. Let S be a set of representatives of isomorphism classes of simple H-modules. For $V \in S$, let a_V and b_V be the multiplicities of V as a composition factor of M and N. Since $\mathrm{ch}([M]) = \mathrm{ch}([N])$, one has $(a_V - b_V)\cdot 1_K = 0$. If $\mathcal{O}$ has characteristic zero, this implies $a_V = b_V$, hence $[M] = [N]$. Otherwise, let p be the characteristic of K, $p > 0$. One has $a_V \equiv b_V \bmod p$.

Let us assume that ρ is an injection for modules of dimension at most n and assume that M and N have dimension $n+1$. If there is a $V \in S$ which is a submodule of M and N, then the modules M/V and N/V are isomorphic, since their dimension is less than n. Hence, we may assume that for every $V \in S$, $a_V = 0$ or $b_V = 0$. Let $M' = \oplus_V V^{\frac{a_V}{p}}$ and $N' = \oplus_V V^{\frac{b_V}{p}}$. Again, M' and N' have dimension less than n; hence $M' \simeq N'$,*i.e.*, $M' = N' = 0$ and $M = N = 0$, which gives a contradiction. □

Lemma 2.6 *The subgroup $\bar{R}_0(H)$ of $R_0(K'H)$ consists of those elements f such that $\rho_{K'}(f)(h) \in K[X]$ for all $h \in H$.*

Proof. It follows from the construction of $\bar{R}_0(H)$ that, for $f \in \bar{R}_0(H)$ and $h \in H$, $\rho(f)(h) \in K$.

Let now $f \in R_0(K'H)$ such that $\rho(f)(h) \in K$ for all $h \in H$. We can clearly assume that H is semisimple, since $\rho(f)(h+r) = \rho(f)(h)$ for $h \in H$ and $r \in J(H)$. We can also assume that H is simple. Let $V_1, \ldots, V_s$ be a complete set of representatives of isomorphism classes of simple $K'H$-modules. Then the Galois group of K' over K acts transitively on $\{\mathrm{ch}([V_i])\}$, hence $\rho(f) = \alpha \sum_{i=1}^{s} \mathrm{ch}([V_i])$ for some $\alpha \in K$. Since ρ is injective, by Proposition 2.5, we have $f = \alpha \sum_{i=1}^{s} [V_i]$ for some integer α and the lemma is proved. □

We put $\overline{\mathrm{Irr}}(H) = \{\mathrm{ch}(\frac{1}{m_V}[V])\}$ where V runs over the simple H-modules.

Proposition 2.7 *The map $1_K \otimes \mathrm{ch} : K \otimes \bar{R}_0(H) \to \mathrm{CF}(H)$ is an injection,* i.e., *the elements of $\overline{\mathrm{Irr}}(H)$ are linearly independent. If H is semisimple, then the map above is an isomorphism.*

Proof. One can assume that H is simple and the proposition follows then from Lemma 2.4. □

Assume now that for every simple H-module V, the canonical map $K \to Z(\mathrm{End}_H(V))$ is an isomorphism, *i.e.*, $\mathrm{End}_H(V)$ is a central K-algebra; we say that H is a *quasicentral* K-algebra.

Proposition 2.8 *Assume that H is quasicentral.*

(1) *The form $\langle\cdot,\cdot\rangle$ induces a perfect pairing between $\bar{K}_0(H)$ and $\bar{R}_0(H)$: if V, V' are two simple H-modules and P_V is a projective cover of V, then we have $\langle\frac{1}{m_V}[P_V], \frac{1}{m_{V'}}[V']\rangle = \delta_{[V],[V']}$.*

(2) *If K' is an extension of K, then the maps $t_{K'}^{R_0} : \bar{R}_0(H) \to \bar{R}_0(K'H)$ and $t_{K'}^{K_0} : \bar{K}_0(H) \to \bar{K}_0(K'H)$ are isomorphisms.*

(3) *If K' is a neutralizing field for H, then $\bar{R}_0(H) = R_0(K'H)$ and* $\operatorname{rank} R_0(H) = \operatorname{rank} R_0(K'H)$.

Proof. Since $\operatorname{Hom}_H(P_V, V') \simeq \operatorname{Hom}_H(V, V')$ and $\dim_K \operatorname{Hom}_H(V, V') = m_V^2 \delta_{[V],[V']}$ we have $\langle[P_V], [V']\rangle = m_V^2 \delta_{[V],[V']}$. This implies that the pairing induced by $\langle\cdot,\cdot\rangle$ is perfect. The other statements are clear. □

There is an easy characterization of quasicentral algebras:

Proposition 2.9 *The following statements are equivalent:*

(1) *the K-algebra H is quasicentral;*

(2) *$\bar{R}_0(H) = \bar{R}_0(K'H)$ for any extension K' of K;*

(3) $\operatorname{rank} R_0(H) = \operatorname{rank} R_0(K'H)$ *for any extension K' of K;*

(4) *there is a finite Galois extension L of K which is a neutralizing field for H and such that* $\operatorname{rank} R_0(H) = \operatorname{rank} R_0(LH)$.

Proof. Let L be a finite Galois extension of K which is a neutralizing field for H. Let V be a simple H-module and $D = \operatorname{End}_H(V)$. Then, $V \otimes_K L$ is a direct sum of $[Z(D) : K]$ non-isomorphic simple LH-modules. If $Z(D) \neq K$, it implies $\operatorname{rank} R_0(H) < \operatorname{rank} R_0(LH)$. Hence, (4) ⇒ (1).

By Proposition 2.8, (1) implies (2). Finally, (2) ⟹ (3) ⟹ (4) is clear. □

2.3 The Brauer-Cartan square

From now on, $\mathcal{O}$ is an integrally closed local domain, K its field of fractions and k its residue field. We assume K and k are perfect. Let H be an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module.

Lemma 2.10 *The image of* $\operatorname{ch} : \bar{R}_0(KH) \to \operatorname{CF}(KH)$ *is contained in the $\mathcal{O}$-submodule* $\operatorname{CF}(H)$ *and the image of $\rho_K : \bar{R}_0^+(KH) \to$* $\operatorname{Maps}(KH, K[X])$ *is contained in the $\mathcal{O}$-submodule* $\operatorname{Maps}(H, \mathcal{O}[X])$.

Proof. Let V be a simple $K'H$-module where K' is a finite extension of K, with $\mathrm{ch}(V)(x) \in K$ for all $x \in KH$. Then, for $h \in H$, the characteristic polynomial $\rho(V)(h)$ divides (in $K'[X]$) the characteristic polynomial $\rho(KH)(h) = \rho(H)(h)$ associated to the regular representation. Since $\rho(H)(h) \in \mathcal{O}[X]$, the roots of $\rho(V)(h)$ are algebraic over $\mathcal{O}$, hence $\mathrm{ch}(V)(h)$ is algebraic over $\mathcal{O}$. Since $\mathrm{ch}(V)(h) \in K$ and $\mathcal{O}$ is integrally closed in K, this implies $\mathrm{ch}(V)(h) \in \mathcal{O}$. Similarly, the coefficients of $\rho(V)(h)$ are algebraic over $\mathcal{O}$ and are in K, hence are in $\mathcal{O}$. □

The following proposition establishes the existence of the decomposition map:

Proposition 2.11 *There exists a unique map $d_{\mathcal{O}} : \bar{R}_0(KH) \to \bar{R}_0(kH)$ which makes the following diagram commutative:*

$$\begin{array}{ccccc} \bar{R}_0(KH) & \supset & \bar{R}_0^+(KH) & \xrightarrow{\rho_K} & \mathrm{Maps}(H, \mathcal{O}[X]) \\ \downarrow{\scriptstyle d_{\mathcal{O}}} & & \downarrow & & \downarrow{\scriptstyle t_k} \\ \bar{R}_0(kH) & \supset & \bar{R}_0^+(kH) & \xrightarrow{\rho_k} & \mathrm{Maps}(kH, k[X]) \end{array}$$

Proof. Note first that the unicity follows from the injectivity of ρ_k (cf Proposition 2.5).

Let K' be a finite extension of K such that $K'H$ is split. Let $\mathcal{O}'$ be a valuation ring, $\mathcal{O} \subset \mathcal{O}' \subset K'$ with maximal ideal I such that $I \cap \mathcal{O} = \mathfrak{m}$, the maximal ideal of $\mathcal{O}$, and residue field k'. Let V be a simple $K'H$-module with $[V] \in \bar{R}_0(KH)$. Since finitely generated torsion-free $\mathcal{O}'$-modules are free [Go, §5.2], there exists an $\mathcal{O}'H$-lattice V' such that $K'V' \simeq V$. Then, $\rho_{k'}([k'V'])$ is the reduction mod I of $\rho_K([V]) \in \mathrm{Maps}(\mathcal{O}'H, \mathcal{O}'[X])$. Since $\rho_K([V])$ is actually in $\mathrm{Maps}(\mathcal{O}H, \mathcal{O}[X])$ by assumption, we have also $\rho_{k'}([k'V'])$ in $\mathrm{Maps}(kH, k[X])$, hence $[k'V'] \in \bar{R}_0(kH)$ by Lemma 2.6 and we put $d([V]) = [k'V']$. □

Note that the decomposition map exists not only for $\mathcal{O}$ integrally closed but, more generally, when the image of $\rho_K : \bar{R}_0^+(KH) \to \mathrm{Maps}(KH, K[X])$ is contained in the $\mathcal{O}$-submodule $\mathrm{Maps}(H, \mathcal{O}[X])$. The following proposition is a direct consequence of the definition:

Proposition 2.12 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ such that $k_{\mathfrak{p}} = (\mathcal{O}_{\mathfrak{p}})/\mathfrak{p}$ is perfect and $\mathcal{O}/\mathfrak{p}$ is integrally closed. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \bar{R}_0(KH) & \xrightarrow{d_{\mathcal{O}}} & \bar{R}_0(kH) \\ \downarrow{\scriptstyle d_{\mathcal{O}_{\mathfrak{p}}}} & & \uparrow{\scriptstyle d_{\mathcal{O}/\mathfrak{p}}} \\ \bar{R}_0(k_{\mathfrak{p}}H) & = & \bar{R}_0(k_{\mathfrak{p}}H) \end{array}$$

Suppose now that KH is a quasicentral K-algebra. We define the map $e : \bar{K}_0(kH) \to \bar{K}_0(KH)$ as the map dual to d with respect to the pairing $\langle \cdot, \cdot \rangle$, *i.e.*, for $\eta \in \bar{K}_0(kH)$ and $\chi \in \bar{R}_0(KH)$, we have

$$\langle \eta, d(\chi) \rangle_k = \langle e(\eta), \chi \rangle_K.$$

Let us now give an alternative definition of e, without using d.

Let $\tilde{\mathcal{O}}$ be a strict henselisation of $\mathcal{O}$ [Ray, Chapitre VIII]: this is an henselian (local) ring, local extension of $\mathcal{O}$, faithfully flat as an $\mathcal{O}$-module, with residue field $\bar{k}$ a separable closure of k. Furthermore, since $\mathcal{O}$ is integrally closed, the ring $\tilde{\mathcal{O}}$ is an integral domain [Ray, Chapitre IX, corollaire 1]. Let now $\mathcal{O}'$ be a valuation ring, local extension of $\tilde{\mathcal{O}}$, contained in the field of fractions $\widetilde{K}$ of $\tilde{\mathcal{O}}$, with residue field k'.

Since $\tilde{\mathcal{O}}$ is henselian, every idempotent of $\bar{k}H$ can be lifted to an idempotent of $\tilde{\mathcal{O}}H$, hence every projective $\bar{k}H$-module can be lifted to a projective $\tilde{\mathcal{O}}H$-module. Let P be a projective $\bar{k}H$-module and V a KH-module. There exists an $\mathcal{O}'H$-lattice M such that $V \otimes \widetilde{K} \simeq M \otimes \widetilde{K}$ and a projective $\mathcal{O}'H$-module Q such that $Q \otimes k' \simeq P \otimes k'$. We have

$$\langle [P], d([V]) \rangle_{\bar{k}} = \langle [Q \otimes k'], [M \otimes k'] \rangle_{k'} = \langle [Q], [M] \rangle_{\mathcal{O}'} = \langle [\widetilde{K}Q], [\widetilde{K}M] \rangle_{\widetilde{K}}.$$

(Note that similarly, $([P], f \otimes 1_{\bar{k}})_{\bar{k}} = (e([P]), f \otimes 1_K)_K \cdot 1_{\bar{k}}$ for $f \in \mathrm{CF}(H)$). Hence, $e([P]) = [\widetilde{K}Q]$ (viewed in $\bar{K}_0(KH)$). Furthermore, one has $d \circ c_K \circ e([P]) = c_k([P])$, hence, the following diagram ("Brauer-Cartan square") is commutative:

$$\begin{array}{ccc} \bar{R}_0(KH) & \xrightarrow{d} & \bar{R}_0(kH) \\ \uparrow{\scriptstyle c_K} & & \uparrow{\scriptstyle c_k} \\ \bar{K}_0(KH) & \xleftarrow{e} & \bar{K}_0(kH) \end{array}$$

With the additional assumption that the algebra KH is semi-simple, one has $\bar{K}_0(KH) = \bar{R}_0(KH)$ and we recover the usual Brauer-Cartan triangle (cf [Se, §15]).

3. On the number of simple modules

We keep the assumptions above: We have $\mathcal{O}$ an integrally closed local domain with residue field k perfect and field of fractions K perfect and H an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module.

The following proposition generalizes a theorem of Hattori [CuRe, Theorem 32.5] about the injectivity of the Cartan map. In Hattori's theorem, it is assumed that $H/[H,H]$ is free as an $\mathcal{O}$-module (and $\mathcal{O}$ is assumed to be a discrete valuation ring).

Proposition 3.1 *Assume that kH is quasicentral. If the image of the canonical map $t_k : \mathrm{CF}(H) \to \mathrm{CF}(kH)$ contains $\mathrm{ch}(\bar{R}_0(kH))$, then the map e is injective (hence, given two projective H-modules M and N, we have $KM \simeq KN$ if and only if $M \simeq N$). In particular, the decomposition map d has finite cokernel and the algebra KH has at least as many simple modules as the algebra kH.*

Proof. Let us prove first that e is injective. Replacing $\mathcal{O}$ by a strict henselisation of $\mathcal{O}$, we can assume that kH is split. Let P, Q be two non-zero projective kH-modules with no common direct summand such that $e([P]-[Q]) = 0$. Assume that P has minimal dimension with this property. Let V be a simple kH-module and put $\varphi = [V]$. By assumption, $\mathrm{ch}\varphi$ is the image of some $f \in \mathrm{CF}(H)$. So we have

$$< [P], \varphi > \cdot 1_k = ([P], \mathrm{ch}\varphi)_k = ([P], f \cdot 1_k)_k = (e[P], f)_K \cdot 1_k.$$

As $e([P]) = e([Q])$, we get $< [P]-[Q], \varphi >_k \cdot 1_k = 0$. If k has characteristic zero, this implies that $[P] = [Q]$. Assume then that the characteristic p of k is positive. Then, the multiplicities of a projective cover P_V of V in P and Q as a direct summand are equal modulo p. Since P and Q have no common direct summand by assumption, it implies that the multiplicities of P_V in P and Q are both divisible by p.

So, there exists P_0 and Q_0 two projective kH-modules with $P \simeq P_0^p$ and $Q \simeq Q_0^p$; hence $e([P_0]-[Q_0]) = 0$. Since P_0 has strictly smaller dimension than P, it implies $P_0 \simeq Q_0$, hence $P \simeq Q$, which is impossible. This completes the proof that $\ker e = 0$.

Now, by definition of e as dual of d, it follows that d has finite cokernel. Since $d : \bar{R}_0(KH) \to \bar{R}_0(kH)$, it is clear that KH has at least as many simple modules as kH. □

Lemma 3.2 *Assume that the canonical map $t_k : \mathrm{CF}(H) \to \mathrm{CF}(kH)$ is surjective and that KH is quasicentral. Then kH also is a quasicentral algebra.*

Proof. Let k' be a finite Galois extension of k neutralizing for H. Let $\mathcal{O}'$ be a local domain, local extension of $\mathcal{O}$ with residue field k' and field of fractions K'. By Lemma 2.3, the canonical map $\mathrm{CF}(\mathcal{O}'H) \to \mathrm{CF}(k'H)$ is also surjective, hence Proposition 3.1 proves that $d_{\mathcal{O}'}$ has finite cokernel.

As KH is quasicentral, one has $\bar{R}_0(K'H) = \bar{R}_0(KH)$ by Proposition 2.9; hence the image by the decomposition map of $\bar{R}_0(K'H)$ is contained in $\bar{R}_0(kH)$. Since the decomposition map $d_{\mathcal{O}'} : \bar{R}_0(K'H) \to$

$R_0(k'H)$ has finite cokernel, we get $\operatorname{rank} R_0(kH) = \operatorname{rank} R_0(k'H)$; hence kH is quasicentral, by Proposition 2.9. □

Assume $\mathcal{O}$ is noetherian and has Krull dimension 2. Let $\mathfrak{p}$ be a height one prime ideal of $\mathcal{O}$ with $\mathcal{O}/\mathfrak{p}$ integrally closed (*i.e.*, $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring) and $k_\mathfrak{p} = (\mathcal{O}_\mathfrak{p})/\mathfrak{p}$ perfect.

For $\mathfrak{q}$ a height one prime ideal of $\mathcal{O}$, let $\tilde{k}_\mathfrak{q}$ be a finite separable extension of $k_\mathfrak{q} = (O_\mathfrak{q})/\mathfrak{q}$ neutralizing for $k_\mathfrak{q}H$ and $\tilde{\mathcal{O}}_\mathfrak{q}$ a discrete valuation ring, (unramified) extension of $\mathcal{O}_\mathfrak{q}$ with residue field $\tilde{k}_\mathfrak{q}$ and field of fractions $\widetilde{K}_\mathfrak{q}$; let $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}$ denote the completion of $\tilde{\mathcal{O}}_\mathfrak{q}$ and $\hat{\widetilde{K}}_\mathfrak{q}$ its field of fractions.

Here is now the crucial result:

Theorem 3.3 *Assume that KH is quasicentral and that the canonical map* $\mathrm{CF}(H) \to \mathrm{CF}(kH)$ *is surjective.*

(1) The map $1_k \otimes d_{\mathcal{O}/\mathfrak{p}} : k \otimes \bar{R}_0(k_\mathfrak{p}H) \to k \otimes \bar{R}_0(kH)$ is an isomorphism if and only if the restriction of the bilinear form $(\cdot,\cdot)_{\hat{\tilde{\mathcal{O}}}_\mathfrak{p}}$ to $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{p}) \times \mathrm{CF}(H)$ has values in $\mathcal{O}$.

(2) If for every height one prime ideal $\mathfrak{q} \neq \mathfrak{p}$ of $\mathcal{O}$, the algebra $k_\mathfrak{q}H$ is semisimple, then the two equivalent statements in (1) hold. In particular, the number of simple kH-modules is equal to the number of simple $k_\mathfrak{p}H$-modules.

Proof. Note that $k_\mathfrak{p}H$ is quasicentral since $k_\mathfrak{p}$ is perfect, by Lemma 3.2. Let $\mathfrak{q}$ be a height one prime ideal of $\mathcal{O}$. Note that $\hat{\tilde{\mathcal{O}}}_\mathfrak{q} \cap K = \mathcal{O}_\mathfrak{q}$ (the intersection is taken in $\hat{\widetilde{K}}_\mathfrak{q}$). Since $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}$ is a complete discrete valuation ring, idempotents can be lifted from $\tilde{k}_\mathfrak{q}H$ to $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}H$. Hence, the canonical map $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{q}H) \to K_0(\tilde{k}_\mathfrak{q}H)$ is an isomorphism. We have the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{CF}(H) & \xrightarrow{surj.} & \mathrm{CF}(kH) & \xrightarrow{inj.} & \mathrm{Hom}_k(kH,k) \\
\big\downarrow{\scriptstyle t_{\mathcal{O}/\mathfrak{p}}} & & \big\uparrow & & \big\uparrow \\
\mathrm{CF}((\mathcal{O}/\mathfrak{p})H) & = & \mathrm{CF}((\mathcal{O}/\mathfrak{p})H) & \xrightarrow{inj.} & \mathrm{Hom}_{\mathcal{O}/\mathfrak{p}}((\mathcal{O}/\mathfrak{p})H, \mathcal{O}/\mathfrak{p})
\end{array}
$$

We have

$$\mathrm{CF}((\mathcal{O}/\mathfrak{p})H) = \mathrm{im}(t_{\mathcal{O}/\mathfrak{p}}) + (\mathfrak{m}/\mathfrak{p})\mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$$

(note that $\mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ is a direct summand of $\mathrm{Hom}_{\mathcal{O}/\mathfrak{p}}((\mathcal{O}/\mathfrak{p})H, \mathcal{O}/\mathfrak{p})$ as $\mathcal{O}/\mathfrak{p}$-modules since $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring); hence $\mathrm{im}(t_{\mathcal{O}/\mathfrak{p}}) = \mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ by Nakayama's lemma. This proves that the canonical map $\mathrm{CF}(H) \to \mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ is surjective, hence the bilinear form $(\cdot,\cdot)_{\hat{\tilde{\mathcal{O}}}_\mathfrak{p}}$ restricted to $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{p}H) \times \mathrm{CF}(H)$ has values in $\mathcal{O}$ if and only if $(\cdot,\cdot)_{k_\mathfrak{p}}$ restricted to $\bar{K}_0(k_\mathfrak{p}H) \times \mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ has values in $\mathcal{O}/\mathfrak{p}$.

Since $\langle\cdot,\cdot\rangle$ induces a perfect pairing between $\bar{K}_0(k_\mathfrak{p}H)$ and $\bar{R}_0(k_\mathfrak{p}H)$, the submodule $\mathrm{ch}\,\bar{R}_0(k_\mathfrak{p}H)$ of $\mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ is pure if and only if the form $(\cdot,\cdot)_{k_\mathfrak{p}}$ restricted to $\bar{K}_0(k_\mathfrak{p}H) \times \mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$ has values in $\mathcal{O}/\mathfrak{p}$. By Proposition 3.1, the decomposition map $d_{\mathcal{O}/\mathfrak{p}} : \bar{R}_0(k_\mathfrak{p}H) \to \bar{R}_0(kH)$ has finite cokernel; hence the map $1_k \otimes d_{\mathcal{O}/\mathfrak{p}}$ is an isomorphism if and only if it is injective, *i.e.*, if and only if $\mathrm{ch}\,\bar{R}_0(k_\mathfrak{p}H)$ is a pure submodule of $\mathrm{CF}((\mathcal{O}/\mathfrak{p})H)$. This completes the proof of (1).

Assume that for $\mathfrak{q} \neq \mathfrak{p}$, the algebra $k_\mathfrak{q}H$ is semisimple. Then, the algebra $\tilde{k}_\mathfrak{q}H$ is split semisimple; hence, the algebra $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}H$ is isomorphic to a direct product of matrix algebras over $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}$. So, the canonical map $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{q}H) \to K_0(\hat{\tilde{K}}_\mathfrak{q}H)$ is an isomorphism and the form $(\cdot,\cdot)_{\hat{\tilde{K}}_\mathfrak{q}}$ restricted to $K_0(\hat{\tilde{K}}_\mathfrak{q}H) \times \mathrm{CF}(H)$ has values in $\hat{\tilde{\mathcal{O}}}_\mathfrak{q}$ and finally the form$(\cdot,\cdot)_K$ restricted to $\bar{K}_0(KH) \times \mathrm{CF}(H)$ has values in $\cap_{\mathfrak{q}\neq\mathfrak{p}}\mathcal{O}_\mathfrak{q}$. Composing the canonical map $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{p}H) \to K_0(\hat{\tilde{K}}_\mathfrak{p}H)$ with the inverse of the canonical map $\bar{K}_0(KH) \to K_0(\hat{\tilde{K}}_\mathfrak{p}H)$ (note that KH is quasicentral), we get that the bilinear form $(\cdot,\cdot)_{\hat{\tilde{\mathcal{O}}}_\mathfrak{p}}$ restricted to $K_0(\hat{\tilde{\mathcal{O}}}_\mathfrak{p}H) \times \mathrm{CF}(H)$ has values in $\bigcap_{\mathfrak{q}\neq\mathfrak{p}} \mathcal{O}_\mathfrak{q} \cap \hat{\tilde{\mathcal{O}}}_\mathfrak{p}$, hence in $\mathcal{O}$ since $\mathcal{O}$ is a Krull ring [Bki2, Chapitre VII, §1, théorème 4]. □

4. Center and class functions for symmetric algebras

Let $\mathcal{O}$ be a commutative ring and H an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module. Let $\tau \in \mathrm{CF}(H)$. We say that τ is a *symmetrizing form* for H (cf [Br]) if the induced map

$$\hat{\tau} : H \to \mathrm{Hom}_\mathcal{O}(H,\mathcal{O}), h \mapsto (h' \mapsto \tau(hh'))$$

is an isomorphism. More concretely, this means that, if $\mathcal{B}$ is an $\mathcal{O}$-basis of H, then the determinant of the matrix $(\tau(hh'))_{h,h'\in\mathcal{B}}$ is a unit in$\mathcal{O}$. When such a symmetrizing form exists, we say that the algebra H is *symmetric*.

Assume now that τ is a symmetrizing form for H. To simplify the notations, for $h \in H$ and $f \in \mathrm{Hom}_\mathcal{O}(H,\mathcal{O})$, we put $h^* = \hat{\tau}(h)$ and $f^* = \hat{\tau}^{-1}(f)$. Note that $\hat{\tau}$ induces an isomorphism of $\mathcal{O}$-modules

$Z(H) \to \mathrm{CF}(H)$: If A is a commutative $\mathcal{O}$-algebra, then the canonical map $\mathrm{CF}(H) \to \mathrm{CF}(AH)$ is surjective if and only if the canonical map $Z(H) \to Z(AH)$ is surjective.

If $\mathcal{B}$ is an $\mathcal{O}$-basis of H, then the dual basis $\{b^\vee\}_{b\in\mathcal{B}}$ is defined by the requirement $\tau(b_1 b_2^\vee) = \delta_{b_1 b_2}$ for $b_1, b_2 \in \mathcal{B}$. For $f \in \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O})$, one has $f^* = \sum_{b\in\mathcal{B}} f(b) b^\vee$.

Let M be an H-module. We have a map $\mathrm{Tr} : \mathrm{End}_{\mathcal{O}}(M) \to \mathrm{End}_H(M)$ given by

$$\mathrm{Tr}(f)(m) = \sum_{b\in\mathcal{B}} b f(b^\vee m).$$

We have Higman's lemma, following [Br] :

Lemma 4.1 *The module M is projective if and only if there is $f \in \mathrm{End}_{\mathcal{O}}(M)$ such that* $\mathrm{Tr}(f)$ *is the identity.*

Assume $\mathcal{O}$ is an integrally closed integral domain with field of fractions K perfect. Let us assume that KH is quasicentral.

Proposition 4.2 *Let $\chi \in \overline{\mathrm{Irr}}(KH)$ and let c_χ be the scalar by which $\chi^* \in Z(KH)$ acts on a simple KH-module V affording a multiple of χ. Then the following hold:*

(1) The element c_χ lies in $\mathcal{O}$.

(2) The module V is projective if and only if $c_\chi \neq 0$. In particular, the algebra KH is semisimple if and only if $c_\chi \neq 0$ for all $\chi \in \overline{\mathrm{Irr}}(KH)$.

*(3) Assume KH semisimple. Then, $\chi^* = c_\chi e_\chi$ where e_χ denotes the central primitive idempotent corresponding to χ (*i.e.*, $\chi^* e_\chi \neq 0$). Moreover,*

$$\tau = \sum_{\chi\in\overline{Irr}(KH)} \frac{1}{c_\chi}\chi.$$

Proof. Let V be a simple module with character a multiple of $\chi \in \overline{\mathrm{Irr}}(KH)$. The polynomial $X - c_\chi$ divides $\rho(H)(\chi^*)$: the roots of this polynomial are algebraic over $\mathcal{O}$, hence c_χ is algebraic over $\mathcal{O}$. Finally, $c_\chi \in K$, hence c_χ lies in the integral closure of $\mathcal{O}$ in K, *i.e.*, in $\mathcal{O}$.

There is a finite extension K' of K such that $K'H$ is split and it is clear that if the parts (2) and (3) of the proposition hold for $K'H$, they hold for KH. Hence, we can assume that KH is split.

Let V be a simple KH-module with character $\chi \in \mathrm{Irr}(KH)$. Note that if i is a primitive idempotent of $\mathrm{End}_{\mathcal{O}}(V)$, then $\mathrm{Tr}(i) = c_\chi 1_V$. Hence, if $c_\chi \neq 0$, it follows from Lemma 4.1 that V is projective.

Now, if V is a projective module and e_χ the central primitive idempotent of KH such that $e_\chi V \neq 0$, then the algebras $e_\chi KH$ and $(1-e_\chi)KH$ are symmetric algebras with symmetrizing form $\tau_{|e_\chi KH}$ and $\tau_{|(1-e_\chi)KH}$ and we have $KH = e_\chi KH \oplus (1-e_\chi)KH$. It is then clear that for the remaining part of the proposition, we can assume that KH is simple and let χ be the unique irreducible character of KH.

Then $Z(KH) = K \cdot 1$, hence $\chi^* = c_\chi$. It implies$\chi = c_\chi \tau$. If i is a primitive idempotent of KH, we have $\chi(i) = 1$, hence $c_\chi \neq 0$ and the proof is complete. □

Let now k be the residue field of $\mathcal{O}$ which we assume to be perfect.

Proposition 4.3 [Ge3] *The algebra kH is semisimple if and only if for every $\chi \in \overline{\mathrm{Irr}}(KH)$, we have $1_k c_\chi \neq 0$. If kH is semisimple, then KH is semisimple and kH is quasicentral.*

Proof. Suppose that kH is semisimple. Then, we have $\dim(kH) \leq \sum_V (\dim V/m_V)^2$ where V runs over a complete set of representatives of simple kH-modules. Since KH is quasicentral, we have $\dim(KH) \geq \sum_S (\dim S/m_S)^2$ where S runs over a complete set of representatives of simple KH-modules. We have $d([KH]) = [kH]$, hence $d(\sum_S [S/m_S]) = \sum_V \alpha_V [V/m_V]$ where S (resp. V) runs over a complete set of representatives of simple KH-modules (resp. kH-modules) and $\alpha_V > 0$ for all V. It follows that $\sum_S (\dim S/m_S)^2 \geq \sum_V (\dim V/m_V)^2$ and we have equality if and only if for all S, there exists V such that $d(S) = d(V)$ and $m_S = m_V$. Now,

$$\sum_S \left(\frac{\dim S}{m_S}\right)^2 \leq \dim(KH) = \dim kH \leq \left(\frac{\dim V}{m_V}\right)^2.$$

Hence we have equalities everywhere above, *i.e.*, kH is quasicentral and KH is semisimple. Futhermore, for $\chi \in \overline{\mathrm{Irr}}(KH)$, then $d(\chi) \in \overline{\mathrm{Irr}}(kH)$. By Proposition 4.2(2), we get $c_\chi 1_k = c_{d(\chi)} \neq 0$.

Suppose now $1_k c_\chi \neq 0$ for all $\chi \in \overline{\mathrm{Irr}}(KH)$. We have $d(\chi)(x) = 0$ for $x \in J(kH)$. Since $\tau = \sum_\chi \frac{1}{c_\chi}\chi$, we get $\bar{\tau}(x) = 0$ for $x \in J(kH)$, where $\bar{\tau} = 1_k \otimes \tau$. Hence, $J(kH)$ is an ideal of kH which is in the kernel of $\bar{\tau}$: since $\bar{\tau}$ is a symmetrizing form for kH, it implies $J(kH) = 0$ and kH is semisimple. □

Let P be a projective H-module. Let e be an idempotent of $M_n(H)$, for some n, such that $eH^n \simeq P$. Let $\eta(P) : Z(H) \to \mathcal{O}$ be the restriction to $Z(H)$ of $\mathrm{Tr}(e)^*$. We have $\eta(P)(z) = \tau(\mathrm{Tr}(e)z) = z^*(\mathrm{Tr}(e)) = ([P], z^*)_{\mathcal{O}}$. Note that it implies that $\eta(P)$ depends only on P.

Assume in addition KH semisimple and split. Given P a projective kH-module, J. Müller suggested considering the map $\psi(P) : Z(KH) \to K$ defined by

$$\psi(P) = \sum_{\chi\in\mathrm{Irr}(KH)} (e([P]),\chi)\,\frac{1}{c_\chi}w_\chi$$

where w_χ is the one dimensional representation of $Z(KH)$ acting (as multiplication by scalars) on a simple KH-module with character χ.

Proposition 4.4 *The map $\psi(P)$ restricts to a map $Z(H) \to \mathcal{O}$ and $1_k \otimes \psi(P) = \eta(P)$. In particular,*

$$\psi(P)(1) = \sum_{\chi\in Irr(KH)} \frac{(e([P]),\chi)}{c_\chi} \in \mathcal{O}.$$

Proof. Let $\mathcal{O}'$ be an henselisation of $\mathcal{O}$; we have $K\cap\mathcal{O}' = \mathcal{O}$, where the intersection is taken in the field of fractions of $\mathcal{O}'$, since $\mathcal{O}'$ is faithfully flat over $\mathcal{O}$. Hence, to prove the proposition, we can assume that $\mathcal{O}$ is henselian. There exists an idempotent e of $M_n(H)$ such that $keH^n \simeq P$. Then, $\eta(eH^n)$ is the restriction to $Z(H)$ of $\mathrm{Tr}(e)^*$. We have $w_{\chi'}(\chi^*) = \delta_{\chi,\chi'}c_\chi$ for $\chi,\chi' \in \mathrm{Irr}(KH)$, hence

$$\eta(eH^n) = \sum_\chi \frac{\chi(\mathrm{Tr}(e))}{c_\chi}w_\chi = \psi(P)$$

and $1_k \otimes \psi(P) = 1_k \otimes \eta(eH^n) = \eta(P)$. □

Note that if $H = \mathcal{O}G$ is a group algebra with its usual symmetrizing form, K having characteristic zero and k characteristic $p > 0$, then the integrality property above is equivalent to the statement that the dimension of a projective kG-module is divisible by the order of a Sylow p-subgroup of G.

When H is an Iwahori–Hecke algebra associated to a Weyl group, with equal parameters and k has characteristic zero, this result was proven in [Ge4, Proposition 2.1] using algebraic groups.

5. Iwahori-Hecke algebras

We fix a finite Weyl group W with a corresponding set $S \subset W$ of simple reflections. Let $\{u_s\}_{s\in S}$ be a set of indeterminates such that $u_s = u_t$ whenever $s,t \in S$ are conjugate in W, and $A = \mathbb{Z}[u_s, u_s^{-1}]_{s\in S}$ be the ring of Laurent polynomials in these indeterminates. The generic Iwahori-Hecke algebra $\mathcal{H}$ is the A-free A-algebra with A-basis $\{T_w\}_{w\in W}$ and relations:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } l(ww') = l(w) + l(w'), \\ (T_s - u_s)(T_s + 1) = 0 & \text{for } s \in S \end{cases}$$

where $w \mapsto l(w)$ is the length function on W with respect to the generating set S. The algebra $\mathcal{H}$ is symmetric, with respect to the form $\tau : \mathcal{H} \to A$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $w \neq 1$. The elements in the dual basis of $\{T_w\}$ are given by $T_w^\vee = \operatorname{ind}(T_w)^{-1} T_{w^{-1}}$, where $\operatorname{ind} : \mathcal{H} \to A$ is the 1-dimensional representation of $\mathcal{H}$ defined by $\operatorname{ind}(T_s) = u_s$ for $s \in S$. Thus, we can apply the results of the previous section to the pair $(\mathcal{H}, \tau)$.

5.1 Centers of Iwahori–Hecke algebras

For each conjugacy class C of W, we denote by C_{min} the set of elements of minimal length in C, and we choose one element $w_C \in C_{min}$. For each class C and each $w \in W$, there exists an element $f_{w,C} \in A$ (called *class polynomial* in [Ge-Pf]) uniquely determined by the property that

$$\varphi(T_w) = \sum_C f_{w,C}\varphi(T_{w_C}) \quad \text{for all } \varphi \in \mathrm{CF}(\mathcal{H}).$$

(It is shown in [Ge-Pf] that, if $w, w' \in C_{min}$ then T_w and $T_{w'}$ are conjugate in $\mathcal{H}$. In particular, every class function of $\mathcal{H}$ has the same values on T_w and $T_{w'}$. Hence the definition of $f_{w,C}$ is independent of the choice of $w_C \in C_{min}$.)

For each class C we define a function $f_C : \mathcal{H} \to A$ by

$$f_C : T_w \mapsto f_{w,C} \quad (w \in W).$$

By inverting the defining formula for $f_{w,C}$ above, we see that f_C is in fact a central function. Hence, given any $\varphi \in \mathrm{CF}(\mathcal{H})$, one has

$$\varphi = \sum_C \varphi(T_{w_C}) f_C,$$

i.e., the set $\{f_C\}$ is a basis of the A-module $\mathrm{CF}(\mathcal{H})$. Using the correspondence between central functions on H and central elements in H, we conclude that the elements $\{z_C := f_C^*\}$ form an A-basis of the centre of $\mathcal{H}$. Explicitly, we have

$$z_C = \sum_{w\in W} \operatorname{ind}(T_w)^{-1} f_C(T_w) T_{w^{-1}}.$$

Note that the class polynomials have the following properties. Let C, C' be conjugacy classes in W, and $w \in C'$. Then

$$f_{w,C} = \delta_{C,C'} \quad \text{if } w \in C'_{min}.$$

Moreover, if we specialize all parameters u_s to 1 then the function f_C specializes to the indicator function of the conjugacy class C and, hence, the element z_C specializes to the class sum of C in AW. Thus, the elements $\{z_C\}$ indeed are "generic" analogues of the class sums.

Lemma 5.1 *Let B be a commutative A-algebra and $z = \sum_{w \in W} a_w T_w \in Z(B\mathcal{H})$. Let C be a conjugacy class in W. Then the following hold:*

(1) If $w, w' \in C_{min}$ then $a_w = a_{w'}$.

(2) If $w \in C$ does not have minimal length, then there exists an element $w' \in C$ and an element $s \in S$ such that $l(w) = l(w')$, $l(sw's) = l(w') - 2$ and $a_w = a_{w'} = (1/u_s)a_{sw's} + (1 - 1/u_s)a_{sw'}$.

Proof. The coefficient a_w of T_w in z is given by $\tau(zT_w^\vee)$. Now assume that $T_v^\vee$ and $T_w^\vee$ (for $v, w \in W$) are conjugate by a unit, say $h \in B\mathcal{H}$. Using that τ is a central function and that z is a central element, we deduce that

$$a_v = \tau(zT_v^\vee) = \tau(zhT_w^\vee h^{-1}) = \tau(h^{-1}zhT_w^\vee) = \tau(zT_w^\vee) = a_w.$$

Now let $w, w' \in C$ and $x \in W$ such that $l(w) = l(w'), w' = xwx^{-1}$, and $l(xw) = l(x) + l(w)$. As in [Ge-Pf], we can compute in $B\mathcal{H}$ that $T_xT_w = T_{w'}T_x$, hence T_w and $T_{w'}$ are conjugate in $B\mathcal{H}$. A similar relation will also hold with w, w', x replaced by their inverses. Thus, $T_w^\vee$ and $T_{w'}^\vee$ are conjugate in $B\mathcal{H}$. Using [Ge-Pf, Theorem 1.1], we conclude that $T_w^\vee$ and$T_{w'}^\vee$ are conjugate in $B\mathcal{H}$, for all $w, w' \in C_{min}$. Hence, (1) is proved using the above argument.

Now let $w, w" \in W$ and $s \in S$ such that $w" = sws$ and $l(w") \leq l(w)$. As in [Ge-Pf] we see that, if $l(w) = l(w")$, the element T_w is conjugate to T_{sws}. If $l(w") = l(w) - 2$ then T_w is conjugate to $u_sT_{sws} + (u_s - 1)T_{ws}$. Again, similar relations hold with w replaced by w^{-1}. Thus, $T_w^\vee$ will be conjugate either to $T_{sws}^\vee$ or to $(1/u_s)T_{sws}^\vee + (1 - 1/u_s)T_{sw}^\vee$. Now, by [Ge-Pf, Theorem 1.1], there exist $w' \in W$ such that $l(w') = l(w)$, $T_{w'}$ is conjugate to T_w and $l(sw's) = l(w') - 2$. Hence, the argument above implies (2). □

Theorem 5.2 *Let B be a commutative A-algebra. Then, the set $\{1_B \otimes z_C\}$ (where C runs over the conjugacy classes of W) forms a B-basis for* *the centre of $B\mathcal{H}$. In particular, the centre of $B\mathcal{H}$ is free as a B-module of rank equal to the number of conjugacy classes in W and the canonical morphism $B \otimes Z(\mathcal{H}) \to Z(B\mathcal{H})$ is an isomorphism.*

Proof. Since $f_{w_C,C'} = \delta_{C,C'}$, the elements $1_B \otimes z_C$ are linearly independent in $B\mathcal{H}$. So we must show that they generate $Z(B\mathcal{H})$. The strategy for the following proof is taken from [Di-Ja].

Let $z = \sum_{w \in W} a_w T_w \in Z(B\mathcal{H})$ (where $a_w \in B$). Assume that $z \neq 0$ and let $w \in W$ be of minimal possible length such that $a_w \neq 0$. Then w lies in some conjugacy class C and we claim that $w \in C_{min}$. This can be seen as follows. Assume, if possible, that w does not have miminal length in C. By Lemma 5.1(2), there exist some $w' \in W$ and $s \in S$ such that $a_w = a_{w'} = (1/u_s)a_{sw's} + (1 - 1/u_s)a_{sw'}$. Since $l(w) = l(w')$ and $l(sw's) = l(w') - 2$, both $sw's$ and ws' have length strictly smaller than w. By the minimality of w, we conclude that $a_{sw's} = a_{sw'} = 0$ and, hence, also $a_w = 0$, a contradiction. Thus, $a_w \neq 0$ for some $w \in C_{min}$. Moreover, Lemma 5.1(1) shows that $a_w = a_{w'}$ for all $w' \in C_{min}$.

We now consider the element $z' := z - \sum_C a_{w_C} \mathrm{ind}(T_{w_C})(1_B \otimes z_C) \in Z(B\mathcal{H})$. The above mentioned properties of the elements $f_{w,C}$ show that the coefficient of T_w in z' is zero, for any element w of minimal length in any conjugacy class of W. Thus, $z' = 0$ and we are done. □

5.2 Number of simple modules for Iwahori–Hecke algebras

Consider the following polynomials associated to irreducible finite Weyl groups:

$$
\begin{aligned}
Q_{A_n} &= \prod_{i=1}^{n} [i]_x \\
Q_{B_n} &= \prod_{i=0}^{n-1} [2]_{x^i y} (x^i + y)[i]_x \\
Q_{D_n} &= 2[n]_x \prod_{i=1}^{n-1} [2i]_x \\
Q_{E_6} &= 6[2]_x[5]_x[6]_x[8]_x[9]_x[12]_x \\
Q_{E_7} &= 6[2]_x[6]_x[8]_x[10]_x[12]_x[14]_x[18]_x \\
Q_{E_8} &= 30[2]_x[8]_x[12]_x[14]_x[18]_x[20]_x[24]_x[30]_x \\
Q_{F_4} &= 6[6]_x[6]_y[2]_{xy^2}[2]_{x^2y}[2]_{xy}[2]_{x^2y^2}[2]_{x^3y^3}(x+y^2)(x^2+y)(x+y)\cdot \\
&\quad \cdot(x^2+y^2)(x^3+y^3) \\
Q_{G_2} &= 2[2]_x[2]_y[3]_{xy}(x^2+xy+y^2)
\end{aligned}
$$

where $[i]_q = 1 + q + \cdots + q^{i-1}$.

The following proposition gives a criterion for a multi-parameter Iwahori-Hecke algebra to be semisimple. It generalizes [Gy-Uno] which deals with the equal parameters case and characteristic zero fields.

Assume W is irreducible and let $\{s_1, s_2\}$ be a set of representatives of conjugacy classes of elements in S, with s_1 corresponding to a long root in type B_n. Let $B = \mathbb{Z}[\sqrt{u_s}, \sqrt{u_s}^{-1}]_{s\in S} = A[\sqrt{u_s}]_{s\in S}$ and $K = \mathbb{Q}(\sqrt{u_s})_{s\in S}$ the field of fractions of B. Note that, by [Di-Mi, théorème 3.1], the algebra $K\mathcal{H}$ is quasicentral (actually, by [Ge2], the algebra $K\mathcal{H}$ is even split, but we won't need it here).

Proposition 5.3 [Ge3] *Let k be a perfect field which is a B-algebra. Then, the algebra $k\mathcal{H}$ is semisimple if and only if $1_k \cdot Q_W(u_{s_1}, u_{s_2}) \neq 0$.*

Proof. A criterion to decide when the specialized Iwahori-Hecke algebra $k\mathcal{H}$ is semisimple is given by Proposition 4.3. One has to check that $B[\left\{\frac{1}{c_\chi}\right\}_{\chi\in\overline{\mathrm{Irr}}(K\mathcal{H})}] = B[\frac{1}{Q_W}]$. Let us define $P_W = \sum_w \mathrm{ind}(T_w)$. For χ an irreducible character of H, we put $D_\chi = P_W/c_\chi$: this is the generic degree of χ. The generic degrees are given in [Ca] and one checks easily the property above. □

To simplify the exposition, we assume now that there is a set of integers (not all zero) $\{a_s\}_{s\in S}$ (with greatest common divisor 1) such that $u_s = t^{a_s}$ where t is an indeterminate. In this case, $B = \mathbb{Z}[\sqrt{t}, \sqrt{t}^{-1}]$, and the polynomial Q_W is a product of cyclotomic polynomials in t, up to a power of t.

Let ℓ be a prime and q an integer with $\ell \not| q$. Let $e \geq 1$ be minimal with $1+q+\cdots+q^{e-1} = 0 \bmod \ell$. If d is an integer, we put $\Phi'_d = \Phi_{2d}(\sqrt{t})$. Let $\mathcal{O} = B_\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of B generated by ℓ and by $\sqrt{t} - \sqrt{q}$ if q is a square or by $t - q$ otherwise.

Let I be a prime ideal of $\mathcal{O}/(\Phi'_e)$ containing ℓ. Let $k = \mathcal{O}/\mathfrak{m} \simeq \mathbb{F}_\ell(\sqrt{q})$, $\mathfrak{p}$ be the (height one) prime ideal of $\mathcal{O}$ generated by Φ'_e, $\mathcal{O}' = (\mathcal{O}/\mathfrak{p})_I$ and $k' = (\mathcal{O}_\mathfrak{p})/\mathfrak{p}$ the field of fractions of $\mathcal{O}'$. Note that $\mathcal{O}$ is integrally closed in K and $\mathcal{O}'$ is integrally closed in k'. Furthermore, the fields K, k' and k are perfect. As explained above, the algebra KH is quasicentral.

Theorem 5.4 *Assume that ℓ and $\Phi_{e\ell^r}$ do not divide Q_W for $r > 0$. Then, the number of simple $k\mathcal{H}$-modules is equal to the number of simple $k'\mathcal{H}$-modules. More precisely, the map $1_k \otimes d_{\mathcal{O}'} : k \otimes \bar{R}_0(k'\mathcal{H}) \to k \otimes \bar{R}_0(k\mathcal{H})$ is an isomorphism.*

Proof. This is a direct consequence of Theorem 3.3; we need to check its assumptions. The canonical map $Z(\mathcal{H}) \to Z(k\mathcal{H})$ is an isomorphism by

Theorem 5.2, hence the canonical map $\mathrm{CF}(\mathcal{H}) \to \mathrm{CF}(k\mathcal{H})$ is surjective, since $\mathcal{H}$ is a symmetric algebra (cf §4).

Let now $\mathfrak{q}$ be a height one prime ideal of $\mathcal{O}$, $\mathfrak{q} \neq \mathfrak{p}$. The algebra $((\mathcal{O}_\mathfrak{q})/\mathfrak{q})\mathcal{H}$ is semisimple if and only if $Q_W \notin \mathfrak{q}$, according to Proposition 5.3. If $\ell \in \mathfrak{q}$, then $Q_W \notin \mathfrak{q}$. If $\Phi'_d \in \mathfrak{q}$, then $\Phi'_d = \Phi'_{e\ell^r}$, for some integer r, $r > 0$ since $\mathfrak{q} \neq \mathfrak{p}$. But $\Phi_{e\ell^r}$ doesn't divide Q_W, hence $\Phi'_{e\ell^r}$ doesn't divide Q_W (as a polynomial in $\sqrt{t}$), hence $Q_W \notin \mathfrak{q}$. It follows that for $\mathfrak{q} \neq \mathfrak{p}$, then $Q_W \notin \mathfrak{q}$, *i.e.*, the algebra $((\mathcal{O}_\mathfrak{q})/\mathfrak{q})\mathcal{H}$ is semisimple and we are done. □

Remark 5.5 The decomposition map $d_{\mathcal{O}'}$ is the map denoted d'_ℓ in [Ge1]. In [Ge1], it is conjectured that $d_\mathcal{O} : \bar{R}_0(k'\mathcal{H}) \to \bar{R}_0(k\mathcal{H})$ maps actually classes of simple modules to classes of simple modules.

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