Annals of Mathematics **185** (2017), 609–670
https://doi.org/10.4007/annals.2017.185.2.5

Derived categories and Deligne-Lusztig varieties II

By Cédric Bonnafé, Jean-François Dat, and Raphaël Rouquier

Abstract

This paper is a continuation and a completion of the work of the first and the third author on the Jordan decomposition. We extend the Jordan decomposition of blocks: we show that blocks of finite groups of Lie type in nondescribing characteristic are Morita equivalent to blocks of subgroups associated to isolated elements of the dual group — this is the modular version of a fundamental result of Lusztig, and the best approximation of the character-theoretic Jordan decomposition that can be obtained via Deligne-Lusztig varieties. The key new result is the invariance of the part of the cohomology in a given modular series of Deligne-Lusztig varieties associated to a given Levi subgroup, under certain variations of parabolic subgroups.

We also bring in local block theory methods: we show that the equivalence arises from a splendid Rickard equivalence. Even in the setting of the original work of the first and the third author, the finer homotopy equivalence was unknown. As a consequence, the equivalences preserve defect groups and categories of subpairs. We finally determine when Deligne-Lusztig induced representations of tori generate the derived category of representations. An additional new feature is an extension of the results to disconnected reductive algebraic groups, which is required to handle local subgroups.

Contents

The first author is partly supported by the ANR (Project No ANR-12-JS01-0003-01 ACORT). The second author thanks the Institut Universitaire de France and the project ANR-14-CE25-0002-01 PerCoLaTor for their support. The third author is partly supported by the NSF (grant DMS-1161999) and by a grant from the Simons Foundation (#376202, Raphaël Rouquier).

1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of a finite field, endowed with an endomorphism F, a power of which is a Frobenius endomorphism. Let ℓ be a prime number distinct from the defining characteristic of $\mathbf{G}$ and K a finite extension of $\mathbb{Q}_\ell$, large enough for the finite groups considered. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_\ell$ and k the residue field. We will denote by Λ a ring that is either K, $\mathcal{O}$ or k.

The main tool for the study of representations of $\mathbf{G}^F$ over Λ is the Deligne-Lusztig induction. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ with unipotent radical $\mathbf{V}$ so that $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$. Consider the Deligne-Lusztig variety

$$\mathbf{Y_P} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V} \cdot F(\mathbf{V})\}.$$

It has a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$ by multiplication. The corresponding complex of ℓ-adic cohomology induces a triangulated functor

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : D^b(\Lambda\mathbf{L}^F) \to D^b(\Lambda\mathbf{G}^F),\ M \mapsto R\Gamma_c(\mathbf{Y_P}, \Lambda) \otimes^{\mathbb{L}}_{\Lambda\mathbf{L}^F} M$$

and a morphism

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = [\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}] : G_0(\Lambda\mathbf{L}^F) \to G_0(\Lambda\mathbf{G}^F).$$

This is the usual Harish-Chandra construction when $\mathbf{P}$ is F-stable.

1.A. *Jordan decomposition.* Let $\mathbf{G}^*$ be a group Langlands dual to $\mathbf{G}$, with Frobenius F^*. Consider the set $\mathrm{Irr}(\mathbf{G}^F)$ of characters of irreducible representations of $\mathbf{G}^F$ over K. Deligne and Lusztig gave a decomposition of $\mathrm{Irr}(\mathbf{G}^F)$ into rational series

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_{(s)} \mathrm{Irr}(\mathbf{G}^F, (s)),$$

where (s) runs over the set of $\mathbf{G}^{*F^*}$-conjugacy classes of semi-simple elements of $\mathbf{G}^{*F^*}$. The *unipotent characters* of $\mathbf{G}^F$ are those in $\mathrm{Irr}(\mathbf{G}^F, 1)$.

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ with dual $\mathbf{L}^* \subset \mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$. Lusztig constructed a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (s)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{G}^F, (s)),\ \psi \mapsto \pm R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\psi).$$

If $s \in Z(\mathbf{L}^*)$, then there is a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (1)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{L}^F, (s)),\ \psi \mapsto \eta\psi,$$

where η is the one-dimensional character of $\mathbf{L}^F$ corresponding to s, and we obtain a bijection

$$\mathrm{Irr}(\mathbf{L}^F, (1)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{G}^F, (s)).$$

This provides a description of irreducible characters of $\mathbf{G}^F$ in the rational series (s) in terms of unipotent characters of another group, when $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$.

Let us now consider the modular version of the theory described above. Let s be a semi-simple element of $\mathbf{G}^{*F^*}$ of order prime to ℓ. Let us consider $\coprod_t \mathrm{Irr}(\mathbf{G}^F, (t))$, where (t) runs over conjugacy classes of semi-simple elements of $\mathbf{G}^{*F^*}$ whose ℓ'-part is (s). Broué and Michel [BM89] have shown this is a union of blocks of $\mathcal{O}\mathbf{G}^F$. The sum of the corresponding block idempotents is an idempotent $e_s^{\mathbf{G}^F} \in Z(\mathcal{O}\mathbf{G}^F)$, and we obtain a decomposition

$$\mathcal{O}\mathbf{G}^F\text{-mod} = \bigoplus_{(s)} \mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}\text{-mod},$$

where (s) runs over $\mathbf{G}^{*F^*}$-conjugacy classes of semi-simple ℓ'-elements of $\mathbf{G}^{*F^*}$.

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ with dual $\mathbf{L}^*$ containing $C_{\mathbf{G}^*}(s)$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{V}$ and Levi complement $\mathbf{L}$. Broué [Bro90] conjectured that the $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$-bimodule

$$\mathrm{H}^{\dim \mathbf{Y_P}}(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$$

induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$. This was proven by Broué [Bro90] when $\mathbf{L}$ is a torus and in [BR03] in general.

Broué also conjectured that the truncated complex of cohomology

$$\mathrm{G}\Gamma_c(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$$

(Rickard's refinement of $\mathrm{R}\Gamma_c(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$, well defined in the homotopy category [Ric94]) induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F}$: it induces not only an equivalence of derived categories, but even an equivalence of homotopy categories, and it induces a similar equivalence for centralizers of ℓ-subgroups. One of our main results here is a proof of that conjecture. In order to show that there is a homotopy equivalence, for connected groups, we show that the global functor induces local derived equivalences for centralizers of ℓ-subgroups. Since such centralizers need not be connected, we need to extend the results of [BR03] to disconnected groups. So, part of this work involves working with disconnected groups.

We also extend the "Jordan decomposition equivalences" (Morita and splendid Rickard) to the "quasi-isolated case": assume now only $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, and that $\mathbf{L}^*$ is minimal with respect to this property. We show that the right action of $\mathbf{L}^F$ on $\mathrm{H}^{\dim \mathbf{Y_P}}(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an action of $N = N_{\mathbf{G}^F}(\mathbf{L}, e_s^{\mathbf{L}^F})$ commuting with the action of $\mathbf{G}^F$, and the resulting bimodule induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$. Similarly, the complex

$$\mathrm{G}\Gamma_c(\mathbf{Y_V}, \mathcal{O})e_s^{\mathbf{G}^F}$$

induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$.

As a consequence, we deduce that the bijection between blocks of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}N e_s^{\mathbf{L}^F}$ preserves the local structure and, in particular, preserves defect groups. Cabanes and Enguehard have proven this under some assumptions on ℓ [CE99, Prop. 5.1], and Kessar and Malle in the setting of [BR03], when one of the blocks under consideration has abelian defect groups (modulo a central ℓ-subgroup) [KM13, Th. 1.3], an important step in their proof of half of Brauer's height zero conjecture for all finite groups [KM13, Th. 1.1] and the second half for quasi-simple groups [KM15, Main Theorem].

Let us summarize this.

THEOREM 1.1. *Assume $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$ and that $\mathbf{L}^*$ is minimal with respect to this property.*

The right action of $\mathbf{L}^F$ on $\mathrm{G}\Gamma_c(\mathbf{Y_P}, \mathcal{O})e_s^{\mathbf{L}^F}$ extends to an action of N, and the resulting complex C induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}Ne_s^{\mathbf{L}^F}$. The bimodule $\mathrm{H}^{\dim \mathbf{Y_P}}(C)$ induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}Ne_s^{\mathbf{L}^F}$.

The bijections between blocks of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}Ne_s^{\mathbf{L}^F}$ induced by those equivalences preserve the local structure.

Significant progress has been made recently on counting conjectures for finite groups, using the classification of finite simple groups, and [BR03] has proved very useful. We hope this theorem will lead to simplifications and new results.

The character-theoretic consequence of this theorem is that, for groups with disconnected center, the Jordan decomposition shares many of the properties of that for the connected case. In type A, the Jordan decomposition of characters links all series to unipotent series of smaller groups: even in that case, the good behavior of those correspondences was known only when q is large (Bonnafé [Bon06] for $\mathbf{SL}_n$ and Cabanes [Cab13] for $\mathbf{SU}_n$).

1.B. *Generation of the derived category.* One of the two key steps in [BR03] was the proof that the category of perfect complexes for $\mathcal{O}\mathbf{G}^F$ is generated by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B})$, where $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}$ with an F-stable maximal torus. We show here a more precise result of generation of the derived category of $\mathcal{O}\mathbf{G}^F$. Let $\mathcal{E}$ be the set $\{\mathrm{R}\Gamma_c(\mathbf{Y_B}) \otimes^{\mathbf{L}}_{\mathcal{O}\mathbf{T}^F} M\}$, where $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ over Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$, and M over isomorphism classes of $\mathcal{O}\mathbf{T}^F$-modules.

THEOREM 1.2. *The set $\mathcal{E}$ generates $D^b(\mathcal{O}\mathbf{G}^F)$* (*as a thick subcategory*) *if and only if all elementary abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in tori.*

This, in turn, requires an extension of the results of Broué-Michel [BM89] on the compatibility between Deligne-Lusztig series of characters and the Brauer morphism, to disconnected groups. We are able to achieve this by refining our result on the generation of the category of perfect complexes to a generation of the category of ℓ-permutation modules whose vertices are contained in tori. (The crucial case is that of connected groups.) Such a result allows us to obtain a generating result for the full derived category, under the assumption that all elementary abelian ℓ-subgroups are contained in tori.

Note that the condition is automatically satisfied for $\mathbf{GL}_n$ and $\mathbf{U}_n$ (see Examples 3.17) and when ℓ is *very good* for $\mathbf{G}$.

1.C. *Independence of the Deligne-Lusztig induction of the parabolic in a given series.* It is known in most cases, and conjectured in general, that the map $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ on Grothendieck groups is actually independent of $\mathbf{P}$ ([DL76, Lus78] when $\mathbf{L}$ is a torus and [BM11] when $q > 2$ and F is a Frobenius endomorphism over $\mathbf{F}_q$). On the other hand, the functor $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ does depend on $\mathbf{P}$. Our main new geometrical result proves the independence after truncating by a suitable series.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two parabolic subgroups admitting a common Levi complement $\mathbf{L}$. Denote by $\mathbf{V}_i^*$ the unipotent radical of the parabolic subgroup of $\mathbf{G}^*$ corresponding to $\mathbf{P}_i$.

THEOREM 1.3. *Let s be a semi-simple element of $\mathbf{L}^{*F^*}$ of order prime to ℓ. If*

$$C_{\mathbf{V}_1^*\cap {}^{F^*}\mathbf{V}_1^*}(s) \subset C_{\mathbf{V}_2^*}(s) \quad \textit{and} \quad C_{\mathbf{V}_2^*\cap {}^{F^*}\mathbf{V}_2^*}(s) \subset C_{{}^{F^*}\mathbf{V}_1^*}(s),$$

then there is an isomorphism of functors between

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_1} : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$$

and

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_2}[m] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}),$$

where $m = \dim(\mathbf{Y}^{\mathbf{G}}_{\mathbf{P}_2}) - \dim(\mathbf{Y}^{\mathbf{G}}_{\mathbf{P}_1})$.

For instance, if $C_{\mathbf{V}_1^*}(s) = C_{\mathbf{V}_2^*}(s)$, then the assumption of Theorem 1.3 is satisfied.

This is the key result to prove Theorem 1.1. This result shows that when $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, the $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$-bimodule $\mathrm{H}^{\dim \mathbf{Y}_\mathbf{P}}(\mathbf{Y}_\mathbf{P}, \mathcal{O})e_s^{\mathbf{L}^F}$ is independent of $\mathbf{P}$, a question left open in [BR03]. We deduce that the bimodule is stable under the action of $N = N_{\mathbf{G}^F}(\mathbf{L}^F, e_s^{\mathbf{L}^F})$. Using an embedding in a group with connected center, we show that the obstruction for extending the action of $\mathbf{L}^F$ to N does vanish.

Remark. Theorem 1.3 is used in [Dat16] to construct equivalences of categories between tamely ramified blocks of p-adic general linear groups. Roughly speaking, the main idea of [Dat16] is to “glue” the bimodules giving the Morita equivalences of [BR03] along a suitable building. The gluing process crucially uses the independence of the bimodules on the choice of parabolic subgroups.

1.D. *Structure of the article.* We begin in Section 3 with the study of generation of the category of perfect complexes, then we move to complexes of ℓ-permutation modules, and finally we derive our result on the derived category. A key tool, due to Rickard, is that the Brauer functor applied to the complex of cohomology of a variety is the complex of cohomology of the fixed point variety.

Section 4 is devoted to the study of rational series and their compatibility with local block theory. Broué and Michel proved a commutation formula between generalized decomposition maps and Deligne-Lusztig induction. We need to extend the compatibility between Brauer and Deligne-Lusztig theory to disconnected groups and check that the local blocks obtained from a series satisfying $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$ also satisfy a similar assumption $C^\circ_{(C^\circ_{\mathbf{G}}(Q))^*}(s) \subset (\mathbf{L} \cap C^\circ_{\mathbf{G}}(Q))^*$.

From Section 5 onwards, the group $\mathbf{G}$ is assumed to be connected. Sections 5 and 6 are devoted to the study of the dependence of the Deligne-Lusztig induction with respect to the parabolic subgroup. The first section is devoted to the particular case of varieties associated with Borel subgroups (and generalizations involving sequences of elements). It is convenient there to work with a reference maximal torus. This is the crucial case, from which the general one is deduced in the latter section, where we go back to Levi subgroups that do not necessarily contain that fixed maximal torus.

Section 7 is devoted to the Jordan decomposition. We start by providing an extension of the action of N on the cohomology bimodule by proving that the cocycle obstruction would survive in a similar setting for a group with connected center, where the action does exist. The Rickard equivalence is obtained inductively, and that induction requires working with disconnected groups.

In an appendix, we provide some results on the homotopy category of complexes of ℓ-permutation modules for a general finite group.

We would like to thank the Referee for an extraordinarily thorough list of suggestions, which greatly improved our paper.

2. Notation

2.A. *Modules.* Let ℓ be a prime number, K a finite extension of $\mathbb{Q}_\ell$ large enough for the finite groups considered, $\mathcal{O}$ its ring of integers over $\mathbb{Z}_\ell$ and k its residue field. We will denote by Λ a ring that is either K, $\mathcal{O}$ or k.

Given $\mathcal{C}$ an additive category, we denote by $\mathrm{Comp}^b(\mathcal{C})$ the category of bounded complexes of objects of $\mathcal{C}$ and by $\mathrm{Ho}^b(\mathcal{C})$ its homotopy category.

Let A be a Λ-algebra, finitely generated and projective as a Λ-module. We denote by A^{opp} the algebra opposite to A. We denote by A-mod the category of finitely generated A-modules and by A-proj its full subcategory of projective modules. We denote by $G_0(A)$ the Grothendieck group of A-mod.

We put $\mathrm{Comp}^b(A) = \mathrm{Comp}^b(A\text{-mod})$, $D^b(A) = D^b(A\text{-mod})$ and $\mathrm{Ho}^b(A) = \mathrm{Ho}^b(A\text{-mod})$. We denote by A-perf $\subset D^b(A)$ the thick full subcategory of perfect complexes (complexes quasi-isomorphic to objects of $\mathrm{Comp}^b(A\text{-proj})$).

Let $C \in \mathrm{Comp}^b(A)$. There is a unique (up to a nonunique isomorphism) complex C^{red} that is isomorphic to C in the homotopy category $\mathrm{Ho}^b(A)$ and that has no nonzero direct summand that is homotopy equivalent to 0. Note that $C \simeq C^{\mathrm{red}} \oplus C'$ for some C' homotopy equivalent to zero.

We denote by $\mathrm{End}^\bullet_A(C)$ the total Hom-complex, with degree n term $\bigoplus_{j-i=n} \mathrm{Hom}_A(C^i, C^j)$.

Let B be Λ-algebra, finitely generated and projective as a Λ-module. Let C be a bounded complex of $(A \otimes_\Lambda B^{\mathrm{opp}})$-modules, finitely generated and projective as left A-modules and as right B-modules. We say that C induces a *Rickard equivalence* between A and B if the canonical map $B \to \mathrm{End}^\bullet_A(C)$ is an isomorphism in $\mathrm{Ho}(B \otimes_\Lambda B^{\mathrm{opp}})$ and the canonical map $A \to \mathrm{End}^\bullet_{B^{\mathrm{opp}}}(C)^{\mathrm{opp}}$ is an isomorphism in $\mathrm{Ho}(A \otimes_\Lambda A^{\mathrm{opp}})$.

2.B. *Finite groups.* Let G be a finite group. We denote by G^{opp} the opposite group to G. We put $\Delta G = \{(g, g^{-1}) | g \in G\} \subset G \times G^{\mathrm{opp}}$. Given $g \in G$, we denote by $|g|$ the order of g.

Let H be a subgroup of G and $x \in G$. We denote by x_* the equivalence of categories

$$x_* : \Lambda(x^{-1}Hx)\text{-mod} \xrightarrow{\sim} \Lambda H\text{-mod},$$

where $x_*(M) = M$ as a Λ-module and the action of $h \in H$ on $x_*(M)$ is given by the action of $x^{-1}hx$ on M. We also denote by x_* the corresponding isomorphism of Grothendieck groups

$$x_* : G_0(\Lambda(x^{-1}Hx)) \xrightarrow{\sim} G_0(\Lambda H).$$

We assume $\Lambda = \mathcal{O}$ or $\Lambda = k$ in the remainder of Section 2.B.

An *ℓ-permutation ΛG-module* is defined to be a direct summand of a finitely generated permutation module. We denote by ΛG-perm the full subcategory of ΛG-mod with objects the ℓ-permutation ΛG-modules.

Let Q be an ℓ-subgroup Q of G. We consider the *Brauer functor* $\mathrm{Br}_Q : \Lambda G\text{-perm} \to k[N_G(Q)/Q]\text{-perm}$. Given $M \in \Lambda G$-perm, we define $\mathrm{Br}_Q(M)$ as the image of M^Q in $(kM)_Q$, where $(kM)_Q$ is the largest quotient of $kM = k \otimes_\Lambda M$ on which Q acts trivially.

We denote by $\mathrm{br}_Q : (\Lambda G)^Q \to kC_G(Q)$ the algebra morphism given by $\mathrm{br}_Q(\sum_{g \in G} \lambda_g g) = \sum_{g \in C_G(Q)} \lambda_g g$, where $\lambda_g \in \Lambda$ for $g \in G$. Given $M \in \Lambda G$-perm and $e \in Z(\Lambda G)$ an idempotent, we have $\mathrm{Br}_Q(Me) = \mathrm{Br}_Q(M)\mathrm{br}_Q(e)$.

Let H be a subgroup of G, let b be an idempotent of $Z(\Lambda G)$ and c an idempotent of $Z(\Lambda H)$. Let $C \in \mathrm{Comp}^b(\Lambda G b \otimes (\Lambda H c)^{\mathrm{opp}})$. We say that C is *splendid* if the $(C^{\mathrm{red}})^i$'s are ℓ-permutation modules whose indecomposable direct summands have a vertex contained in ΔH.

2.C. *Varieties.* Let p be a prime number different from ℓ and $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$. By variety, we mean a quasi-projective algebraic variety over $\mathbb{F}$.

Let $\mathbf{X}$ be a variety acted on by a finite group G. There is an object $\mathrm{G}\Gamma_c(\mathbf{X},\Lambda)$ of $\mathrm{Ho}^b(\Lambda G\text{-perm})$, well defined up to a unique isomorphism. It is a representative in the homotopy category of ΛG-modules of the isomorphism class of the complex of étale Λ-cohomology with compact support of $\mathbf{X}$ constructed as $\tau_{\leqslant 2\dim \mathbf{X}}$ of the Godement resolution (cf. [Rou02, §2], [DR14, §1.2], and [Ric94]). We denote by $\mathrm{R}\Gamma_c(\mathbf{X},\Lambda)$ the image of $\mathrm{G}\Gamma_c(\mathbf{X},\Lambda)$ in $D^b(\Lambda G)$.

Assume $\Lambda = \mathcal{O}$ or k, and let Q be an ℓ-subgroup of G. The inclusion $\mathbf{X}^Q \hookrightarrow \mathbf{X}$ induces an isomorphism [Ric94, Th. 4.2]

$$\mathrm{G}\Gamma_c(\mathbf{X}^Q,k) \xrightarrow{\sim} \mathrm{Br}_Q(\mathrm{G}\Gamma_c(\mathbf{X},\Lambda)) \text{ in } \mathrm{Ho}^b(kN_G(Q)\text{-perm}).$$

2.D. *Reductive groups.* Let $\mathbf{G}$ be a (possibly disconnected) reductive algebraic group endowed with an endomorphism F, a power F^δ of which is a Frobenius endomorphism defining a rational structure over a finite field $\mathbb{F}_q$ of characteristic p. We refer to [DM94], [DM15] for basic results on disconnected groups.

Recall that a torus of $\mathbf{G}$ is torus of $\mathbf{G}^\circ$. Following the classical terminology (cf., for example, [Spr98, §6.2]), we define a *Borel subgroup* of $\mathbf{G}$ to be a maximal connected solvable subgroup of $\mathbf{G}$. We define a *parabolic subgroup* of $\mathbf{G}$ to be a subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\mathbf{G}/\mathbf{P}$ is complete. We define the *unipotent radical* $\mathbf{V}$ of a parabolic subgroup $\mathbf{P}$ to be its unique maximal connected unipotent normal subgroup. A *Levi complement* to $\mathbf{V}$ in $\mathbf{P}$ is a subgroup $\mathbf{L}$ of $\mathbf{P}$ such that $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$.

Note that a closed subgroup $\mathbf{P}$ of $\mathbf{G}$ is a parabolic subgroup of $\mathbf{G}$ if and only if $\mathbf{P}^\circ$ is a parabolic subgroup of $\mathbf{G}^\circ$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. We have $\mathbf{P}^\circ = \mathbf{P} \cap \mathbf{G}^\circ$. The unipotent radical $\mathbf{V}$ of $\mathbf{P}$ coincides with that of $\mathbf{P}^\circ$. A Levi complement to $\mathbf{V}$ in $\mathbf{P}$ is a subgroup of the form $\mathbf{L} = N_{\mathbf{P}}(\mathbf{L}_\circ)$, where $\mathbf{L}_\circ$ is a Levi complement of $\mathbf{V}$ in $\mathbf{P}_\circ$. (Then $\mathbf{L}^\circ = \mathbf{L}_\circ$ and $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$.) Note that our definition of parabolic subgroup is more general than that of "parabolic" subgroup of [DM94], which requires $\mathbf{P} = N_{\mathbf{G}}(\mathbf{P}^\circ)$.

We denote by $\nabla(\mathbf{G},F)$ the set of pairs $(\mathbf{T},\theta)$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and θ is an irreducible character of $\mathbf{T}^F$. Note that here $\mathbf{T}$ is a torus of $\mathbf{G}^\circ$.

Given an integer d, we denote by $\nabla_{d'}(\mathbf{G},F)$ the set of pairs $(\mathbf{T},\theta) \in \nabla(\mathbf{G},F)$ such that the order of θ is prime to d. We put $\nabla_\Lambda(\mathbf{G},F) = \nabla(\mathbf{G},F)$ if $\Lambda = K$ and $\nabla_\Lambda(\mathbf{G},F) = \nabla_{\ell'}(\mathbf{G},F)$ if $\Lambda = \mathcal{O}$ or k. (Recall that k is a field of characteristic ℓ.)

2.E. *Deligne-Lusztig varieties.* Given $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{V}$ and F-stable Levi complement $\mathbf{L}$, we define the Deligne-Lusztig variety

$$\mathbf{Y}_{\mathbf{V}} = \mathbf{Y}_{\mathbf{V}}^{\mathbf{G}} = \mathbf{Y}_{\mathbf{P}} = \mathbf{Y}_{\mathbf{P}}^{\mathbf{G}} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V}\cdot F(\mathbf{V})\}.$$

This is a smooth variety, as in the case of connected reductive groups. It has a left action by multiplication of $\mathbf{G}^F$ and a right action by multiplication of $\mathbf{L}^F$. (Note that the left and right actions of $Z(\mathbf{G})^F$ coincide.) This provides a triangulated functor

$$\begin{array}{llcc} \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}: & D^b(\Lambda\mathbf{L}^F) & \longrightarrow & D^b(\Lambda\mathbf{G}^F) \\ & M & \longmapsto & R\Gamma_c(\mathbf{Y}_{\mathbf{V}},\Lambda)\otimes^{\mathbb{L}}_{\Lambda\mathbf{L}^F} M \end{array} \tag{2.1}$$

and a morphism

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = [\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}] : G_0(\Lambda\mathbf{L}^F) \to G_0(\Lambda\mathbf{G}^F).$$

We put $\mathbf{X}_{\mathbf{P}}^{\mathbf{G}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}\cdot F(\mathbf{P})\} = \mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}/\mathbf{L}^F$.

Remark 2.2. Since $\mathbf{Y}_{\mathbf{P}}$ depends only on $\mathbf{V}$, it is endowed with an action of $N_{\mathbf{G}^F}(\mathbf{P}^\circ,\mathbf{L}^\circ)$, which is the group of rational points of the maximal Levi subgroup with connected component $\mathbf{L}^\circ$.

3. **Generation**

The aim of this section is to extend [BR03, Th. A] to the case of disconnected groups and to deduce a generation theorem for the derived category.

In this section, $\mathbf{G}$ is a (possibly disconnected) reductive algebraic group.

3.A. *Centralizers of ℓ-subgroups.* Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi complement $\mathbf{L}$, and let $\mathbf{V}$ denote the unipotent radical of $\mathbf{P}$. It is easily checked [DM94, proof of Prop. 2.3] that

$$\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}} = \coprod_{g\in\mathbf{G}^F/\mathbf{G}^{\circ F}} g\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}^\circ} = \mathbf{G}^F \times_{\mathbf{G}^{\circ F}} \mathbf{Y}_{\mathbf{V}}^{\mathbf{G}^\circ}. \tag{3.1}$$

It follows immediately from (3.1) that

$$\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ \mathrm{Ind}_{\mathbf{L}^{\circ F}}^{\mathbf{L}^F} \simeq \mathcal{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^{\mathbf{G}} \simeq \mathrm{Ind}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \circ \mathcal{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^{\mathbf{G}^\circ}. \tag{3.2}$$

If $\mathbf{G} = \mathbf{P}\cdot\mathbf{G}^\circ$, then the isomorphism $\mathbf{G}^\circ/\mathbf{V} \times_{\mathbf{L}^\circ} \mathbf{L} \simeq \mathbf{G}/\mathbf{V}$ induces an isomorphism

$$\mathcal{R}_{\mathbf{L}^\circ\subset\mathbf{P}^\circ}^{\mathbf{G}^\circ} \circ \mathrm{Res}_{\mathbf{L}^{\circ F}}^{\mathbf{L}^F} \simeq \mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \circ \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}. \tag{3.3}$$

PROPOSITION 3.4. *Let Q be a finite solvable p'-group of automorphisms of $\mathbf{G}$ that commute with F and normalize $(\mathbf{P},\mathbf{L})$.*

(a) *The group $\mathbf{G}^Q$ is reductive.*

(b) *$\mathbf{P}^Q$ is a parabolic subgroup of $\mathbf{G}^Q$ whose unipotent radical is $\mathbf{V}^Q$ and admitting $\mathbf{L}^Q$ as an F-stable Levi complement. In particular, $\mathbf{V}^Q$ is connected.*

(c) *The natural map $\mathbf{G}^Q/\mathbf{V}^Q \to (\mathbf{G}/\mathbf{V})^Q$ of $(\mathbf{G}^Q, N_{\mathbf{G}}(\mathbf{P}^\circ,\mathbf{L}^\circ)^Q)$-varieties is an isomorphism.*

(d) $(\mathbf{V}\cdot {}^F\mathbf{V})^Q = \mathbf{V}^Q\cdot {}^F(\mathbf{V}^Q)$.

(e) *The natural map* $\mathbf{Y}_{\mathbf{V}^Q}^{\mathbf{G}^Q} \to (\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^Q$ *of* $((\mathbf{G}^Q)^F, (N_{\mathbf{G}}(\mathbf{P}^\circ, \mathbf{L}^\circ)^Q)^F)$*-varieties is an isomorphism. If* Q *is an* ℓ*-group, it gives rise to an isomorphism* $\mathrm{Br}_Q(G\Gamma_c(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}}, k)) \xrightarrow{\sim} G\Gamma_c(\mathbf{Y}_{\mathbf{V}^Q}^{\mathbf{G}^Q}, k)$ *in* $\mathrm{Ho}^b(k((\mathbf{G}^Q)^F \times (N_{\mathbf{G}}(\mathbf{P}^\circ, \mathbf{L}^\circ)^Q)^{F\mathrm{opp}})$.

Proof. Assume first that Q is cyclic, generated by an element l. (a) and (b) follow from [DM94, Prop. 1.3, Th. 1.8, Prop. 1.11].

(c) Note that that both varieties are smooth. (For $(\mathbf{G}/\mathbf{V})^Q$, this follows from the fact that Q is a p'-group and $\mathbf{G}/\mathbf{V}$ is smooth.) The injectivity of the map is clear.

Let us prove the surjectivity. Let $g\mathbf{V} \in (\mathbf{G}/\mathbf{V})^Q$. Then, $g^{-1}l(g) \in \mathbf{V}$. Denote by $\mathrm{ad}(g)$ the automorphism $x \mapsto gxg^{-1}$ of $\mathbf{G}$. Since $\mathrm{ad}(g)^{-1} l\, \mathrm{ad}(g)$ is semisimple, it stabilizes a maximal torus of $\mathbf{P}^\circ$ (see [Ste68, Th. 7.5]), and hence it stabilizes the unique Levi complement $\mathbf{L}'$ of $\mathbf{P}^\circ$ containing this maximal torus. Since all Levi complements are conjugate under the action of $\mathbf{V}$, there exists $v \in \mathbf{V}$ such that $v^{-1}\mathbf{L}'v = \mathbf{L}^\circ$. It follows that $(gv)^{-1}l(gv) \in \mathbf{V}$ and $(gv)^{-1}l(gv)$ normalizes $\mathbf{L}^\circ$, hence $(gv)^{-1}l(gv) = 1$, so $gv \in \mathbf{G}^Q$, as desired.

The tangent space at $\mathbf{V}$ of $(\mathbf{G}/\mathbf{V})^Q$ is the Q-invariant part of the tangent space of $\mathbf{G}/\mathbf{V}$ at $\mathbf{V}$. That last tangent space is a quotient of the tangent space of $\mathbf{G}$ at the origin. It follows that the canonical map $\mathbf{G}^Q \to (\mathbf{G}/\mathbf{V})^Q$ induces a surjective map between tangent spaces at the origin. Consequently, the canonical map $\mathbf{G}^Q/\mathbf{V}^Q \to (\mathbf{G}/\mathbf{V})^Q$ induces a surjective map between tangent spaces at the origin. We deduce that the map is an isomorphism.

(d) The number of F-stable maximal tori of $\mathbf{L}$ is a power of p (see [Ste68, Cor. 14.16]). Since Q is a p'-group, it normalizes some F-stable maximal torus. Using now the root system with respect to this maximal torus, we deduce that there exists a Q-stable subgroup $\mathbf{V}'$ of $\mathbf{V}$ such that $\mathbf{V} = \mathbf{V}' \cdot (\mathbf{V} \cap F(\mathbf{V}))$ and $\mathbf{V}' \cap F(\mathbf{V}) = 1$. Therefore, $\mathbf{V} \cdot F(\mathbf{V}) = \mathbf{V}' \cdot F(\mathbf{V})$ and the result follows.

(e) follows immediately from (c) and (d).

We prove now the proposition by induction on $|Q|$. Let Q_1 be a normal subgroup of Q of index a prime number and let $l \in Q$, $l \notin Q_1$. Let Q_2 be the subgroup of Q generated by l. By induction, the proposition holds for Q replaced by Q_1: we have a reductive group $\mathbf{G}_1 = \mathbf{G}^{Q_1}$ and a parabolic subgroup $\mathbf{P}_1 = \mathbf{P}^{Q_1}$ with unipotent radical $\mathbf{V}_1 = \mathbf{V}^{Q_1}$ and an F-stable Levi complement $\mathbf{L}_1 = \mathbf{L}^{Q_1}$. These are all stable under Q_2. The cyclic case of the proposition applied to the action of Q_2 on $(\mathbf{G}_1, \mathbf{P}_1, \mathbf{V}_1, \mathbf{L}_1)$ establishes the proposition for the action of Q on $(\mathbf{G}, \mathbf{P}, \mathbf{V}, \mathbf{L})$. □

Remark 3.5. If Q is a finite solvable p'-subgroup of $\mathbf{G}^F$, then $N_{\mathbf{G}}(Q)$ is reductive. If in addition Q normalizes $(\mathbf{P}, \mathbf{L})$, then $N_{\mathbf{P}}(Q)$ is a parabolic subgroup of $N_{\mathbf{G}}(Q)$ with unipotent radical $\mathbf{V}^Q$ and Levi complement $N_{\mathbf{L}}(Q)$. The maps defined in (e) of Proposition 3.4 are equivariant for the diagonal action of $N_{\mathbf{G}}(\mathbf{P}^\circ, \mathbf{L}^\circ, Q)^F$.

We will need a converse to Proposition 3.4, in the case of tori.

LEMMA 3.6. *Let Q be a finite solvable p'-group of automorphisms of $\mathbf{G}$ that commute with F. We assume Q stabilizes a maximal torus of $\mathbf{G}$ and a Borel subgroup containing that maximal torus.*

Let $\mathbf{T}_Q$ be an F-stable maximal torus of $\mathbf{G}^Q$ contained in a Borel subgroup $\mathbf{B}_Q$ of $\mathbf{G}^Q$. Then, $C_{\mathbf{G}^\circ}(\mathbf{T}_Q)$ is an F-stable maximal torus of $\mathbf{G}$ that is contained in a Q-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $(\mathbf{B}^Q)^\circ = \mathbf{B}_Q$.

Proof. Note that the lemma holds for Q cyclic by [DM94, Th. 1.8]. We proceed by induction on $|Q|$ as in the proof of Proposition 3.4, and we keep the notation of that proof. We know (the lemma for Q_2) that $\mathbf{T}_{Q_1} = C_{\mathbf{G}_1^\circ}(\mathbf{T}_Q)$ is an F-stable maximal torus of $\mathbf{G}_1^\circ$. By induction, $C_{\mathbf{G}^\circ}(\mathbf{T}_Q) = N_{\mathbf{G}^\circ}(\mathbf{T}_{Q_1})^\circ = C_{\mathbf{G}^\circ}(\mathbf{T}_{Q_1})$ is an F-stable maximal torus of $\mathbf{G}$.

The existence of the Borel subgroup can be obtained as in [DM94, p. 350]. Let $\mathbf{T}'$ be a maximal torus of $\mathbf{G}$ stable under Q and $\mathbf{B}'$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}'$ and stable under Q. By Proposition 3.4, $(\mathbf{T}'^Q)^\circ$ is a maximal torus of $\mathbf{G}^Q$ and $(\mathbf{B}'^Q)^\circ$ is a Borel subgroup containing it. So, there is $x \in (\mathbf{G}^Q)^\circ$ such that $\mathbf{T}_Q = {}^x(\mathbf{T}'^Q)^\circ$ and $\mathbf{B}_Q = {}^x(\mathbf{B}'^Q)^\circ$. Let $\mathbf{B} = {}^x\mathbf{B}'$. This is a Q-stable Borel subgroup of $\mathbf{G}$ containing $C_{\mathbf{G}^\circ}(\mathbf{T}_Q)$. By Proposition 3.4, $(\mathbf{B}^Q)^\circ$ is a Borel subgroup of $\mathbf{G}^Q$, hence $(\mathbf{B}^Q)^\circ = \mathbf{B}_Q$. □

To complete Proposition 3.4, note the following result.

LEMMA 3.7. *Let P be an ℓ-subgroup of $\mathbf{G}^F \times N_{\mathbf{G}^F}(\mathbf{P},\mathbf{L})^{\mathrm{opp}}$ such that $(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^P \neq \varnothing$. Then P is $(\mathbf{G}^F \times 1)$-conjugate to a subgroup of $\Delta N_{\mathbf{G}^F}(\mathbf{P},\mathbf{L})$.*

Proof. Replacing $\mathbf{L}$ by $N_{\mathbf{G}}(\mathbf{P},\mathbf{L})$, we can assume that $N_{\mathbf{G}}(\mathbf{P},\mathbf{L}) = \mathbf{L}$. Let $Q \subset \mathbf{L}^F$ (resp. $R \subset \mathbf{G}^F$) denote the image of P through the second (resp. first) projection, and let $y\mathbf{V} \in (\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^P$.

If $g \in R$, then there exists $l \in Q$ such that $(g,l) \in P$. Therefore, $gyl\mathbf{V} = y\mathbf{V}$, hence $y^{-1}gy\mathbf{V} = l^{-1}\mathbf{V}$. This implies that $y^{-1}Ry \subset Q\mathbf{V}$. We denote by $\eta : R \to Q$ the composition $R \xrightarrow{\sim} y^{-1}Ry \hookrightarrow Q\mathbf{V} \twoheadrightarrow Q$. Since R (resp. Q) acts freely on $\mathbf{G}/\mathbf{V}$ as they are ℓ-groups, the previous computation shows that η is an isomorphism and that

$$P = \{(g,\eta(g)) \mid g \in R\}.$$

Now, there exists a positive integer m such that $F^m(\mathbf{P}) = \mathbf{P}$ and $y^{-1}Ry \subset \mathbf{P}^{F^m}$. So $y^{-1}Ry$ acts by left translation on $\mathbf{P}^{F^m}/\mathbf{L}^{F^m}$. Since $y^{-1}Ry$ is a finite ℓ-group and $|\mathbf{P}^{F^m}/\mathbf{L}^{F^m}| = |\mathbf{V}^{F^m}|$ is a power of p, it follows that $y^{-1}Ry$ has a fixed point in $\mathbf{P}^{F^m}/\mathbf{L}^{F^m}$. Consequently, there exists $v \in \mathbf{V}$ such that $y^{-1}lyv\mathbf{L} = v\mathbf{L}$ for all $l \in R$. In other words, $(yv)^{-1}R(yv) \subset \mathbf{L}$. This means that, by replacing y by yv if necessary, we may assume that $y^{-1}Ry \subset \mathbf{L}$. Therefore, $y^{-1}Ry = Q$ and $P = \{(yly^{-1}, l) \mid l \in Q\}$.

Now, $y^{-1}F(y) \in \mathbf{V} \cdot F(\mathbf{V})$ but, since $F(yly^{-1}) = yly^{-1}$ for all $l \in Q$, we deduce that $y^{-1}F(y) \in C_{\mathbf{G}}(Q)$. So

$$y^{-1}F(y) \in (\mathbf{V} \cdot F(\mathbf{V})) \cap C_{\mathbf{G}}(Q) = C_{\mathbf{V}}(Q) \cdot F(C_{\mathbf{V}}(Q)) \subset C_{\mathbf{G}}^{\circ}(Q);$$

see Proposition 3.4(b) and (d). So, by Lang's Theorem, there exists $x \in C_{\mathbf{G}}^{\circ}(Q)$ such that $y^{-1}F(y) = x^{-1}F(x)$. This implies that $h = yx^{-1} \in \mathbf{G}^F$, and

$$P = \{(hlh^{-1}, l) \mid l \in Q\},$$

as expected. □

COROLLARY 3.8. *The indecomposable direct summands of the complex of* $\mathcal{O}(\mathbf{G}^F \times N_{\mathbf{G}^F}(\mathbf{P}, \mathbf{L})^{\mathrm{opp}})$*-module* $\mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}}, \mathcal{O})^{\mathrm{red}}$ *have a vertex contained in* $\Delta N_{\mathbf{G}^F}(\mathbf{P}, \mathbf{L})$.

Proof. Let Q be an ℓ-subgroup of $\mathbf{G}^F \times N_{\mathbf{G}^F}(\mathbf{P}, \mathbf{L})^{\mathrm{opp}}$ that is not $(\mathbf{G}^F \times 1)$-conjugate to a subgroup of $\Delta N_{\mathbf{G}^F}(\mathbf{P}, \mathbf{L})$. We have $\mathrm{Br}_Q(\mathrm{G}\Gamma_c(\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}}, \mathcal{O})) \simeq \mathrm{G}\Gamma_c((\mathbf{Y}_{\mathbf{V}}^{\mathbf{G}})^Q, k) \simeq 0$ in $\mathrm{Ho}^b(kN_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(Q))$ by Lemma 3.7. The result follows now from Lemma A.2. □

3.B. *Perfect complexes and disconnected groups.* Given M a simple $\Lambda\mathbf{G}^F$-module, we denote by $\mathcal{Y}(M)$ the set of pairs $(\mathbf{T}, \mathbf{B})$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\mathbf{B}$ is a (connected) Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $\mathrm{RHom}^{\bullet}_{\Lambda\mathbf{G}^F}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}, \Lambda), M) \neq 0$. We then set $d(M) = \min_{(\mathbf{T},\mathbf{B}) \in \mathcal{Y}(M)} \dim(\mathbf{Y}_{\mathbf{B}})$. The following two theorems are proved in [BR03, Th. A] whenever $\mathbf{G}$ is connected.

THEOREM 3.9. *Let M be a simple $\Lambda\mathbf{G}^F$-module. Then $\mathcal{Y}(M) \neq \varnothing$. Moreover, given $(\mathbf{T}, \mathbf{B}) \in \mathcal{Y}(M)$ such that $d(M) = \dim(\mathbf{Y}_{\mathbf{B}})$, we have*

$$\mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^F)}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}, \Lambda), M[-i]) = 0$$

for all $i \neq d(M)$.

Proof. By (3.2), we have

$$\mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^F)}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}}, \Lambda), M[-i]) \\ = \mathrm{Hom}_{\mathrm{D}^b(\Lambda\mathbf{G}^{\circ F})}(R\Gamma_c(\mathbf{Y}_{\mathbf{B}}^{\mathbf{G}^{\circ}}, \Lambda), \mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M[-i]).$$

Since M is simple and $\mathbf{G}^{\circ F} \lhd \mathbf{G}^F$, it follows that $\mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M$ is semisimple. Since the theorem holds in $\mathbf{G}^{\circ F}$ (see [BR03, proof of Th. A]), we know that $\mathcal{Y}(M)$ is not empty. The second statement follows from the fact that, if two simple $\Lambda\mathbf{G}^{\circ F}$-modules M_1 and M_2 occur in the semisimple module $\mathrm{Res}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} M$, then they are conjugate under $\mathbf{G}^F$, and so $d(M_1) = d(M_2) = d(M)$. □

THEOREM 3.10. *The triangulated category* $\Lambda\mathbf{G}^F$-perf *is generated by the complexes* $R\Gamma_c(\mathbf{Y_B},\Lambda)$, *where* $\mathbf{T}$ *runs over the set of* F*-stable maximal tori of* $\mathbf{G}$ *and* $\mathbf{B}$ *runs over the set of Borel subgroups of* $\mathbf{G}$ *containing* $\mathbf{T}$.

3.C. *Generation of the derived category.* In this subsection, we assume $\Lambda = \mathcal{O}$ or k. We refer to Appendix A for the needed facts about ℓ-permutation modules.

Let Q be an ℓ-subgroup of $\mathbf{G}^F$, and let M be an indecomposable ℓ-permutation $\Lambda[\mathbf{G}^F\times Q^{\mathrm{opp}}]$-module with vertex ΔQ. We denote by $\mathcal{Y}[M]$ the set of pairs $(\mathbf{T},\mathbf{B})$ satisfying the following conditions:

- $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that Q normalizes $(\mathbf{T},\mathbf{B})$;
- M is a direct summand of a term of the complex

$$\big(\mathrm{Res}^{\mathbf{G}^F\times N_{\mathbf{G}^F}(\mathbf{B},\mathbf{T})^{\mathrm{opp}}}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}\ \mathrm{G}\Gamma_c(\mathbf{Y_B},\Lambda)\big)^{\mathrm{red}}.$$

We set $d[M]=\min_{(\mathbf{T},\mathbf{B})\in\mathcal{Y}[M]}\dim(\mathbf{Y}^{C^\circ_{\mathbf{G}}(Q)}_{C^\circ_{\mathbf{B}}(Q)})$.

LEMMA 3.11. *If* Q *normalizes a pair* $(\mathbf{T}\subset\mathbf{B})$, *where* $\mathbf{T}$ *is an* F*-stable maximal torus and* $\mathbf{B}$ *a Borel subgroup of* $\mathbf{G}$, *then* $\mathcal{Y}[M]\neq\varnothing$. *Moreover, given* $(\mathbf{T},\mathbf{B})\in\mathcal{Y}[M]$ *such that* $d[M]=\dim(\mathbf{Y}_{C^\circ_{\mathbf{B}}(Q)})$, *the degree* i *term of the complex* $\big(\mathrm{Res}^{\mathbf{G}^F\times N_{\mathbf{G}^F}(\mathbf{B},\mathbf{T})^{\mathrm{opp}}}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}\ \mathrm{G}\Gamma_c(\mathbf{Y_B},\Lambda)\big)^{\mathrm{red}}$ *has no direct summand isomorphic to* M *if* $i\neq d[M]$.

Proof. Note that $N_{\mathbf{G}^F\times Q^{\mathrm{opp}}}(\Delta Q)=(C_{\mathbf{G}}(Q)^F\times 1)\Delta Q$. We identify $C_{\mathbf{G}}(Q)^F$ with $N_{\mathbf{G}^F\times Q^{\mathrm{opp}}}(\Delta Q)/\Delta Q$ via the first projection. Let $V=\mathrm{Br}_{\Delta Q}(M)$, an indecomposable projective $kC_{\mathbf{G}}(Q)^F$-module. Let L be the simple quotient of V.

Now, let $\mathbf{B}_Q$ be a Borel subgroup of $C_{\mathbf{G}}(Q)$ admitting an F-stable maximal torus $\mathbf{T}_Q$. By Lemma 3.6, $C_{\mathbf{G}}(\mathbf{T}_Q)^\circ$ is an F-stable maximal torus of $\mathbf{G}$ and it is contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $\mathbf{B}_Q=C^\circ_{\mathbf{B}}(Q)$.

We set $D=\big(\mathrm{Res}^{C_{\mathbf{G}}(Q)^F\times\mathbf{T}_Q^{F\mathrm{opp}}}_{C_{\mathbf{G}}(Q)^F\times 1}\ \mathrm{G}\Gamma_c(\mathbf{Y}^{C_{\mathbf{G}}(Q)}_{\mathbf{B}_Q},k)\big)^{\mathrm{red}}$. By Proposition 3.4(e), we have

$$\mathrm{Br}_{\Delta Q}(\mathrm{G}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}},\Lambda))\simeq\mathrm{G}\Gamma_c\big((\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}})^{\Delta Q},k\big)\simeq\mathrm{G}\Gamma_c(\mathbf{Y}^{C_{\mathbf{G}}(Q)}_{\mathbf{B}_Q},k)\simeq D$$

in $\mathrm{Ho}^b(kC_{\mathbf{G}}(Q)^F)$. It follows from Lemma A.2 that M is a direct summand of the i-th term of $\big(\mathrm{Res}^{\mathbf{G}^F\times\mathbf{T}_Q^{F\mathrm{opp}}}_{\mathbf{G}^F\times Q^{\mathrm{opp}}}\ \mathrm{G}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}},\Lambda)\big)^{\mathrm{red}}$ if and only if V is a direct summand of D^i. So the result follows from Theorem 3.9. Note that $d[M]=d[V]=d(L)=\dim\mathbf{Y}^{C_{\mathbf{G}}(Q)}_{\mathbf{B}_Q}$. ☐

Recall that given a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ with an F-stable maximal torus $\mathbf{T}$, the variety $\mathbf{Y_B}$ has a right action of $N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$ (cf. Remark 2.2).

Let $\mathcal{A}$ be the thick subcategory of $\mathrm{Ho}^b(\Lambda\mathbf{G}^F)$ generated by the complexes of the form

$$G\Gamma_c(\mathbf{Y_B},\Lambda)\otimes_{\Lambda Q} L,$$

where

- $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$;
- $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$;
- Q is an ℓ-subgroup of $N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$;
- and L is a ΛQ-module, free of rank 1 over Λ.

Let $\mathcal{B}$ be the full subcategory of $\Lambda\mathbf{G}^F$-mod consisting of modules whose indecomposable direct summands have a one-dimensional source and a vertex Q that normalizes a pair $(\mathbf{T}\subset\mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.

THEOREM 3.12. *We have* $\mathcal{A}=\mathrm{Ho}^b(\mathcal{B})$.

Proof. Given N an indecomposable $\Lambda\mathbf{G}^F$-module with a one-dimensional source L and a vertex Q that normalizes a pair $(\mathbf{T}\subset\mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup, we set $d[N]$ to be the minimum of the numbers $d[M]$, where M runs over the set of indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times Q^{\mathrm{opp}})$-modules with vertex ΔQ and such that N is a direct summand of $M\otimes_{\Lambda Q} L$.

Note that if M is an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times Q^{\mathrm{opp}})$-module with vertex properly contained in ΔQ, then the indecomposable direct summands of $M\otimes_{\Lambda Q} L$ have vertices of size $<|Q|$ and a one-dimensional source. Since the $\Lambda(\mathbf{G}^F\times Q^{\mathrm{opp}})$-module ΛG is a direct sum of indecomposable modules with vertices contained in ΔQ, we deduce that there is an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times Q^{\mathrm{opp}})$-module M with vertex ΔQ and such that N is a direct summand of $M\otimes_{\Lambda Q} L$.

We now proceed by induction on the pair $(|Q|,d[N])$ (ordered lexicographically) to show that $N\in\mathcal{A}$. Fix M an indecomposable ℓ-permutation $\Lambda(\mathbf{G}^F\times Q^{\mathrm{opp}})$-module M with vertex ΔQ and such that N is a direct summand of $M\otimes_{\Lambda Q} L$, with $d[N]=d[M]$. Let $(\mathbf{T},\mathbf{B})\in\mathcal{Y}[M]$ be such that $\dim(\mathbf{Y_B})=d[M]$, and let $D=\left(\mathrm{Res}^{\mathbf{G}^F\times N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})^{\mathrm{opp}}}_{\mathbf{G}^F\times Q^{\mathrm{opp}}} G\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}},\Lambda)\right)^{\mathrm{red}}$.

If $i\neq d[M]$, then Lemma 3.11 and Corollary 3.8 show that the indecomposable direct summands M' of D^i have vertices of size $<|Q|$, or have vertex ΔQ and satisfy $d[M']<d[M]$. Therefore, the indecomposable direct summands N' of $D^i\otimes_{\Lambda Q} L$ have vertices of size $<|Q|$ or have vertex Q and satisfy $d[N']<d[N]$. We deduce from the induction hypothesis that $D^i\otimes_{\Lambda Q} L\in\mathcal{A}$ for $i\neq d[N]$. Since N is a direct summand of $D^{d[N]}\otimes_{\Lambda Q} L$ and $D\otimes_{\Lambda Q} L\in\mathcal{A}$ by construction, we deduce that $N\in\mathcal{A}$. $\square$

Corollary 3.13. *Assume every elementary abelian ℓ-subgroup of $\mathbf{G}^F$ normalizes a pair $(\mathbf{T} \subset \mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$. Then $D^b(\Lambda\mathbf{G}^F)$ is generated, as a triangulated category closed under direct summands, by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B}, \Lambda) \otimes_{\Lambda Q} L$, where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ runs over the set of Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$, Q runs over the set of ℓ-subgroups of $N_{\mathbf{G}^F}(\mathbf{T}, \mathbf{B})$ and L runs over the set of (isomorphism classes) of ΛQ-modules that are free of rank 1 over Λ.*

Proof. Since the category $\mathrm{D}^b(\Lambda\mathbf{G}^F)$ is generated, as a triangulated category closed under taking direct summands, by indecomposable modules with elementary abelian vertices and one-dimensional source [Rou14, Cor. 2.3], the statement follows from Theorem 3.12. □

Remark 3.14. It is easy to show conversely that if $D^b(\Lambda\mathbf{G}^F)$ is generated by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B}, \Lambda) \otimes_{\Lambda Q} L$ as in Corollary 3.13, then $\mathrm{D}^b(\Lambda\mathbf{G}^F)$ is generated by indecomposable modules with a one-dimensional source and an elementary abelian vertex that normalizes a pair $(\mathbf{T} \subset \mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup.

In particular, the generation assumption for $\Lambda = k$ implies that all elementary abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in maximal tori.

The particular case $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$ (for arbitrary n) is enough to ensure that $\mathrm{D}^b(H)$ is generated by indecomposable modules with elementary abelian vertices and one-dimensional source, for any finite group H — this fact is a straightforward consequence of Serre's product of Bockstein's Theorem, but we know of no other proof. It would be interesting to find a direct proof of that result for $\mathbf{GL}_n(\mathbb{F}_q)$.

Recall that an element of $G_0(\Lambda\mathbf{G}^F)$ is *uniform* if it is in the image of $\sum_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(G_0(\Lambda\mathbf{T}^F))$, where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$.

One can actually describe exactly which complexes are "uniform".

Corollary 3.15. *Let $\mathcal{T}$ be the full triangulated subcategory of $D^b(\Lambda\mathbf{G}^F)$ generated by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B}, \Lambda) \otimes_{\Lambda N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})} M$ where $\mathbf{T}$ runs over the set of F-stable maximal tori of $\mathbf{G}$, $\mathbf{B}$ runs over the set of Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}$ and M runs over the set of (isomorphism classes) of finitely generated $\Lambda N_{\mathbf{G}^F}(\mathbf{T}, \mathbf{B})$-modules. Assume that every elementary abelian ℓ-subgroup of $\mathbf{G}^F$ normalizes a pair $(\mathbf{T} \subset \mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.*

An object C of $\mathrm{D}^b(\Lambda\mathbf{G}^F)$ is in $\mathcal{T}$ if and only if $[C] \in G_0(\Lambda\mathbf{G}^F)$ is uniform.

Proof. The statement follows from Corollary 3.13 and from Thomason's classification of full triangulated dense subcategories [Tho97, Th. 2.1]. □

Remark 3.16. Note that Corollary 3.15 holds also for $\Lambda = K$: in the proof, Corollary 3.13 is replaced by Theorem 3.10.

Examples 3.17. (1) If $\mathbf{G} = \mathbf{GL}_n(\mathbb{F})$ or $\mathbf{SL}_n(\mathbb{F})$, then all abelian subgroups consisting of semisimple elements are contained in maximal tori. This just amounts to the classical result in linear algebra that says a family of commuting semisimple elements always admits a basis of common eigenvectors.

(2) Assume $\mathbf{G}$ is connected. Let $\pi(\mathbf{G})$ denote the set of prime numbers that are bad for $\mathbf{G}$ or divide $|(\mathrm{Z}(\mathbf{G}^*)/\mathrm{Z}(\mathbf{G}^*)^\circ)^{F^*}|$. If $\ell \notin \pi$ and if t is an ℓ-element of $\mathbf{G}^F$, then $C_{\mathbf{G}}(t)$ is a Levi subgroup of $\mathbf{G}$ and $\pi(C_{\mathbf{G}}(t)) \subset \pi(\mathbf{G})$ [CE04, Prop. 13.12(iii)]. An induction argument shows the following fact.

(3.18)

If $\ell \notin \pi$, then all abelian ℓ-subgroups of $\mathbf{G}^F$ are contained in maximal tori.

So Corollary 3.13 can be applied if $\ell \notin \pi$. This generalizes (1).

Counter-example 3.19. Assume here, and only here, that $\ell = 2$ (so that $p \neq 2$) and that $\mathbf{G} = \mathbf{PGL}_2(\mathbb{F})$. Let t (resp. t') denote the class of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$) in $\mathbf{G}$. Then $\langle t, t' \rangle$ is an elementary abelian 2-subgroup of $\mathbf{G}$ which is not contained in any maximal torus of $\mathbf{G}$. (Indeed, since $\mathbf{G}$ has rank 1, all finite subgroups of maximal tori of $\mathbf{G}$ are cyclic.)

4. **Rational series**

4.A. *Rational series in connected groups.* We assume in this subsection that $\mathbf{G}$ is connected.

Let d be a positive integer divisible by δ and such that $(wF)^d(t) = t^{q^{d/\delta}}$ for all $t \in \mathbf{T}$ and $w \in N_{\mathbf{G}}(\mathbf{T})$. Let ζ be a generator of $\mathbb{F}_{q^{d/\delta}}^\times$. Recall [DM91, Prop. 13.7] that the map

$$\begin{array}{rccc} \mathrm{N}: & Y(\mathbf{T}) & \longrightarrow & \mathbf{T}^F \\ & \lambda & \longmapsto & N_{F^d/F}(\lambda(\zeta)) = \lambda(\zeta)\, {}^F(\lambda(\zeta)) \cdots {}^{F^{d-1}}(\lambda(\zeta)) \end{array}$$

is surjective and it induces an isomorphism $Y(\mathbf{T})/(F-1)(Y(\mathbf{T})) \xrightarrow{\sim} \mathbf{T}^F$. The morphism

$$Y(\mathbf{Y}) \times X(\mathbf{T}) \to K^\times,\ (\lambda, \mu) \mapsto \zeta^{\langle \mu, \lambda + F(\lambda) + \cdots + F^{d-1}(\lambda)\rangle}$$

factors through $N \times 1$ and induces a morphism $\mathbf{T}^F \times X(\mathbf{T}) \to K^\times$. The corresponding morphism $X(\mathbf{T}) \to \mathrm{Hom}(\mathbf{T}^F, K^\times) = \mathrm{Irr}(\mathbf{T}^F)$ is surjective and induces an isomorphism $X(\mathbf{T})/(F-1)(X(\mathbf{T})) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{T}^F)$ [DM91, Prop. 13.7].

Let $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ be a triple dual to $(\mathbf{G}, \mathbf{T}, F)$ [DL76, Def. 5.21]. The isomorphisms $X(\mathbf{T})/(F-1)(X(\mathbf{T})) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{T}^F)$ and $X(\mathbf{T})/(F-1)(X(\mathbf{T})) = Y(\mathbf{T}^*)/(F^*-1)(Y(\mathbf{T}^*)) \xrightarrow{\sim} \mathbf{T}^{*F}$ induce an isomorphism $\mathrm{Irr}(\mathbf{T}^F) \xrightarrow{\sim} \mathbf{T}^{*F^*}$.

Let $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F)$, and let Φ (resp. $\Phi^\vee$) denote the root (resp. coroot) system of $\mathbf{G}$ relative to $\mathbf{T}$.

We set $\theta^Y = \theta \circ \mathrm{N} : Y(\mathbf{T}) \to K^\times$ and

$$\Phi^\vee(\theta) = \Phi^\vee \cap \mathrm{Ker}(\theta^Y).$$

Note that $\Phi^\vee(\theta)$ is closed and symmetric, hence it defines a root system. We denote by $W^\circ_{\mathbf{G}}(\mathbf{T},\theta)$ its Weyl group. It is a subgroup of the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, and it is contained in the stabilizer $W_{\mathbf{G}}(\mathbf{T},\theta)$ of θ^Y.

This can be translated as follows in the dual group [DM91, Prop. 2.3]. Let $s \in \mathbf{T}^{*F^*}$ be the element corresponding to θ. Identifying the coroot system $\Phi^\vee$ with the root system of $\mathbf{G}^*$, we obtain that

$$\Phi^\vee(\theta) = \{\alpha^\vee \in \Phi^\vee \mid \alpha^\vee(s) = 1\}$$

is the root system of $C_{\mathbf{G}^*}(s)$. If $\mathbf{V}^*$ is a unipotent subgroup of $\mathbf{G}^*$ normalized by $\mathbf{T}^*$, then $C_{\mathbf{V}^*}(s)$ is generated by the one-parameter subgroups of $\mathbf{G}^*$ normalized by $\mathbf{T}^*$, contained in $\mathbf{V}^*$, and corresponding to elements of $\Phi^\vee(\theta)$.

The group $W^\circ_{\mathbf{G}}(\mathbf{T},\theta)$ is identified with the Weyl group $W^\circ(\mathbf{T}^*,s)$ of $C^\circ_{\mathbf{G}^*}(s)$ relative to $\mathbf{T}^*$, while $W_{\mathbf{G}}(\mathbf{T},\theta)$ is identified with the Weyl group $W(\mathbf{T}^*,s)$ of $C_{\mathbf{G}^*}(s)$.

Recall that $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are in the same *geometric series* if there exists $x \in \mathbf{G}$ such that $(\mathbf{T}_2,\theta_2^Y) = {}^x(\mathbf{T}_1,\theta_1^Y)$ and $x^{-1}F(x)\mathbf{T}_1 \in W_{\mathbf{G}}(\mathbf{T}_1,\theta_1)$. The pairs are in the same *rational series* if, in addition, the element $s_2 \in \mathbf{T}_1^{*F^*}$ corresponding to ${}^{x^{-1}}\theta_2$ is $\mathbf{G}^{*F^*}$-conjugate to s_1. We have now a direct description of rational series.

PROPOSITION 4.1. *The pairs $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are in the same rational series if and only if there exists $x \in \mathbf{G}$ such that $(\mathbf{T}_2,\theta_2^Y) = {}^x(\mathbf{T}_1,\theta_1^Y)$ and $x^{-1}F(x)\mathbf{T}_1 \in W^\circ_{\mathbf{G}}(\mathbf{T}_1,\theta_1)$.*

Proof. Note that given $x \in \mathbf{G}$ such that ${}^x\mathbf{T}_1$ is F-stable, then $x^{-1}F(x) \in N_{\mathbf{G}_1}(\mathbf{T}_1)$.

Let $\mathbf{T}_i^*$ be an F^*-stable maximal torus of $\mathbf{G}^*$, and let $s_i \in \mathbf{T}_i^{*F^*}$ be such that the $\mathbf{G}^{*F^*}$-orbit of $(\mathbf{T}_i^*,s_i)$ corresponds to the $\mathbf{G}^F$-orbit of $(\mathbf{T}_i,\theta_i)$. Then the statement of the proposition is equivalent to the following:

(∗) *s_1 and s_2 are $\mathbf{G}^{*F^*}$-conjugate if and only if there exists $x \in \mathbf{G}^*$ such that $(\mathbf{T}_2^*,s_2) = {}^x(\mathbf{T}_1^*,s_1)$ and $x^{-1}F^*(x)\mathbf{T}_1^* \in W^\circ(\mathbf{T}_1^*,s_1)$.*

So let us prove (∗).

First, if s_1 and s_2 are $\mathbf{G}^{*F^*}$-conjugate, then there exists $x \in \mathbf{G}^{*F^*}$ such that $s_2 = xs_1x^{-1}$. Then $\mathbf{T}_1^*$ and $x^{-1}\mathbf{T}_2^*x$ are two maximal tori of $C^\circ_{\mathbf{G}^*}(s_1)$, so there exists $y \in C^\circ_{\mathbf{G}^*}(s_1)$ such that $y\mathbf{T}_1^*y^{-1} = x^{-1}\mathbf{T}_2^*x$. Then $(\mathbf{T}_2^*,s_2) = {}^{xy}(\mathbf{T}_1^*,s_1)$ and

$$(xy)^{-1}F^*(xy) = y^{-1}F^*(y) \in C^\circ_{\mathbf{G}^*}(s_1),$$

as desired.

Conversely, assume that there exists $x \in \mathbf{G}^*$ such that $(\mathbf{T}_2^*,s_2) = {}^x(\mathbf{T}_1^*,s_1)$ and $x^{-1}F^*(x)\mathbf{T}_1^* \in W^\circ(\mathbf{T}_1^*,s_1)$. By Lang's Theorem applied to the connected group $C^\circ_{\mathbf{G}^*}(s_1)$, there exists $y \in C^\circ_{\mathbf{G}^*}(s_1)$ such that $x^{-1}F^*(x) = y^{-1}F^*(y)$. Then $xy^{-1} \in \mathbf{G}^{*F^*}$ and $s_2 = xy^{-1}s_1yx^{-1}$. The proof of $(*)$ is complete. □

We can now translate the properties of regularity and super-regularity defined in [BR03, §11.4]. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, and let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$. We assume that $\mathbf{L}$ is F-stable. Let $\mathcal{X} \subset \nabla(\mathbf{L}, F)$ be a rational series.

PROPOSITION 4.2. *The rational series* $\mathcal{X}$ *is* $(\mathbf{G}, \mathbf{L})$*-regular* (*resp.* $(\mathbf{G}, \mathbf{L})$*-super-regular*) *if and only if* $W^\circ_{\mathbf{G}}(\mathbf{T}, \theta) \subset \mathbf{L}$ (*resp.* $W_{\mathbf{G}}(\mathbf{T}, \theta) \subset \mathbf{L}$) *for some* (*or any*) *pair* $(\mathbf{T}, \theta) \in \mathcal{X}$.

Proof. This follows immediately from [BR03, Lemma 11.6]. □

4.B. *Coroots of fixed points subgroups.* Now we again consider a not necessarily connected reductive group $\mathbf{G}$.

We fix an element $g \in \mathbf{G}$ that stabilizes a pair $(\mathbf{T}, \mathbf{B})$, where $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ and $\mathbf{T}$ is a maximal torus of $\mathbf{B}$. Such an element is called *quasi-semisimple* in [DM94] and [DM15]. For instance, any semisimple element of $\mathbf{G}$ is quasi-semisimple. Recall from [DM94, Th. 1.8] that $C_{\mathbf{G}}(g)^\circ$ is a reductive group, that $C_{\mathbf{B}}(g)^\circ = \mathbf{B} \cap C_{\mathbf{G}}(g)^\circ$ is a Borel subgroup of $C_{\mathbf{G}}(g)$ and that $C_{\mathbf{T}}(g)^\circ = \mathbf{T} \cap C_{\mathbf{G}}(g)^\circ$ is a maximal torus of $C_{\mathbf{B}}(g)$. We shall be interested in determining the coroot system of the fixed points subgroup $C_{\mathbf{G}}(g)^\circ$.

Let Φ (resp. $\Phi^\vee$) be the root (resp. coroot) system of $\mathbf{G}^\circ$ relative to $\mathbf{T}$. Let $\Phi(g)$ (resp. $\Phi^\vee(g)$) denote the root (resp. coroot) system of $C_{\mathbf{G}}(g)^\circ$ relative to $C_{\mathbf{T}}(g)^\circ$. If Ω is a g-orbit in Φ, we denote by $c_\Omega \in \mathbb{F}^\times$ the scalar by which $g^{|\Omega|}$ acts on the one-parameter unipotent subgroup associated with α (through any identification of this one-parameter subgroup with the additive group $\mathbb{F}$). We denote by $(\Phi/g)^a$ the set of g-orbits Ω in Φ such that there exist α, $\beta \in \Omega$ such that $\alpha + \beta \in \Phi$. We denote by $(\Phi/g)^b$ the set of other orbits. We set

$$\Phi[g] = \{\Omega \in (\Phi/g)^a \mid c_\Omega = 1 \text{ and } p{\neq}2\} \cup \{\Omega \in (\Phi/g)^b \mid c_\Omega = 1\}.$$

Finally, if $\Omega \in (\Phi/g)^a$ (resp. $\Omega \in (\Phi/g)^b$), then let $\overline{\Omega}^\vee = 2\sum_{\alpha\in\Omega} \alpha^\vee$ (resp. $\overline{\Omega}^\vee = \sum_{\alpha\in\Omega} \alpha^\vee$). Note that $\overline{\Omega}^\vee$ is g-invariant, so it belongs to $Y(\mathbf{T})^g = Y(C_{\mathbf{T}}(g)^\circ)$.

PROPOSITION 4.3. $\Phi^\vee(g) = \{\overline{\Omega}^\vee \mid \Omega \in \Phi[g]\}$.

Proof. The statement depends only on the automorphism induced by g on $\mathbf{G}^\circ$ and can be proved with assuming that $\mathbf{G}^\circ$ is semisimple. Since this automorphism can then be lifted uniquely to the simply-connected covering of $\mathbf{G}^\circ$ (see [Ste68, 9.16]), we may also assume that $\mathbf{G}^\circ$ is simply-connected. Therefore, g permutes the irreducible components of $\mathbf{G}^\circ$, and an easy reduction argument shows that we may assume that $\mathbf{G}^\circ$ is quasi-simple. Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{B}$, $\mathbf{U}^-$ the unipotent radical of the opposite Borel subgroup and, if $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ denote the corresponding one-parameter unipotent subgroup. We also denote by $\mathbf{G}_\alpha$ the subgroup generated by $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$: it is isomorphic to $\mathbf{SL}_2(\mathbb{F})$ because $\mathbf{G}^\circ$ is simply-connected.

If $\Omega \in \Phi/g$, we denote by $\mathbf{U}_\Omega$ the unipotent subgroup generated by $(\mathbf{U}_\alpha)_{\alpha\in\Omega}$. We follow the proof of [Ste68, Th. 8.2]. According to this proof, any one-parameter unipotent subgroup $\mathbf{V}$ normalized by $C_{\mathbf{T}}(g)^\circ$ is contained in some of these $\mathbf{U}_\Omega$'s, and one of the following holds:

(1) $\Omega \in (\Phi/g)^b$ and $c_\Omega = 1$;
(2) $\Omega \in (\Phi/g)^a$ and $c_\Omega = -1$;
(3) $\Omega \in (\Phi/g)^a$, $c_\Omega = 1$ and $p \neq 2$;

In all cases, $\mathbf{V} = C_{\mathbf{U}_\Omega}(g)$. Let $\mathbf{V}^- = C_{\mathbf{U}_\Omega}(g)$.

In case (1), as $[\mathbf{U}_\alpha, \mathbf{U}_\beta] = 1$ if α, $\beta \in \Omega$, the group $\langle \mathbf{U}_\Omega, \mathbf{U}_{-\Omega}\rangle$ is a direct product of groups isomorphic to $\mathbf{SL}_2(\mathbb{F})$ that are permuted by g. It then follows that the coroot corresponding to the one-parameter subgroup $\mathbf{V} = C_{\mathbf{U}_\Omega}(g)$ (since $c_\Omega = 1$) is equal to $\omega^\vee = \overline{\Omega}^\vee$.

In cases (2) or (3), it follows from the classification that $|\Omega| = 2$. (This case only occurs in type A_{2n}.) Let $\alpha \in \Omega$. Then $\mathbf{U}_\Omega = \mathbf{U}_\alpha \mathbf{U}_{g(\alpha)} \mathbf{U}_{\alpha+g(\alpha)}$. In case (2), the computations done in [Ste68, proof of Th. 8.2(2′′′′)] show that $\mathbf{V} = \mathbf{U}_{\alpha+g(\alpha)}$. Therefore $\mathbf{V} \subset \mathbf{U}_{\Omega'}$, where Ω' is the g-orbit (of cardinality 1) of $\alpha + g(\alpha)$ and $\Omega' \in (\Phi/g)^b$, and we are back to case (1).

In case (3), the computations done in [Ste68, proof of Th. 8.2,(2′′′′)] show that $\langle \mathbf{V}, \mathbf{V}^-\rangle \simeq \mathbf{SO}_3(\mathbb{F}) \simeq \mathbf{PGL}_2(\mathbb{F})$ and that the associated coroot is $2(\alpha^\vee + g(\alpha^\vee)) = \overline{\Omega}^\vee$. □

Remark 4.4. If $\Omega \in (\Phi/g)^a$, then it follows from the classification that $|\Omega|$ is even, and so the order of g is even.

4.C. *Centralizers and rational series.* Let $g \in \mathbf{G}^F$ be a quasi-semisimple element of $\mathbf{G}$. Let $(\mathbf{S}, \theta) \in \nabla(C^\circ_{\mathbf{G}}(g), F)$. We then set $\mathbf{S}^+ = C_{\mathbf{G}^\circ}(\mathbf{S})$. It follows from [DM94, Th. 1.8(iv)] that $\mathbf{S}^+$ is a maximal torus of $\mathbf{G}^\circ$ (containing $\mathbf{S}$). It is stable under the action of g, so we have a map $\mathcal{L}_g : \mathbf{S}^+ \to \mathbf{S}^+$, $t \mapsto t^{-1}gtg^{-1} = [g, t]$ (which is a morphism of groups because $\mathbf{S}^+$ is abelian). If $t = \mathcal{L}_g(s)$, then $t\,{}^gt\,{}^{g^2}t \cdots {}^{g^{m-1}}t = \mathcal{L}_{g^m}(s)$. In particular, if $t \in C_{\mathbf{S}^+}(g) = \operatorname{Ker} \mathcal{L}_g$, then $t^m = \mathcal{L}_{g^m}(s)$. This shows that any element of $C_{\mathbf{S}^+}(g) \cap \mathcal{L}_g(\mathbf{S}^+)$ has order dividing the order of g. Note further that $C_{\mathbf{S}^+}(g)^\circ = \mathbf{S}$ (see [DM94, Th. 1.8(iii)]). We have

$$\begin{aligned}\dim(\mathbf{S} \cdot \mathcal{L}_g(\mathbf{S}^+)) &= \dim(\mathbf{S}) + \dim(\mathcal{L}_g(\mathbf{S}^+)) = \dim(\mathbf{S}) + \dim(\mathbf{S}^+) - \dim(\ker \mathcal{L}_g)^\circ \\ &= \dim(\mathbf{S}) + \dim(\mathbf{S}^+) - \dim(\mathbf{S}) = \dim(\mathbf{S}^+).\end{aligned}$$

We deduce that

(4.5)
$$\mathbf{S}^+ = \mathbf{S} \cdot \mathcal{L}_g(\mathbf{S}^+) \text{ and } \mathbf{S} \cap \mathcal{L}_g(\mathbf{S}^+) \text{ is finite of exponent dividing the order of } g.$$

Now, if H is a g-stable finite subgroup of $\mathbf{S}^+$ of order prime to the order of g, then $C_H(g) \subset \mathbf{S}$ (because the order of $C_{\mathbf{S}^+}(g)/\mathbf{S}$ divides the order of g by [DM94, Prop. 1.28]) and

$$H = C_H(g) \times \mathcal{L}_g(H). \tag{4.6}$$

So, if the linear character θ of $\mathbf{S}^F$ has order prime to the order of g, then it can be extended canonically to a linear character θ^+ of $\mathbf{S}^{+F}$ as follows (we use the discussion above with H the subgroup of $\mathbf{S}^{+F}$ of elements with order dividing a power of the order of θ): θ^+ is trivial on $\mathcal{L}_g(\mathbf{S}^{+F})$, is trivial on elements of $\mathbf{S}^{+F}$ of order prime to the order of θ and coincides with θ on $\mathbf{S}^F$. The fact that θ^+ is trivial on $\mathcal{L}_g(\mathbf{S}^{+F})$ is equivalent to

$$\theta^+ \text{ is } g\text{-stable.} \tag{4.7}$$

Note that, since $\mathbf{S}^+ \cap C^\circ_{\mathbf{G}}(g) = \mathbf{S}$ by [DM94, Th. 1.8], we may identify the Weyl group of $C^\circ_{\mathbf{G}}(g)$ relative to $\mathbf{S}$ to a subgroup of the Weyl group of $\mathbf{G}^\circ$ relative to $\mathbf{S}^+$. Through this identification, we get

Lemma 4.8. *If the order of θ is prime to the order of g, then $W_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta) \subset W_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$ and $W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta) \subset W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$.*

Proof. Let $w \in W_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta)$. Then w stabilizes $\mathbf{S}^+ = C_{\mathbf{G}^\circ}(\mathbf{S})$ and its action on $\mathbf{S}$ commutes with the action of g. So it follows from the construction of θ^+ that w stabilizes θ^+.

Let us now prove the second statement. Let $\alpha^\vee$ be a coroot of $C^\circ_{\mathbf{G}}(g)$ relative to $\mathbf{S}$ such that $\theta^Y(\alpha^\vee) = 1$. Let $s_{g,\alpha}$ denote the corresponding reflection in $W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S},\theta)$. It is sufficient to prove that $s_{g,\alpha} \in W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+)$. Then it follows from Proposition 4.3 that there exists a coroot $\beta^\vee$ of $\mathbf{G}^\circ$ relative to $\mathbf{S}^+$ and $m \in \{1,2\}$ such that

$$\alpha^\vee = m \sum_{i=0}^{r-1} g^i(\beta^\vee),$$

where $r \geqslant 1$ is minimal such that $g^r(\beta^\vee) = \beta^\vee$. It follows from Remark 4.4 that, if $m = 2$, then g has even order. Now,

$$1 = \theta^{+Y}(\alpha^\vee) = \prod_{i=1}^{r-1} \theta^{+Y}(g^i(\beta^\vee))^m = \theta^{+Y}(\beta^\vee)^{mr},$$

because θ^+ is g-stable. Since m and r divide the order of g, mr is prime to the order of θ^+, so this implies that $\theta^{+Y}(\beta^\vee) = 1$. In particular,

$$s_\beta, s_{g(\beta)}, \ldots, s_{g^{r-1}(\beta)} \in W^\circ_{\mathbf{G}^\circ}(\mathbf{S}^+,\theta^+).$$

It follows from [Ste68, Proof of Th. 8.2(2‴)] that then $s_{g,\alpha}$ belongs to the subgroup generated by s_β, $s_{g(\beta)}$,..., $s_{g^{r-1}(\beta)}$. □

Let $(\mathbf{T}_1,\theta_1),(\mathbf{T}_2,\theta_2)\in\nabla(\mathbf{G},F)$. We say that $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are *geometrically conjugate* (resp. in the same *rational series*) if there is $t\in N_{\mathbf{G}^F}(\mathbf{T}_1)$ such that $(\mathbf{T}_1,{}^t\theta_1)$ and $(\mathbf{T}_2,\theta_2)$ are geometrically conjugate (resp. in the same rational series) for $\nabla(\mathbf{G}^\circ,F)$. We denote by $\nabla(\mathbf{G},F)/\equiv$ the set of rational series.

Let Q be the subgroup of $\mathbf{G}$ generated by g, and let $\mathbf{N}$ be a subgroup of $N_{\mathbf{G}}(Q)$ containing $C_{\mathbf{G}}(Q)$.

Corollary 4.9. *Let* $(\mathbf{S}_1,\theta_1),(\mathbf{S}_2,\theta_2)\in\nabla_{|g|'}(\mathbf{N},F)$.

(a) *If* $(\mathbf{S}_1,\theta_1)$ *and* $(\mathbf{S}_2,\theta_2)$ *are geometrically conjugate in* $\mathbf{N}$, *then* $(\mathbf{S}_1^+,\theta_1^+)$ *and* $(\mathbf{S}_2^+,\theta_2^+)$ *are geometrically conjugate in* $\mathbf{G}$.

(b) *If* $(\mathbf{S}_1,\theta_1)$ *and* $(\mathbf{S}_2,\theta_2)$ *are in the same rational series of* $\mathbf{N}$, *then* $(\mathbf{S}_1^+,\theta_1^+)$ *and* $(\mathbf{S}_2^+,\theta_2^+)$ *are in the same rational series of* $\mathbf{G}$.

So, the injective map $\nabla_{|g|'}(\mathbf{N},F)\to\nabla_{|g|'}(\mathbf{G},F)$, $(\mathbf{S},\theta)\mapsto(\mathbf{S}^+,\theta^+)$ *induces a map*

$$i_Q^{\mathbf{G}}:\nabla_{|g|'}(\mathbf{N},F)/\equiv\;\longrightarrow\;\nabla_{|g|'}(\mathbf{G},F)/\equiv .$$

Proof. (a) If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are geometrically conjugate in $\mathbf{N}^\circ=C^\circ_{\mathbf{G}}(g)$ then, by definition, there exists $x\in C^\circ_{\mathbf{G}}(g)$ such that $\mathbf{S}_2={}^x\mathbf{S}_1$ and $\theta_2^Y={}^x\theta_1^Y={}^{F(x)}\theta_1^Y$ (as linear characters of $Y(\mathbf{S}_2)$). Since x commutes with g, it sends $\mathcal{L}_g(\mathbf{S}_1^+)$ to $\mathcal{L}_g(\mathbf{S}_2^+)$, so it is immediately checked that $\theta_2^{+Y}={}^x\theta_1^{+Y}={}^{F(x)}\theta_1^{+Y}$. The case of geometric conjugacy in $\mathbf{N}$ and $\mathbf{G}$ follows immediately.

(b) If $(\mathbf{S}_1,\theta_1)$ and $(\mathbf{S}_2,\theta_2)$ are in the same rational series of $C^\circ_{\mathbf{G}}(g)$, then, by Proposition 4.1, there exists $x\in C^\circ_{\mathbf{G}}(g)$ such that $\mathbf{T}_2={}^x\mathbf{T}_1$, $\theta_2^Y={}^x\theta_1^Y$ (as linear characters of $Y(\mathbf{S}_2)$) and $x^{-1}F(x)\in W^\circ_{C^\circ_{\mathbf{G}}(g)}(\mathbf{S}_1,\theta_1)$. So the result follows from (a) and from Propositions 4.1 and 4.3. The case of rational series in $\mathbf{N}$ and $\mathbf{G}$ follows immediately. □

Let $\mathbf{L}$ be an F-stable Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing g. Then $C_{\mathbf{L}}(g)$ is an F-stable Levi complement of $C_{\mathbf{P}}(g)$ [DM94, Prop. 1.11].

Corollary 4.10. *Let* $\mathcal{X}\in\nabla_{|g|'}(C^\circ_{\mathbf{L}}(g),F)/\equiv$ *be a rational series. If* $i_Q^{\mathbf{L}}(\mathcal{X})$ *is* $(\mathbf{G}^\circ,\mathbf{L}^\circ)$*-regular (resp.* $(\mathbf{G}^\circ,\mathbf{L}^\circ)$*-super regular), then* $\mathcal{X}$ *is* $(C^\circ_{\mathbf{G}}(g),C^\circ_{\mathbf{L}}(g))$*-regular (resp.* $(C^\circ_{\mathbf{G}}(g),C^\circ_{\mathbf{L}}(g))$*-super regular).*

Proof. This follows from Proposition 4.2 and Lemma 4.8. □

The results above extend by induction to general nilpotent p'-subgroups. Let Q be a nilpotent subgroup of $\mathbf{G}^F$ of order prime to p. Fix a sequence $1=Q_0\subset Q_1\subset\cdots\subset Q_r=Q$ of normal subgroups of Q such that Q_i/Q_{i-1} is cyclic for $1\leqslant i\leqslant r$. Let $\mathbf{G}_i=N_{\mathbf{G}}(Q_1\subset\cdots\subset Q_i)$.

The construction above defines a map

$$\nabla_{|Q|'}(\mathbf{G}_{i+1}/Q_i, F) = \nabla_{|Q|'}(N_{\mathbf{G}_i/Q_i}(Q_{i+1}/Q_i), F) \to \nabla_{|Q|'}(\mathbf{G}_i/Q_i, F) \tag{4.10}$$

that preserves rational and geometric series.

Fix $0 \leqslant j \leqslant i \leqslant r$. Let $(\mathbf{T}, \theta) \in \nabla_{|Q|'}(\mathbf{G}_i, F)$. The kernel of the canonical map $\mathbf{T}^F \to (\mathbf{T}/(\mathbf{T} \cap Q_j))^F$ has order a divisor of a power of the order of Q_j, and so does its cokernel, since it is isomorphic to $H^1(F, \mathbf{T} \cap Q_j)$. Since θ is trivial on $\mathbf{T}^F \cap Q_j$, it factors through a character of $\mathbf{T}^F/(\mathbf{T}^F \cap Q_j)$ that comes by restriction from a (unique) character θ' of $(\mathbf{T}/(\mathbf{T} \cap Q_j))^F$. We obtain a pair $(\mathbf{T}/(\mathbf{T} \cap Q_j), \theta') \in \nabla_{|Q|'}(\mathbf{G}_i/Q_j, F)$. This correspondence defines a bijection $\nabla_{|Q|'}(\mathbf{G}_i, F) \xrightarrow{\sim} \nabla_{|Q|'}(\mathbf{G}_i/Q_j, F)$ that preserves rational and geometric series.

Composing those bijections with the map in (4.10), we obtain a map

$$\nabla_{|Q|'}(\mathbf{G}_{i+1}, F) \to \nabla_{|Q|'}(\mathbf{G}_i, F),$$

and composing all those maps, we obtain a map

$$\nabla_{|Q|'}(N_{\mathbf{G}}(Q_1 \subset \cdots \subset Q_r), F) \to \nabla_{|Q|'}(\mathbf{G}, F).$$

Finally, composing with the canonical map $\nabla_{|Q|'}(C_{\mathbf{G}}(Q), F) \to \nabla_{|Q|'}(N_{\mathbf{G}}(Q_1 \subset \cdots \subset Q_r), F)$, we obtain a map

$$\nabla_{|Q|'}(C_{\mathbf{G}}(Q), F) \to \nabla_{|Q|'}(\mathbf{G}, F)$$

that preserves rational and geometric series. Note that this map depends not only on Q, but also on the filtration $Q_1 \subset \cdots \subset Q_r$. Summarizing, we have the following proposition.

Proposition 4.11. *Let Q be a nilpotent subgroup of $\mathbf{G}^F$ of order prime to p. Fix a sequence $1 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q$ of normal subgroups of Q such that Q_i/Q_{i-1} is cyclic for $1 \leqslant i \leqslant r$.*

The constructions above define a map

$$i_{Q_\bullet}^{\mathbf{G}} : \nabla_{|Q|'}(C_{\mathbf{G}}(Q), F)/\equiv \;\to\; \nabla_{|Q|'}(\mathbf{G}, F)/\equiv .$$

Let $\mathbf{L}$ be an F-stable Levi complement of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing Q. Let $\mathcal{X} \in \nabla_{|Q|'}(C_{\mathbf{L}}(Q), F)/\equiv$ be a rational series. Then

- *$C_{\mathbf{L}}(Q)$ is an F-stable Levi complement of $C_{\mathbf{P}}(Q)$;*
- *if $i_{Q_\bullet}^{\mathbf{L}}(\mathcal{X})$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-regular, then $\mathcal{X}$ is $(C_{\mathbf{G}}^\circ(Q), C_{\mathbf{L}}^\circ(Q))$-regular;*
- *if $i_{Q_\bullet}^{\mathbf{L}}(\mathcal{X})$ is $(\mathbf{G}^\circ, \mathbf{L}^\circ)$-super regular, then $\mathcal{X}$ is $(C_{\mathbf{G}}^\circ(Q), C_{\mathbf{L}}^\circ(Q))$-super regular.*

The map $i_{Q_\bullet}^{\mathbf{G}}$ is actually independent of the choice of the filtration of Q; cf. Remark 4.15 below.

4.D. *Generation and series.* Given $(\mathbf{T},\theta)\in\nabla_\Lambda(\mathbf{G},F)$, we denote by e_θ° the block idempotent of $\Lambda\mathbf{T}^F$ not vanishing on θ.

We have now a generalization of [BR03, Th. A].

Given $\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$, let $\mathcal{C}_\mathcal{X}$ be the thick subcategory of $(\Lambda\mathbf{G}^F)$-perf generated by the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B})e_\theta^\circ$ where $(\mathbf{T},\theta)$ runs over $\mathcal{X}$ and $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}^\circ$ containing $\mathbf{T}$.

Note that, by definition of rational series for nonconnected groups, we obtain the same thick subcategory by taking instead the complexes $\mathrm{R}\Gamma_c(\mathbf{Y_B})e_\theta$, where $e_\theta=\sum_{t\in N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})/C_{N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})}(\theta)} e_{t\theta}^\circ$.

Theorem 4.12. *Let $\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$. There is a (unique) central idempotent $e_\mathcal{X}$ of $\Lambda\mathbf{G}^F$ such that $\mathcal{C}_\mathcal{X}=(\Lambda\mathbf{G}^F e_\mathcal{X})$-perf.*

We have a decomposition in central orthogonal idempotents of $\Lambda\mathbf{G}^F$:

$$1=\sum_{\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv} e_\mathcal{X}.$$

Proof. Note first that the theorem holds for $\mathbf{G}^\circ$ by [BR03, Th. A]. Let $(\mathbf{T}_i,\theta_i)\in\nabla_\Lambda(\mathbf{G},F)$, and let $\mathbf{B}_i$ be a Borel subgroup of $\mathbf{G}^\circ$ containing $\mathbf{T}_i$ for $i\in\{1,2\}$. By (3.2) and (3.3), we have

$$\begin{aligned}\mathrm{Hom}^\bullet_{\Lambda\mathbf{G}^F}&(\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}_1})e^\circ_{\theta_1},\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{B}_2})e^\circ_{\theta_2})\\ &\simeq \mathrm{Hom}^\bullet_{\Lambda\mathbf{G}^{\circ F}}(\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{\mathbf{B}_1})e^\circ_{\theta_1},\bigoplus_{t\in N_{\mathbf{G}^F}(\mathbf{T}_2,\mathbf{B}_2)/\mathbf{T}_2^F}\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathbf{G}^\circ}_{\mathbf{B}_2})e^\circ_{{}^t\theta_2}).\end{aligned}$$

The connected case of the theorem shows this is 0 unless $(\mathbf{T}_1,\theta_1)$ and $(\mathbf{T}_2,{}^t\theta_2)$ are in the same rational series of $(\mathbf{G}^\circ,F)$ for some t.

We have shown that the categories $\mathcal{C}_{\mathcal{X}_1}$ and $\mathcal{C}_{\mathcal{X}_2}$ are othogonal for $\mathcal{X}_1\neq\mathcal{X}_2$. The theorem follows now from [BR03, Prop. 9.2] and Theorem 3.10. ☐

Let $\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$. Let $\mathcal{A}_\mathcal{X}$ be the thick subcategory of $\mathrm{Ho}^b(\Lambda\mathbf{G}^F)$ generated by the complexes of the form

$$\mathrm{G}\Gamma_c(\mathbf{Y_B},\Lambda)e_\theta\otimes_{\Lambda Q}L,$$

where

- $(\mathbf{T},\theta)$ runs over $\mathcal{X}$;
- $\mathbf{B}$ runs over Borel subgroups of $\mathbf{G}^\circ$ containing $\mathbf{T}$;
- Q is an ℓ-subgroup of $N_{\mathbf{G}^F}(\mathbf{T},\mathbf{B})$;
- and L is a ΛQ-module, free of rank 1 over Λ.

Let $\mathcal{B}_\mathcal{X}$ be the full subcategory of $\Lambda\mathbf{G}^F e_\mathcal{X}$-mod consisting of modules whose indecomposable direct summands have a one-dimensional source and a vertex Q that normalizes a pair $(\mathbf{T}\subset\mathbf{B})$, where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$.

Theorem 4.13. *Let $\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv$. We have $\mathcal{A}_\mathcal{X}=\mathrm{Ho}^b(\mathcal{B}_\mathcal{X})$.*

Proof. By Theorem 4.12, we have $G\Gamma_c(\mathbf{Y}_\mathbf{V},\Lambda)e_\theta\otimes_{\Lambda Q}L\in \mathrm{Ho}^b(\mathcal{B}_\mathcal{X})$ if $(\mathbf{T},\theta)\in\mathcal{X}$. It follows that $\mathcal{A}_\mathcal{X}\subset \mathrm{Ho}^b(\mathcal{B}_\mathcal{X})$. Since $\mathcal{A}=\bigoplus_{\mathcal{X}\in\nabla_\Lambda(\mathbf{G},F)/\equiv}\mathcal{A}_\mathcal{X}$, the theorem follows from Theorem 3.12. □

4.E. *Decomposition map and Deligne-Lusztig induction.* The following result generalizes [BM89, Th. 3.2] to noncyclic ℓ-subgroups and to disconnected groups (needed to handle the noncyclic case by induction).

THEOREM 4.14. *Let Q be an ℓ-subgroup of $\mathbf{G}^F$. The map $i^\mathbf{G}_{Q_\bullet}$ (cf. Proposition 4.11) is independent of the filtration of Q, and we denote it by $i_Q=i^\mathbf{G}_Q$.*
Let $\mathcal{X}\in\nabla_{\ell'}(\mathbf{G},F)/\equiv$. We have

$$\mathrm{br}_Q(e_\mathcal{X})=\sum_{\mathcal{Y}\in i_Q^{-1}(\mathcal{X})}e_\mathcal{Y}.$$

Proof. Assume first that Q is cyclic. Let $\mathcal{Y}\in i_Q^{-1}(\mathcal{X})$, and let $(\mathbf{S},\theta)\in\mathcal{Y}$. Let $\mathbf{B}_Q$ be a Borel subgroup of $C_\mathbf{G}(Q)$ containing $\mathbf{S}$. Note that $G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta$ is not acyclic, because its class in $G_0(kC_\mathbf{G}(Q)^F)$ is nonzero. We have

$$G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta\simeq e_\mathcal{Y}G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta.$$

Let $(\mathbf{S}^+,\theta^+)=i_Q(\mathbf{T},\theta)\in\mathcal{X}$ and let $\mathbf{B}$ be a Q-stable Borel subgroup of $\mathbf{G}$ containing $\mathbf{S}^+$ (cf. Lemma 3.6). We have

$$\begin{aligned}\mathrm{Br}_{\Delta Q}(G\Gamma_c(\mathbf{Y}_\mathbf{B},k)e_{\theta+})&\simeq \mathrm{Br}_{\Delta Q}(e_\mathcal{X}G\Gamma_c(\mathbf{Y}_\mathbf{B},k)e_{\theta+})\\&\simeq \mathrm{br}_{\Delta Q}(e_\mathcal{X}\otimes 1)G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)\mathrm{br}_{\Delta Q}(1\otimes e_{\theta+})\\&\simeq \mathrm{br}_Q(e_\mathcal{X})G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta\simeq \mathrm{br}_Q(e_\mathcal{X})e_\mathcal{Y}G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta.\end{aligned}$$

Similarly,

$$\mathrm{Br}_{\Delta Q}(G\Gamma_c(\mathbf{Y}_\mathbf{B},k)e_{\theta+})\simeq G\Gamma_c(\mathbf{Y}_{\mathbf{B}_Q},k)e_\theta\neq 0.$$

It follows that $\mathrm{br}_Q(e_\mathcal{X})e_\mathcal{Y}\neq 0$. Since

$$\sum_{\mathcal{X}'\in\nabla_{\ell'}(\mathbf{G},F)/\equiv}\mathrm{br}_Q(e_{\mathcal{X}'})=1=\sum_{\mathcal{Y}'\in\nabla_{\ell'}(C_\mathbf{G}(Q),F)/\equiv}e_{\mathcal{Y}'},$$

we deduce that $\mathrm{br}_Q(e_\mathcal{X})=\sum_{\mathcal{Y}\in i_Q^{-1}(\mathcal{X})}e_\mathcal{Y}$.

By transitivity of br_Q, we obtain the formula for br_Q for a general Q by induction on $|Q|$, with i_Q replaced by $i_{Q_\bullet}$. This shows that actually $i_{Q_\bullet}$ is independent of the chosen filtration of Q. □

Remark 4.15. Let $Q=Q'\times Q''$ be a product of two cyclic groups of coprime orders. Fix a filtration $Q_1=Q'$ and $Q_2=Q$. We have $i_{Q_\bullet}=i_Q$. It is easy to deduce now from Theorem 4.14 that $i_{Q_\bullet}$ is independent of Q for any nilpotent p'-group Q.

Broué-Michel's proof of Theorem 4.14 for $\mathbf{G}$ connected and Q cyclic relies on the compatibility of Deligne-Lusztig induction with generalized decomposition maps. This does generalize to disconnected groups, as we explain below. A direct approach along the lines of Broué-Michel is possible, based on the results of [DM15]. While we will not use the results in the remaining part of this section, they might be useful for character theoretic questions.

Let π be a set of prime numbers not containing p. An element of finite order of $\mathbf{G}$ is a π-element (resp. a π'-element) if its order is a product of primes in π (resp. not in π).

Let g be an automorphism of finite order of an algebraic variety $\mathbf{X}$. Write $g = lx = xl$, where l is a π-element and x a π'-element. The following result is an immediate consequence of [DL76, Th. 3.2]:

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(g, \mathrm{H}^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)) = \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(x, \mathrm{H}^i_c(\mathbf{X}^l, \overline{\mathbb{Q}}_\ell)). \tag{4.16}$$

Proof. Write $x = su = us$, where s has order prime to p and u has order a power of p. Then l, s and u commute and have coprime orders. By [DL76, Th. 3.2], we have

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(g, \mathrm{H}^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)) = \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(u, \mathrm{H}^i_c(\mathbf{X}^{ls}, \overline{\mathbb{Q}}_\ell))$$

and

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(x, \mathrm{H}^i_c(\mathbf{X}^l, \overline{\mathbb{Q}}_\ell)) = \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(u, \mathrm{H}^i_c((\mathbf{X}^l)^s, \overline{\mathbb{Q}}_\ell)).$$

So the result follows from the fact that $\mathbf{X}^{ls} = (\mathbf{X}^l)^s$ because $\langle ls \rangle = \langle l, s \rangle$. $\square$

Given H a finite group and $h \in H$ a π-element, we have a generalized decomposition map from the vector space of class functions $H \to K$ to the vector space of class functions on π'-elements of $C_H(h)$ given by $d_h^H(f)(u) = f(hu)$ for u a π'-element of $C_H(h)$.

The following result generalizes the character formula for $\mathrm{R}^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}}$ [DM94, Prop. 2.6], which corresponds to the case where π is the set of all primes distinct from p,

Proposition 4.17. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, let $\mathbf{V}$ be its unipotent radical, let $\mathbf{L}$ be a Levi complement of $\mathbf{P}$, and assume that $\mathbf{L}$ is F-stable. Let $g \in \mathbf{G}^F$ be a π-element. We have*

$$d_g^{\mathbf{G}^F} \circ \mathrm{R}^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}} = \sum_{\substack{x \in C_{\mathbf{G}}(g)^F \backslash \mathbf{G}^F / \mathbf{L}^F \\ g \in {}^x\mathbf{L}}} \mathrm{R}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{L}}(g) \subset C_{^x\mathbf{P}}(g)} \circ d_g^{{}^x\mathbf{L}^F} \circ x_*.$$

Proof. Given H a finite group, we denote by H_π (resp. $H_{\pi'}$) the set of π-elements (resp. π'-elements) of H. The proof follows essentially the same argument as the proof of the character formula (see, for instance, [DM91, Prop. 12.2]). Let λ be a class function on $\mathbf{L}^F$, and let $u \in C_{\mathbf{G}}(g)^F_{\pi'}$ be a π'-element. By definition of the Deligne-Lusztig induction and by using (4.16), we get

$$\begin{aligned}\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda)(gu) &= \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F_\pi}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(lv)\sum_{i\geqslant 0}(-1)^i\,\mathrm{Tr}((gu,lv),\mathrm{H}^i_c(\mathbf{Y}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell)).\\ &= \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F_\pi}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(lv)\sum_{i\geqslant 0}(-1)^i\,\mathrm{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{(g,l)}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell)).\end{aligned}$$

But it follows from Lemma 3.7 that $\mathbf{Y}^{(g,l)}_{\mathbf{V}} \neq \varnothing$ if and only if there exists $x \in \mathbf{G}^F$ such that $x^{-1}gx = l$. Moreover, in this case, then $\mathbf{Y}^{(g,l)}_{\mathbf{V}} \simeq \mathbf{Y}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{V}}(g)}$ by Proposition 3.4. Therefore,

$$\begin{aligned}\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\lambda)(gu) &= \frac{1}{|\mathbf{L}^F|\cdot|C_{\mathbf{G}}(g)^F|}\sum_{\substack{x\in\mathbf{G}^F\\ g\in{}^x\mathbf{L}}}\sum_{v\in C_{\mathbf{L}}(l)^F_{\pi'}}\lambda(x^{-1}gxv)\\ &\quad\times\sum_{i\geqslant 0}(-1)^i\,\mathrm{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{(g,x^{-1}gx)}_{\mathbf{V}},\overline{\mathbb{Q}}_\ell))\\ &= \frac{1}{|\mathbf{L}^F|\cdot|C_{\mathbf{G}}(g)^F|}\sum_{\substack{x\in\mathbf{G}^F\\ g\in{}^x\mathbf{L}}}\sum_{v\in C_{^x\mathbf{L}}(g)^F_{\pi'}}d^{^x\mathbf{L}}_g(x_*(\lambda))(v)\\ &\quad\times\sum_{i\geqslant 0}(-1)^i\,\mathrm{Tr}((u,v),\mathrm{H}^i_c(\mathbf{Y}^{C_{\mathbf{G}}(g)}_{C_{^x\mathbf{V}}(g)},\overline{\mathbb{Q}}_\ell)).\end{aligned}$$

Now, if $x \in \mathbf{G}^F$ is such that $g \in {}^x\mathbf{L}$, then

$$|C_{\mathbf{G}}(g)^F x\mathbf{L}^F| = \frac{|C_{\mathbf{G}}(g)^F|\cdot|\mathbf{L}^F|}{|C_{^x\mathbf{L}}(g)^F|}.$$

So the result follows. $\square$

5. Comparing Y-varieties

From now on, and until the end of this article, we assume $\mathbf{G}$ is connected.

Deligne-Lusztig varieties can be associated to sequences of elements of W, and there is a canonical isomorphism $\mathbf{X}(v,w) \xrightarrow{\sim} \mathbf{X}(vw)$ when $l(vw) = l(v) + l(w)$. We will show in this section that while such an isomorphism fails when $l(vw) \neq l(v) + l(w)$, its consequence on cohomology remains true for local systems associated to characters of tori satisfying certain regularity conditions with respect to (v,w).

In this section we will prove the preliminary statements necessary for our proof of Theorem 1.3. Roughly speaking, the main result of this section (Theorem 5.16) is almost equivalent to Theorem 1.3 whenever $\mathbf{L}$ is a maximal torus. As Theorem 1.3 will be proved by reduction to this case, Theorem 5.16 may be seen as the crucial step.

In this section, we fix an F-stable maximal torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$, and we denote by $\mathbf{U}$ its unipotent radical. We put $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We denote by Φ the associated root system, by Φ^+ the set of positive roots and by Δ the basis of Φ. Let $\alpha \in \Phi$. We denote by $s_\alpha \in W$ the corresponding reflection, and by $\alpha^\vee \in \Phi^\vee$ the corresponding coroot. We put $\mathbf{T}_{\alpha^\vee} = \mathrm{Im}(\alpha^\vee) \subset \mathbf{T}$ and we denote by $\mathbf{U}_\alpha$ the one-parameter subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$ and associated with α. We define $\mathbf{G}_\alpha$ as the subgroup of $\mathbf{G}$ generated by $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$.

5.A. *Dimension estimates and further.* In this section we fix four parabolic subgroups $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ admitting a common Levi complement $\mathbf{L}$. We denote by $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ the unipotent radicals of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ respectively.

We define the varieties

$$\begin{aligned}\boldsymbol{\mathcal{Y}}_{1,2,3} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_2 \times \mathbf{G}/\mathbf{V}_3 \mid g_1^{-1}g_2 \in \mathbf{V}_1 \cdot \mathbf{V}_2 \\ \text{and } g_2^{-1}g_3 \in \mathbf{V}_2 \cdot \mathbf{V}_3\},\end{aligned}$$

$$\boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \boldsymbol{\mathcal{Y}}_{1,2,3} \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\},$$

and

$$\boldsymbol{\mathcal{Y}}_{1,3} = \{(g_1\mathbf{V}_1, g_3\mathbf{V}_3) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_3 \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\}.$$

We denote by $i_{1,3} : \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3} \hookrightarrow \boldsymbol{\mathcal{Y}}_{1,2,3}$ the closed immersion, and we define

$$\begin{array}{rccc}\pi_{1,3} : & \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3} & \longrightarrow & \boldsymbol{\mathcal{Y}}_{1,3} \\ & (g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) & \longmapsto & (g_1\mathbf{V}_1, g_3\mathbf{V}_3).\end{array}$$

All these varieties are endowed with a diagonal action of $\mathbf{G}$, and the morphisms $i_{1,3}$ and $\pi_{1,3}$ are $\mathbf{G}$-equivariant.

Proposition 5.1. *We have*

(a) $\dim(\mathbf{V}_1) = \dim(\mathbf{V}_2) = \dim(\mathbf{V}_3)$;
(b) $\dim(\boldsymbol{\mathcal{Y}}_{1,2,3}) - \dim(\boldsymbol{\mathcal{Y}}_{1,3}) = \dim(\mathbf{V}_1) + \dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2) - \dim(\mathbf{V}_2 \cap \mathbf{V}_3)$;
(c) $\dim(\boldsymbol{\mathcal{Y}}_{1,2,3}) - \dim(\boldsymbol{\mathcal{Y}}_{1,3}) = 2\big(\dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3)\big)$.

Proof. (a) is well known. Also,

$$\begin{aligned}\dim(\boldsymbol{\mathcal{Y}}_{1,2,3}) &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1 \cdot \mathbf{V}_2/\mathbf{V}_2) + \dim(\mathbf{V}_2 \cdot \mathbf{V}_3/\mathbf{V}_3) \\ &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2) \\ &\quad + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_2 \cap \mathbf{V}_3)\end{aligned}$$

while

$$\begin{aligned}\dim(\boldsymbol{\mathcal{Y}}_{1,3}) &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1 \cdot \mathbf{V}_3/\mathbf{V}_3)\\ &= \dim(\mathbf{G}/\mathbf{V}_1) + \dim(\mathbf{V}_1) - \dim(\mathbf{V}_1 \cap \mathbf{V}_3).\end{aligned}$$

So (b) follows from the two equalities (and from (a)).

Let us now prove (c). For this, we may assume that $\mathbf{T} \subset \mathbf{L}$. Let Φ_i denote the set of roots $\alpha \in \Phi$ such that $\mathbf{U}_\alpha \subset \mathbf{V}_i$. Then $\Phi_1 \cup -\Phi_1 = \Phi_2 \cup -\Phi_2 = \Phi_3 \cup -\Phi_3 = \Phi \setminus \Phi_{\mathbf{L}}$. In particular,

$$\Phi_1 \cup -\Phi_1 = (\Phi_1 \cup \Phi_2 \cup \Phi_3) \cup -(\Phi_1 \cap \Phi_2 \cap \Phi_3).$$

Therefore,

$$2|\Phi_1| = |\Phi_1 \cup \Phi_2 \cup \Phi_3| + |\Phi_1 \cap \Phi_2 \cap \Phi_3|.$$

On the other hand, by general facts about the cardinality of a union of finite sets,

$$|\Phi_1 \cup \Phi_2 \cup \Phi_3| = |\Phi_1| + |\Phi_2| + |\Phi_3| - |\Phi_1 \cap \Phi_2| - |\Phi_1 \cap \Phi_3| - |\Phi_2 \cap \Phi_3| + |\Phi_1 \cap \Phi_2 \cap \Phi_3|.$$

Hence (c) follows from (a), (b) and from these last two equalities. □

Let $d_{1,3} = \dim(\mathbf{V}_1 \cap \mathbf{V}_3) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3)$. By Proposition 5.1, we have

$$d_{1,3} = \frac{1}{2}\big(\dim(\boldsymbol{\mathcal{Y}}_{1,2,3}) - \dim(\boldsymbol{\mathcal{Y}}_{1,3})\big).$$

Let

$$\begin{array}{rccc}\kappa_{1,3}: & \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_3) & \longrightarrow & \boldsymbol{\mathcal{Y}}_{1,3}\\ & g(\mathbf{V}_1 \cap \mathbf{V}_3) & \longmapsto & (g\mathbf{V}_1, g\mathbf{V}_3)\end{array}$$

and

$$\begin{array}{rccc}\kappa^{\mathrm{cl}}_{1,2,3}: & \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3) & \longrightarrow & \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3}\\ & g(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3) & \longmapsto & (g\mathbf{V}_1, g\mathbf{V}_2, g\mathbf{V}_3).\end{array}$$

Both maps are $\mathbf{G}$-equivariant morphisms of varieties.

PROPOSITION 5.2. *The maps $\kappa_{1,3}$ and $\kappa^{\mathrm{cl}}_{1,2,3}$ are isomorphisms of varieties.*

Proof. The fact that $\kappa_{1,3}$ is an isomorphism is clear. It is also clear that $\kappa^{\mathrm{cl}}_{1,2,3}$ is a closed immersion. It is thus sufficient to prove that $\kappa^{\mathrm{cl}}_{1,2,3}$ is surjective.

So, let $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) \in \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3}$. Using the $\mathbf{G}$-action and the fact that $\kappa_{1,3}$ is an isomorphism, we may assume that $g_1 = g_3 = 1$. Therefore,

$$g_2 \in (\mathbf{V}_1 \cdot \mathbf{V}_2) \cap (\mathbf{V}_3 \cdot \mathbf{V}_2).$$

Given $i \in \{1,3\}$, the multiplication map $(\mathbf{V}_i \cap \mathbf{V}_2^-) \times (\mathbf{V}_i \cap \mathbf{V}_2) \to \mathbf{V}_i$ is an isomorphism of varieties, since $\mathbf{V}_i$ and $\mathbf{V}_2$ have a common Levi complement. Here, $\mathbf{V}_2^-$ denotes the unipotent radical of the parabolic subgroup opposite to $\mathbf{P}_2$. It follows that

$$(\mathbf{V}_1 \cdot \mathbf{V}_2) \cap (\mathbf{V}_3 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cap \mathbf{V}_3 \cap \mathbf{V}_2^-) \cdot \mathbf{V}_2 = (\mathbf{V}_1 \cap \mathbf{V}_3) \cdot \mathbf{V}_2.$$

So there exists $h \in \mathbf{V}_1 \cap \mathbf{V}_3$ such that $h\mathbf{V}_2 = g_2\mathbf{V}_2$. It is then clear that $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3) = \kappa^{\mathrm{cl}}_{1,2,3}(h)$, as desired. □

Corollary 5.3. *The map $\pi_{1,3}$ is a smooth morphism with fibers isomorphic to the affine space of dimension $d_{1,3}$. Moreover,*

$$\dim(\boldsymbol{\mathcal{Y}}_{1,2,3}) - \dim(\boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3}) = \dim(\boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3}) - \dim(\boldsymbol{\mathcal{Y}}_{1,3}) = d_{1,3}.$$

Proof. Using the isomorphisms $\kappa_{1,3}$ and $\kappa^{\mathrm{cl}}_{1,2,3}$ of Proposition 5.2, the map $\pi_{1,3}$ may be identified with the canonical projection $\mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_2 \cap \mathbf{V}_3) \longrightarrow\!\!\!\!\rightarrow \mathbf{G}/(\mathbf{V}_1 \cap \mathbf{V}_3)$. The corollary follows. □

Let us now define

$$\begin{aligned}\boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \mathbf{G}/\mathbf{V}_1 \times \mathbf{G}/\mathbf{V}_2 \times \mathbf{G}/\mathbf{V}_3 \times \mathbf{G}/\mathbf{V}_4 \mid \\ g_1^{-1}g_2 \in \mathbf{V}_1 \cdot \mathbf{V}_2,\ g_2^{-1}g_3 \in \mathbf{V}_2 \cdot \mathbf{V}_3,\ g_3^{-1}g_4 \in \mathbf{V}_3 \cdot \mathbf{V}_4 \\ \text{and } g_1^{-1}g_4 \in \mathbf{V}_1 \cdot \mathbf{V}_4\},\end{aligned}$$

$$\boldsymbol{\mathcal{Y}}^{\mathrm{cl},2}_{1,2,3,4} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4} \mid g_1^{-1}g_3 \in \mathbf{V}_1 \cdot \mathbf{V}_3\},$$

and $$\boldsymbol{\mathcal{Y}}^{\mathrm{cl},3}_{1,2,3,4} = \{(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4} \mid g_2^{-1}g_4 \in \mathbf{V}_2 \cdot \mathbf{V}_4\}.$$

Then:

Corollary 5.4. *Assume that at least one of the following holds*:

(1) $\mathbf{V}_1 \subset \mathbf{V}_4 \cdot \mathbf{V}_2$;
(2) $\mathbf{V}_2 \subset \mathbf{V}_1 \cdot \mathbf{V}_3$;
(3) $\mathbf{V}_3 \subset \mathbf{V}_2 \cdot \mathbf{V}_4$;
(4) $\mathbf{V}_4 \subset \mathbf{V}_3 \cdot \mathbf{V}_1$.

Then $\boldsymbol{\mathcal{Y}}^{\mathrm{cl},2}_{1,2,3,4} = \boldsymbol{\mathcal{Y}}^{\mathrm{cl},3}_{1,2,3,4} = \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4}$.

Proof. Using the fact that the map

$$(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \mapsto (g_4\mathbf{V}_4, g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3)$$

induces an isomorphism $\boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4} \xrightarrow{\sim} \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{4,1,2,3}$ (with obvious notation), we see that it is sufficient to prove only one of the statements.

So let us assume that $\mathbf{V}_2 \subset \mathbf{V}_1 \cdot \mathbf{V}_3$. Let $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4}$. Then $g_1^{-1}g_3 = (g_1^{-1}g_2)(g_2^{-1}g_3) \in \mathbf{V}_1 \cdot \mathbf{V}_2 \cdot \mathbf{V}_3 = \mathbf{V}_1 \cdot \mathbf{V}_3$ and so $\boldsymbol{\mathcal{Y}}^{\mathrm{cl},2}_{1,2,3,4} = \boldsymbol{\mathcal{Y}}^{\mathrm{cl}}_{1,2,3,4}$. So it remains to prove that $(g_1\mathbf{V}_1, g_2\mathbf{V}_2, g_3\mathbf{V}_3, g_4\mathbf{V}_4) \in \boldsymbol{\mathcal{Y}}^{\mathrm{cl},3}_{1,2,3,4}$. Using the action of $\mathbf{G}$ and the isomorphism $\kappa^{\mathrm{cl}}_{1,3,4}$ of Proposition 5.2, we may assume that $g_1 = g_3 = g_4 = 1$. Since $\mathbf{V}_2 \subset \mathbf{V}_1 \cdot \mathbf{V}_3$, we have $\mathbf{V}_1 \cap \mathbf{V}_3 \subset \mathbf{V}_2$, hence $g_2^{-1}g_4 = g_2^{-1} \in (\mathbf{V}_2 \cdot \mathbf{V}_3) \cap (\mathbf{V}_2 \cdot \mathbf{V}_1) \subset \mathbf{V}_2 \cdot (\mathbf{V}_1 \cap \mathbf{V}_3) \subset \mathbf{V}_2$, as desired. □

Remark 5.5. Let w_1, w_2 and w_3 be three elements of W, and let us assume here that $\mathbf{V}_1 = \mathbf{U}$, $\mathbf{V}_2 = {}^{w_1}\mathbf{V}_1$, $\mathbf{V}_3 = {}^{w_1w_2}\mathbf{V}_1$ and $\mathbf{V}_4 = {}^{w_1w_2w_3}\mathbf{V}_1$. Then the

conditions (1), (2), (3) and (4) of Corollary 5.4 are respectively equivalent to the following:
(1) $l(w_2w_3) = l(w_1w_2w_3) + l(w_1)$;
(2) $l(w_1w_2) = l(w_1) + l(w_2)$;
(3) $l(w_2w_3) = l(w_2) + l(w_3)$;
(4) $l(w_1w_2) = l(w_1w_2w_3) + l(w_3)$.

5.B. *Setting.* We fix a positive integer r. Given a family $m_1, \ldots, m_r$ of objects belonging to a structure acted on by F, we put $m_{j+er} = F^e(m_j)$ for $1 \leqslant j \leqslant r$ and $e \geqslant 0$.

Let $\mathbf{n} = (n_1, \ldots, n_r)$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$. We denote by w_i the image of n_i in W, and we put $w = w_1 \cdots w_r$.

We define

$$\mathbf{Y}(\mathbf{n}) = \{(g_1\mathbf{U}, g_2\mathbf{U}, \ldots, g_r\mathbf{U}) \in (\mathbf{G}/\mathbf{U})^r \mid g_j \xrightarrow{\ n_j\ } g_{j+1} \quad \forall 1 \leqslant j \leqslant r\},$$

where $g_j \xrightarrow{\ n_j\ } g_{j+1}$ means $g_j^{-1}g_{j+1} \in \mathbf{U}n_j\mathbf{U}$. This variety has a left action by multiplication of $\mathbf{G}^F$ and a right action of $\mathbf{T}^{wF}$, where $t \in \mathbf{T}^{wF}$ acts by right multiplication by $(t, t^{n_1}, \ldots, t^{n_1\cdots n_{r-1}})$.

We define the functor $\mathcal{R}_{\mathbf{n}} = \mathrm{R}\Gamma_c(\mathbf{Y}(\mathbf{n}), \Lambda) \otimes^{\mathbf{L}}_{\Lambda\mathbf{T}^{wF}} - : \mathrm{D}^b(\Lambda\mathbf{T}^{wF}) \to \mathrm{D}^b(\Lambda\mathbf{G}^F)$, and we put $\mathrm{R}_{\mathbf{n}} = [\mathcal{R}_{\mathbf{n}}] : G_0(\Lambda\mathbf{T}^{wF}) \to G_0(\Lambda\mathbf{G}^F)$ as in [BR03, §5.2].

We fix a positive integer m such that $F^m(n_i) = n_i$ for all i. The action of F^m on $(\mathbf{G}/\mathbf{U})^r$ restricts to an action on $\mathbf{Y}(\mathbf{n})$.

Given $\mathbf{Z}$ a variety of pure dimension n, we will consider

$$\mathrm{R}\Gamma_c^{\dim}(\mathbf{Z}, \Lambda) = \mathrm{R}\Gamma_c(\mathbf{Z}, \Lambda)[n](n/2),$$

where $(n/2)$ denotes a Tate twist.

Given $2 \leqslant j \leqslant r$, we denote by $\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n})$ the F^m-stable closed subvariety of $\mathbf{Y}(\mathbf{n})$ defined by

$$\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) = \{(g_1\mathbf{U}, g_2\mathbf{U}, \ldots, g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-1} \xrightarrow{\ n_{j-1}n_j\ } g_{j+1}\},$$

and we denote by $\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n})$ its open complement. They are both stable under the action of $\mathbf{G}^F \times \mathbf{T}^{wF}$. We denote by $\pi_j : (\mathbf{G}/\mathbf{U})^r \to (\mathbf{G}/\mathbf{U})^{r-1}$ the morphism of varieties that forgets the j-th component, and we set

$$c_j(\mathbf{n}) = (n_1, n_2, \ldots, n_{j-2}, n_{j-1}n_j, n_{j+1}, \ldots, n_r)$$

and

$$d_j(\mathbf{n}) = \frac{l(w_{j-1}) + l(w_j) - l(w_{j-1}w_j)}{2}.$$

Let $i_{\mathbf{n},j} : \mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) \hookrightarrow \mathbf{Y}(\mathbf{n})$ denote the closed immersion and

$$\pi_{\mathbf{n},j} : \mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) \to \mathbf{Y}(c_j(\mathbf{n}))$$

denote the restriction of π_j. Note that $\pi_{\mathbf{n},j}$ is $(\mathbf{G}^F, \mathbf{T}^{wF})$-equivariant and commutes with the action of F^m.

LEMMA 5.6. *If $2 \leqslant j \leqslant r$, then $\pi_{\mathbf{n},j}$ is a smooth morphism with fibers isomorphic to an affine space of dimension $d_j(\mathbf{n})$. Moreover, the codimension of $\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n})$ in $\mathbf{Y}(\mathbf{n})$ is also equal to $d_j(\mathbf{n})$.*

Proof. Let $\mathbf{L} = \mathbf{T}$, $\mathbf{V}_1 = \mathbf{U}$, $\mathbf{V}_2 = {}^{n_{j-1}}\mathbf{U}$ and $\mathbf{V}_3 = {}^{n_{j-1}n_j}\mathbf{U}$. There is a cartesian square

$$\begin{tikzcd} \mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}) \arrow[rrr, "{(g_1\mathbf{U},\ldots,g_r\mathbf{U})\mapsto(g_{j-1}\mathbf{V}_1,g_j n_{j-1}^{-1}\mathbf{V}_2,g_{j+1}n_j^{-1}n_{j-1}^{-1}\mathbf{V}_3)}"] \arrow[d, "\pi_{\mathbf{n},j}"'] & & & \boldsymbol{\mathcal{Y}}_{1,2,3}^{\mathrm{cl}} \arrow[d, "\pi_{1,3}"] \\ \mathbf{Y}(c_j(\mathbf{n})) \arrow[rrr, "{(h_1\mathbf{U},\ldots,h_{r-1}\mathbf{U})\mapsto(g_{j-1}\mathbf{V}_1,g_{j+1}n_j^{-1}n_{j-1}^{-1}\mathbf{V}_3)}"'] & & & \boldsymbol{\mathcal{Y}}_{1,3}. \end{tikzcd}$$

The lemma follows from now from Corollary 5.3 by base change. □

The map $\pi_{\mathbf{n},j}$ induces a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{T}^{wF})$-bimodules

$$\mathrm{R\Gamma_c}(\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}), \Lambda) \xrightarrow{\sim} \mathrm{R\Gamma_c}(\mathbf{Y}(c_j(\mathbf{n})), \Lambda)[-2d_j(\mathbf{n})](-d_j(\mathbf{n})). \tag{5.7}$$

Composing this isomorphism with the morphism

$$i_{\mathbf{n},j}^* : \mathrm{R\Gamma_c}(\mathbf{Y}(\mathbf{n}), \Lambda) \to \mathrm{R\Gamma_c}(\mathbf{Y}_j^{\mathrm{cl}}(\mathbf{n}), \Lambda),$$

we obtain a morphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{T}^{wF})$-bimodules

$$\Psi_{\mathbf{n},j} : \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(\mathbf{n}), \Lambda) \longrightarrow \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(c_j(\mathbf{n})), \Lambda)$$

that commutes with the action of F^m and whose cone is quasi-isomorphic to $\mathrm{R\Gamma_c^{dim}}(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}), \Lambda)[1]$.

5.C. *Preliminaries.* We first recall some results from [BR03]. We denote by B the braid group of W, and by $\sigma : W \to B$ the unique map (not a group morphism) that is a right inverse to the canonical map $B \to W$ and that preserves lengths. We extend it to sequences of elements of W by $\sigma(w_1, \ldots, w_r) = \sigma(w_1) \cdots \sigma(w_r)$.

We denote by $n \mapsto \bar{n} : N_{\mathbf{G}}(\mathbf{T}) \to W$ the quotient map. We fix $\dot{\sigma} : N_{\mathbf{G}}(\mathbf{T}) \to B \ltimes \mathbf{T}$ a map (not a group morphism) such that $\dot{\sigma}(nt) = \dot{\sigma}(n)t$ for all $t \in \mathbf{T}$ and such that the image of $\dot{\sigma}(n)$ in $B = (B \ltimes \mathbf{T})/\mathbf{T}$ is equal to $\sigma(\bar{n})$. We extend it to sequences of elements of $N_{\mathbf{G}}(\mathbf{T})$ by $\dot{\sigma}(n_1, \ldots, n_r) = \dot{\sigma}(n_1) \cdots \dot{\sigma}(n_r)$.

The following result is [BR03, Prop. 5.4].

LEMMA 5.8. *Let $\mathbf{n}'$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$. Then*

(a) *if $\dot{\sigma}(\mathbf{n}) = \dot{\sigma}(\mathbf{n}')$ (they are elements of $B \ltimes \mathbf{T}$), then the varieties $\mathbf{Y}(\mathbf{n})$ and $\mathbf{Y}(\mathbf{n}')$ are canonically isomorphic $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$;*

(b) *if $\sigma(\overline{\mathbf{n}}) = \sigma(\overline{\mathbf{n}}')$ (they are elements of B), then the varieties $\mathbf{Y}(\mathbf{n})$ and $\mathbf{Y}(\mathbf{n}')$ are (noncanonically) isomorphic $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$.*

Proof. (a) is proved in [BR03, 5.5], while (b) is [BR03, Prop. 5.4]. □

Using Lemma 5.8(a), we shall now write $\mathbf{Y}(\mathbf{n}) = \mathbf{Y}(\mathbf{n}')$ when $\dot\sigma(\mathbf{n}) = \dot\sigma(\mathbf{n}')$. Strictly speaking, Lemma 5.8(a) says that these two varieties are only isomorphic but, since this isomorphism is canonical, we shall use the symbol $=$ to simplify the exposition.

We define the *cyclic shift* $\mathrm{sh}(\mathbf{n})$ of $\mathbf{n}$ by

$$\mathrm{sh}(\mathbf{n}) = (n_2, \ldots, n_r, F(n_1)).$$

The next result is proved in [DMR07, Prop. 3.1.6] for the varieties $\mathbf{X}(\mathbf{w})$ and $\mathbf{X}(\mathbf{w}')$. The same proof shows the more precise result below.

LEMMA 5.9. *The map*

$$\begin{array}{ccc} \mathbf{Y}(\mathbf{n}) & \longrightarrow & \mathbf{Y}(\mathrm{sh}(\mathbf{n})) \\ (g_1\mathbf{U}, \ldots, g_r\mathbf{U}) & \longmapsto & (g_2\mathbf{U}, \ldots, g_r\mathbf{U}, F(g_1)\mathbf{U}) \end{array}$$

induces an equivalence of étale sites. Moreover, it is a morphism of $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$, where $t \in \mathbf{T}^{wF}$ acts on $\mathbf{Y}(\mathrm{sh}(\mathbf{n}))$ by right multiplication by $n_1^{-1} t n_1$. Consequently, the diagram

$$\begin{tikzcd} \mathrm{D}^b(\Lambda \mathbf{T}^{w_1^{-1} w F(w_1) F}) \arrow[rr, "n_{1,*}"] \arrow[dr, "\mathcal{R}_{\mathrm{sh}(\mathbf{n})}"'] & & \mathrm{D}^b(\Lambda \mathbf{T}^{wF}) \arrow[dl, "\mathcal{R}_{\mathbf{n}}"] \\ & \mathrm{D}^b(\Lambda \mathbf{G}^F) & \end{tikzcd}$$

is commutative.

Assume in the remaining part of Section 5.C that $3 \leqslant j \leqslant r$ (in particular, $r \geqslant 3$). Note that $c_{j-1}(c_j(\mathbf{n})) = c_{j-1}(c_{j-1}(\mathbf{n}))$. Consider the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}), \Lambda) \arrow[r, "\Psi_{\mathbf{n},j}"] \arrow[d, "\Psi_{\mathbf{n},j-1}"'] & \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})), \Lambda) \arrow[d, "\Psi_{c_j(\mathbf{n}),j-1}"] \\ \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_{j-1}(\mathbf{n})), \Lambda) \arrow[r, "\Psi_{c_{j-1}(\mathbf{n}),j-1}"'] & \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_{j-1}(c_j(\mathbf{n}))), \Lambda). \end{tikzcd} \tag{5.10}$$

It does not seem reasonable to expect that the diagram (5.10) is commutative in general. However, it is in some cases.

Let us first define the following two varieties:

$$\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_j(\mathbf{n})) \times_{\mathbf{Y}(c_j(\mathbf{n}))} \mathbf{Y}^{\mathrm{cl}}_j(\mathbf{n})$$

and

$$\mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_{j-1}(\mathbf{n})) \times_{\mathbf{Y}(c_{j-1}(\mathbf{n}))} \mathbf{Y}^{\mathrm{cl}}_{j-1}(\mathbf{n}).$$

More concretely, they are the closed subvarieties of $\mathbf{Y}(\mathbf{n})$ defined by

$$\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \{(g_1\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-2} \overset{n_{j-2}n_{j-1}n_j}{\overline{\hspace{4em}}} g_{j+1} \text{ and } g_{j-1} \overset{n_{j-1}n_j}{\overline{\hspace{3em}}} g_{j+1}\}$$

and

$$\mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) = \{(g_1\mathbf{U},\dots,g_r\mathbf{U}) \in \mathbf{Y}(\mathbf{n}) \mid g_{j-2} \overset{n_{j-2}n_{j-1}n_j}{\overline{\hspace{4em}}} g_{j+1} \text{ and } g_{j-2} \overset{n_{j-2}n_{j-1}}{\overline{\hspace{3em}}} g_j\}.$$

LEMMA 5.11. *If* $\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n})$, *then the diagram* (5.10) *is commutative.*

Proof. There is a commutative diagram, in which all the arrows of the form $\hookrightarrow$ are closed immersions and all the arrows of the form $\twoheadrightarrow$ are smooth morphisms with fibers isomorphic to an affine space:

(5.12)
$$\begin{tikzcd}
 & & \mathbf{Y}(\mathbf{n}) & & \\
\mathbf{Y}^{\mathrm{cl}}_{j-1}(\mathbf{n}) \arrow[urr, hook, "i_{\mathbf{n},j-1}"] \arrow[dd, two heads, "\pi_{\mathbf{n},j-1}"'] & & & & \mathbf{Y}^{\mathrm{cl}}_{j}(\mathbf{n}) \arrow[ull, hook', "i_{\mathbf{n},j}"'] \arrow[dd, two heads, "\pi_{\mathbf{n},j}"] \\
 & \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n}) \arrow[ul, hook', "i"'] \arrow[rr, equal] \arrow[dd, two heads, "\pi"'] \arrow[dl, phantom, "\blacksquare"] & & \mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) \arrow[ur, hook, "i'"] \arrow[dd, two heads, "\pi'"] \arrow[dr, phantom, "\blacksquare"] & \\
\mathbf{Y}(c_{j-1}(\mathbf{n})) & & & & \mathbf{Y}(c_j(\mathbf{n})) \\
 & \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_{j-1}(\mathbf{n})) \arrow[ul, hook', "i_{c_{j-1}(\mathbf{n}),j-1}"] \arrow[dr, two heads, "\pi_{c_{j-1}(\mathbf{n}),j-1}"'] & & \mathbf{Y}^{\mathrm{cl}}_{j-1}(c_j(\mathbf{n})) \arrow[ur, hook, "i_{c_j(\mathbf{n}),j-1}"'] \arrow[dl, two heads, "\pi_{c_j(\mathbf{n}),j-1}"] & \\
 & & \mathbf{Y}(c_{j-1}(c_j(\mathbf{n}))) & &
\end{tikzcd}$$

Note that the two squares marked with the symbol ■ are cartesian by definition. By the proper base change theorem, the composition $\Psi_{c_{j-1}(\mathbf{n}),j-1} \circ \Psi_{\mathbf{n},j-1}$ is obtained as the composition of $(i_{\mathbf{n},j-1} \circ i)^*$ with the inverse of the isomorphism induced by $(\pi_{c_{j-1}(\mathbf{n}),j-1} \circ \pi)^*$. Similarly, the composition $\Psi_{c_j(\mathbf{n}),j-1} \circ \Psi_{\mathbf{n},j}$ is equal to the composition of $(i_{\mathbf{n},j} \circ i')^*$ with the inverse of the isomorphism induced by $(\pi_{c_j(\mathbf{n}),j-1} \circ \pi')^*$. The lemma follows. □

LEMMA 5.13. *Assume that one of the following holds*:

(1) $l(w_{j-2}w_{j-1}) = l(w_{j-2}) + l(w_{j-1})$;
(2) $l(w_{j-1}w_j) = l(w_{j-1}) + l(w_j)$;
(3) $l(w_{j-2}w_{j-1}) = l(w_{j-2}w_{j-1}w_j) + l(w_j)$;
(4) $l(w_{j-1}w_j) = l(w_{j-2}) + l(w_{j-2}w_{j-1}w_j)$.

Then the diagram (5.10) *is commutative.*

Proof. It is sufficient, by Lemma 5.11, to prove that, if (1), (2), (3) or (4) holds, then $\mathbf{Y}^{\mathrm{cl}}_{j,j-1}(\mathbf{n}) = \mathbf{Y}^{\mathrm{cl}}_{j-1,j}(\mathbf{n})$. This follows, after base change, from Corollary 5.4. □

5.D. *Comparison of complexes.* We start with the description of varieties of the form $\mathbf{Y}^{\mathrm{op}}_1(\mathbf{n})$ in a very special case, which will be the fundamental step in the proof of Theorem 5.16.

Let $\mathbf{w} = \bar{\mathbf{n}} = (w_1, \ldots, w_r)$. Given $\alpha \in \Delta$, we define a subgroup of $\mathbf{T}^{r+1}$

$$\begin{aligned}\mathbf{S}(\alpha, \mathbf{w}) = \{(a_1, \ldots, a_{r+1}) \in \mathbf{T}^{r+1} \mid a_1^{-1} s_\alpha a_2 s_\alpha^{-1} \\ \in \mathbf{T}_{\alpha^\vee},\ a_i^{-1} w_{i-1} a_{i+1} w_{i-1}^{-1} = 1 \text{ for } 2 \leqslant i \leqslant r \\ \text{and } a_{r+1}^{-1} w_r F(a_1) w_r^{-1} = 1\}.\end{aligned}$$

Let $x \in \{1, s_\alpha\}$. The group morphism

$$\mathbf{T} \to \mathbf{T}^{r+1},\ a \mapsto (a, x^{-1}ax, w_1^{-1}x^{-1}axw_1, \ldots, w_{r-1}^{-1} \cdots w_1^{-1}x^{-1}axw_1 \cdots w_{r-1})$$

restricts to an embedding of $\mathbf{T}^{xwF}$ in $\mathbf{S}(\alpha, \mathbf{w})$.

Given $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ two sequences, we denote the concatenation of the sequences by $\mathbf{a} \bullet \mathbf{b} = (a_1, \ldots, a_m, b_1, \ldots, b_n)$.

LEMMA 5.14. *Let $\alpha \in \Delta$, and let $\dot{s}$ be a representative of s_α in $N_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{G}_\alpha$. We assume that $\mathbf{G}_\alpha \simeq \mathbf{SL}_2(\mathbb{F})$. There exist a closed immersion $\mathbf{Y}(\dot{s} \bullet \mathbf{n}) \hookrightarrow \mathbf{Y}^{\mathrm{op}}_2((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ and an action of $\mathbf{S}(\alpha, \mathbf{w})$ on $\mathbf{Y}^{\mathrm{op}}_2((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n})$ such that*

$$\mathbf{Y}^{\mathrm{op}}_2((\dot{s}, \dot{s}^{-1}) \bullet \mathbf{n}) \simeq \mathbf{Y}(\dot{s} \bullet \mathbf{n}) \times_{\mathbf{T}^{s_\alpha wF}} \mathbf{S}(\alpha, \mathbf{w}),$$

as $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$.

Proof. Given $i \in \{1, \ldots, r\}$, consider a reduced decomposition $w_i = s_{i,1} \cdots s_{i,d_i}$. We put $\tilde{\mathbf{w}} = (s_{1,1}, \ldots, s_{1,d_1}, s_{2,1}, \ldots, s_{2,d_2}, \ldots, s_{r,1}, \ldots, s_{r,d_r})$. Note that $\mathbf{S}(\alpha,\mathbf{w})$ is isomorphic to the group $\mathbf{S}(s_\alpha\bullet\tilde{\mathbf{w}},1\bullet\tilde{\mathbf{w}})$ defined in [BR03, §4.4.3]:

$$\mathbf{S}(s_\alpha\bullet\tilde{\mathbf{w}},1\bullet\tilde{\mathbf{w}})\xrightarrow{\sim}\mathbf{S}(\alpha,\mathbf{w}),\ (a_1,\ldots,a_{1+d_1+\cdots+d_r})$$
$$\mapsto(a_1,a_2,a_{2+d_1},a_{2+d_1+d_2},\ldots,a_{2+d_1+\cdots+d_{r-1}}).$$

The following computation in $\mathbf{SL}_2(\mathbb{F})\simeq\mathbf{G}_\alpha$,

(#),
$$\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&y\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&z\\0&1\end{pmatrix}=\begin{pmatrix}1-xy&x+z-xyz\\-y&1-yz\end{pmatrix}$$

shows that the map

$$\begin{array}{ccc}\mathbf{U}_\alpha\times(\mathbf{U}_\alpha\setminus\{1\})\times\mathbf{U}_\alpha&\longrightarrow&\mathbf{U}_\alpha\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}_\alpha=\mathbf{G}_\alpha\setminus\mathbf{B}\cap\mathbf{G}_\alpha\\(u_1,u_2,u_3)&\longmapsto&u_1\dot{s}u_2\dot{s}^{-1}u_3\end{array}$$

is an isomorphism of varieties. Let $\mathbf{U}^\alpha=\mathbf{U}\cap{}^{\dot{s}}\mathbf{U}$. Let $(g_1\mathbf{U},\ldots,g_{r+2}\mathbf{U})\in\mathbf{Y}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$. We have $(g_1\mathbf{U},\ldots,g_{r+2}\mathbf{U})\in\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ if and only if

$$\begin{aligned}g_1^{-1}g_3\in(\mathbf{U}\dot{s}\mathbf{U}\dot{s}^{-1}\mathbf{U})\setminus\mathbf{U}&=\mathbf{U}^\alpha\cdot(\mathbf{U}_\alpha\dot{s}\mathbf{U}_a\dot{s}^{-1}\mathbf{U}_\alpha\setminus\mathbf{U}_\alpha)\\&=\mathbf{U}^\alpha\cdot(\mathbf{U}_\alpha\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}_\alpha)=\mathbf{U}\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}.\end{aligned}$$

Furthermore, if $(g_1\mathbf{U},\ldots,g_{r+2}\mathbf{U})\in\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$, then $g_2\mathbf{U}$ is determined by $g_1\mathbf{U}$ and $g_3\mathbf{U}$.

Therefore, one may forget the second coordinate in the definition of the variety $\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ and we get

(5.15)
$$\begin{aligned}\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})\ \simeq\ &\{(g\mathbf{U},g_1\mathbf{U},\ldots,g_r\mathbf{U})\mid g^{-1}g_1\in\mathbf{U}\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}\\&\text{and } g_1\xrightarrow{\ n_1\ }g_2\xrightarrow{\ n_2\ }\cdots\xrightarrow{\ n_{r-1}\ }g_r\xrightarrow{\ n_r\ }F(g)\}.\end{aligned}$$

This description shows that the group $\mathbf{S}(\alpha,\mathbf{w})$ acts on $\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ (as the restriction of the action by right multiplication of $\mathbf{T}^{r+1}$ on $(\mathbf{G}/\mathbf{U})^{r+1}$). Also, since $\mathbf{U}\dot{s}\mathbf{U}$ is closed in $\mathbf{U}\mathbf{T}_{\alpha^\vee}\dot{s}\mathbf{U}$, it follows that the natural map $\mathbf{Y}(\dot{s}\bullet\mathbf{n})\hookrightarrow\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n})$ is a closed immersion. We have embeddings $\mathbf{T}^{s_\alpha wF}\hookrightarrow\mathbf{S}(\alpha,\mathbf{w})$ and $\mathbf{T}^{wF}\hookrightarrow\mathbf{S}(\alpha,\mathbf{w})$ and

$$\mathbf{S}(\alpha,\mathbf{w})=\mathbf{T}^{s_\alpha wF}\cdot\mathbf{S}(\alpha,\mathbf{w})^\circ=\mathbf{S}(\alpha,\mathbf{w})^\circ\cdot\mathbf{T}^{wF}$$

(see [BR03, Prop. 4.11]). The stabilizer of the closed subvariety $\mathbf{Y}(\dot{s}\bullet\mathbf{n})$ under this action is $\mathbf{T}^{s_\alpha wF}$, so it is readily checked that the componentwise multiplication induces an isomorphism of $\mathbf{G}^F$-varieties-$\mathbf{T}^{wF}$

$$\mathbf{Y}(\dot{s}\bullet\mathbf{n})\times_{\mathbf{T}^{s_\alpha wF}}\mathbf{S}(\alpha,\mathbf{w})\xrightarrow{\sim}\mathbf{Y}_2^{\mathrm{op}}((\dot{s},\dot{s}^{-1})\bullet\mathbf{n}),$$

as desired. $\square$

The next theorem is the main result of this section. It provides a sufficient condition for $\Psi_{\mathbf{n},j}$ to induce a quasi-isomorphism

$$\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}),\Lambda)e_\theta \xrightarrow{\sim} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})),\Lambda)e_\theta.$$

Given $x,y\in W$, we put

$$\Phi^+(x,y)=\{\alpha\in\Phi^+ \mid x^{-1}(\alpha)\in -\Phi^+ \text{ and } (xy)^{-1}(\alpha)\in\Phi^+\}.$$

We define $N_w: Y(\mathbf{T})\to\mathbf{T}^{wF},\ \lambda\mapsto N_{F^d/wF}(\lambda(\zeta))$ (cf. §4.A).

THEOREM 5.16. *Let $\theta:\mathbf{T}^{wF}\to\Lambda^\times$ be a character. Let $j\in\{2,3,\dots,r\}$, and assume that $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))\neq 1$ for all $\alpha\in\Phi^+(w_{j-1},w_j)$. We have $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$, and*

$$\Psi_{\mathbf{n},j,\theta}:\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(\mathbf{n}),\Lambda)e_\theta \xrightarrow{\sim} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}(c_j(\mathbf{n})),\Lambda)e_\theta$$

is a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{T}^{wF})$-bimodules commuting with the action of F^m.

Proof. If $2\leqslant j\leqslant r$, we denote by $\mathcal{P}(\mathbf{n},j,\theta)$ the following property:

$\mathcal{P}(\mathbf{n},j,\theta)$ for all $\alpha\in\Phi^+(w_{j-1},w_j)$, we have $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))\neq 1$.

We want to prove that $\mathcal{P}(\mathbf{n},j,\theta)$ implies that $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$. By [BR03, Prop. 5.19, Rem. 5.21], it is sufficient to prove it whenever $[\mathbf{G},\mathbf{G}]$ is simply connected, and we will assume this holds.

So assume from now that $\mathcal{P}(\mathbf{n},j,\theta)$ holds. We will prove by induction on $l(w_{j-1})$ that $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$. Note that the induction hypothesis does not depend on r. But first, note that if $j\geqslant 3$, then $\mathcal{P}(\mathbf{n},j,\theta)$ is equivalent to $\mathcal{P}(\mathrm{sh}(\mathbf{n}),j-1,\theta\circ n_1)$ and that the morphism constructed in Lemma 5.9 sends $\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n})$ to $\mathbf{Y}_{j-1}^{\mathrm{op}}(\mathrm{sh}(\mathbf{n}))$. Thus $\mathrm{R}\Gamma_c(\mathbf{Y}_j^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=0$ is equivalent to $\mathrm{R}\Gamma_c(\mathbf{Y}_{j-1}^{\mathrm{op}}(\mathrm{sh}(\mathbf{n})),\Lambda)e_{\theta\circ n_1}=0$. By successive applications of this remark, this shows that we may assume that $j=2$.

First case: *Assume that $l(w_1)=0$.* This means that $n_1\in\mathbf{T}$, and it follows from Lemma 5.6 (or Lemma 5.8(a)) that $\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n})=\varnothing$. So the result follows in this case.

Second case: *Assume that $l(w_1)=1$ and $n_1n_2=1$.* Let $\alpha\in\Delta$ be such that $w_1=s_\alpha$. By Lemma 5.8, we may assume that $n_1=\dot{s}$ is a representative of s_α lying in $\mathbf{G}_\alpha$. Note also that, since $[\mathbf{G},\mathbf{G}]$ is simply connected, we have $\mathbf{G}_\alpha\simeq\mathbf{SL}_2(\mathbb{F})$. Define $\mathbf{S}=\mathbf{S}(\alpha,(w_3,\dots,w_r))$. Lemma 5.14 shows that

$$\mathrm{R}\Gamma_c(\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta=\mathrm{R}\Gamma_c(\mathbf{Y}(\dot{s},n_3,\dots,n_r),\Lambda)\otimes_{\Lambda\mathbf{T}^{s_\alpha wF}}\mathrm{R}\Gamma_c(\mathbf{S},\Lambda)e_\theta.$$

But $\Phi^+(w_1,w_2)=\Phi^+(s_\alpha,s_\alpha)=\{\alpha\}$, so $\theta(N_w(\alpha^\vee))\neq 1$ by hypothesis. Note also that $\mathbf{T}^{wF}\cap\mathbf{S}^\circ$ acts trivially on the cohomology groups of the complex $\mathrm{R}\Gamma_c(\mathbf{S})$, as its action extends to the connected group $\mathbf{S}^\circ$. Since $N_w(\alpha^\vee)\in\mathbf{S}^\circ$

(see [BR03, proof of Prop. 4.11, equality (a)]), this proves that $\mathrm{R\Gamma_c}(\mathbf{S},\Lambda)e_\theta = 0$ and so $\mathrm{R\Gamma_c}(\mathbf{Y}_2^{\mathrm{op}}(\mathbf{n}),\Lambda)e_\theta = 0$, as desired.

Last case: *Assume that* $l(w_1) \geqslant 1$. Let $\alpha \in \Delta$ be such that $w_1 = s_\alpha w_1'$, with $l(w_1') = l(w_1) - 1$. Let $\dot{s}$ be a representative of s_α in $\mathbf{G}_\alpha$, and let $n_1' = \dot{s}^{-1}n_1$. We will write $\mathbf{n}' = (n_1', n_2, \dots, n_r)$. Then n_1' is a representative of w_1' and, by Lemma 5.8(a), we have $\mathbf{Y}(\mathbf{n}) = \mathbf{Y}(\dot{s} \bullet \mathbf{n}')$. (See also the remark following Lemma 5.8.)

It is well known that

$$\Phi^+(w_1, w_1^{-1}) = \Phi^+ \cap w_1(-\Phi^+) = \{\alpha\} \coprod s_\alpha(\Phi^+(w_1', w_1^{-1})).$$

Therefore,

$$\begin{aligned}\Phi^+(w_1, w_2) &= \Phi^+ \cap w_1(-\Phi^+) \cap w_1w_2(\Phi^+)\\ &= \big(\{\alpha\} \cap w_1w_2(\Phi^+)\big) \coprod s_\alpha\big(\Phi^+ \cap w_1'(-\Phi^+) \cap w_1'w_2(\Phi^+)\big)\\ &= \big(\{\alpha\} \cap w_1w_2(\Phi^+)\big) \coprod s_\alpha(\Phi^+(w_1', w_2)),\end{aligned}$$

and hence

$$(\#) \qquad \Phi^+(w_1, w_2) = \begin{cases} s_\alpha\big(\Phi^+(w_1', w_2)\big) & \text{if } l(w_1'w_2) < l(w_1w_2),\\ \{\alpha\} \coprod s_\alpha\big(\Phi^+(w_1', w_2)\big) & \text{if } l(w_1'w_2) > l(w_1w_2).\end{cases}$$

Let us now consider the diagram (5.12) with $\mathbf{n}$ replaced by $\dot{s} \bullet \mathbf{n}'$ and j replaced by 3. Since $c_2(\dot{s} \bullet \mathbf{n}') = \mathbf{n}$, it follows from Lemma 5.13(1) that (5.10) gives a commutative diagram

$$\begin{tikzcd} \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(\dot{s} \bullet \mathbf{n}'))e_\theta \arrow[r] \arrow[d] & \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(c_3(\dot{s} \bullet \mathbf{n}')))e_\theta \arrow[d] \\ \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(\mathbf{n}))e_\theta \arrow[r] & \mathrm{R\Gamma_c^{dim}}(\mathbf{Y}(c_2(\mathbf{n})))e_\theta. \end{tikzcd}$$

The left vertical map is an isomorphism since $\mathbf{Y}(\dot{s} \bullet \mathbf{n}') \simeq \mathbf{Y}(\mathbf{n})$. By (#), we have $\Phi^+(w_1', w_2) \subset s_\alpha(\Phi^+(w_1, w_2))$, hence Property $\mathcal{P}(\dot{s} \bullet \mathbf{n}', 3, \theta)$ is fulfilled. So, the top horizontal map is an isomorphism by induction.

In order to show that the bottom horizontal map is an isomorphism, it remains to show that the right vertical map is an isomorphism. Note that $c_3(\dot{s} \bullet \mathbf{n}') = \dot{s} \bullet c_2(\mathbf{n}')$. Two cases may occur:

- Assume first that $l(w_1'w_2) < l(w_1w_2)$. Then $\mathbf{Y}_2^{\mathrm{op}}(c_3(\dot{s} \bullet \mathbf{n}')) = \varnothing$, and the result follows.
- Assume now that $l(w_1'w_2) > l(w_1w_2)$. Then, again by Lemma 5.8(a), we have $\mathbf{Y}(\dot{s} \bullet c_2(\mathbf{n}')) = \mathbf{Y}((\dot{s}, \dot{s}^{-1}) \bullet c_2(\mathbf{n}))$ and, through this identification, $\mathbf{Y}_2^{\mathrm{op}}(c_3(\dot{s} \bullet \mathbf{n}'))$ is identified with $\mathbf{Y}_2^{\mathrm{op}}((\dot{s}, \dot{s}^{-1}) \bullet c_2(\mathbf{n}))$. So the result now follows from the *second case* (thanks to (#)). $\square$

Remark 5.17. Theorem 5.16 provides a comparison of modules, together with the Frobenius action. Consider the case $\Lambda = K$. We have an isomorphism of $K\mathbf{G}^F$-modules, compatible with the Frobenius action

$$H^i_c(\mathbf{Y}(\mathbf{n}), K) \otimes_{K\mathbf{T}^{wF}} K_\theta \simeq H^{i-2r}_c(\mathbf{Y}(c_j(\mathbf{n})), K) \otimes_{K\mathbf{T}^{wF}} K_\theta(-r),$$

where $r = d_j(\mathbf{n})$.

Following the same lines as in the proof of Theorem 5.16, we obtain a new proof of the following classical result.

THEOREM 5.18. *If Λ is a field, then $\mathrm{R}_{\mathbf{n}} = \mathrm{R}_w$.*

Proof. By [BR03, Prop. 5.19, Rem. 5.21], it is sufficient to prove the Theorem whenever $[\mathbf{G}, \mathbf{G}]$ is simply connected, and we assume this holds. Also, by proceeding step-by-step, it is enough to prove that $\mathrm{R}_{\mathbf{n}} = \mathrm{R}_{c_j(\mathbf{n})}$. For this, let $\mathrm{R}\Gamma^{\mathrm{op}}_{\mathbf{n},j}$ denote the class of the complex $\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{op}}_j(\mathbf{n}), \Lambda)$ in $G_0(\Lambda\mathbf{G}^F \otimes \Lambda\mathbf{T}^{wF})$. We only need to prove that $\mathrm{R}\Gamma^{\mathrm{op}}_{\mathbf{n},j} = 0$.

Proceeding by induction on $l(w_j)$ as in the proof of Theorem 5.16, and following the same strategy and arguments, we see that it is enough to prove Theorem 5.18 whenever $j = 1$, $n_1 = \dot{s} = n_2^{-1}$, where $\dot{s}$ is a representative in $\mathbf{G}_\alpha$ of s_α (for some $\alpha \in \Delta$). By Lemma 5.14, it is sufficient to prove that the class $\mathrm{R}\Gamma_{\alpha,\mathbf{w}}$ of the complex $\mathrm{R}\Gamma_c(\mathbf{S}(\alpha, \mathbf{w}), \Lambda)$ in $G_0(\Lambda\mathbf{T}^{s_\alpha wF} \otimes \Lambda\mathbf{T}^{wF})$ is equal to 0.

Now, let T denote the subgroup of $\mathbf{T}^{s_\alpha wF} \times \mathbf{T}^{wF}$ consisting of pairs (t_1, t_2) such that $t_1 t_2 \in \mathbf{S}(\alpha, \mathbf{w})^\circ$, and let $\mathrm{R}\Gamma^\circ$ denote the class of the complex $\mathrm{R}\Gamma_c(\mathbf{S}(\alpha, \mathbf{w})^\circ)$ in $G_0(\Lambda T)$. Then $\mathrm{R}\Gamma_{\alpha,\mathbf{w}} = \mathrm{Ind}_T^{\mathbf{T}^{s_\alpha wF} \times \mathbf{T}^{wF}} \mathrm{R}\Gamma^\circ$. But the action of T on $\mathbf{S}(\alpha, \mathbf{w})$ extends to an action of the connected group $\mathbf{S}(\alpha, \mathbf{w})^\circ$, hence T acts trivially on the cohomology groups of $\mathbf{S}(\alpha, \mathbf{w})^\circ$. Since the Euler characteristic of a torus is equal to 0, this gives $\mathrm{R}\Gamma^\circ = 0$, and consequently $\mathrm{R}\Gamma_{\alpha,\mathbf{w}} = 0$, as desired. $\square$

COROLLARY 5.19. *Let $\mathbf{n}' = (n'_1, n'_2, \ldots, n'_{r'})$ be a sequence of elements of $N_{\mathbf{G}}(\mathbf{T})$, let $x \in W$, and let w' denote the image of $n'_1 n'_2 \cdots n'_{r'}$ in W. We assume that Λ is a field and that $w' = x^{-1}wF(x)$. Then the diagram*

$$\begin{tikzcd}
G_0(\Lambda\mathbf{T}^{\mathbf{n}'F}) \arrow[rr, "x_*"] \arrow[dr, "\mathrm{R}_{\mathbf{n}'}"'] & & G_0(\Lambda\mathbf{T}^{\mathbf{n}F}) \arrow[dl, "\mathrm{R}_{\mathbf{n}}"] \\
& G_0(\Lambda\mathbf{G}^F) &
\end{tikzcd}$$

is commutative.

Proof. Let $\mathbf{n}'' = (\dot{x}^{-1}, n_1, n_2, \ldots, n_r, F(\dot{x}))$. Then, by Lemma 5.9,

$$\mathrm{R}_{\mathbf{n}''} = \mathrm{R}_{\mathbf{n}\bullet(F(\dot{x}),F(\dot{x})^{-1})} \circ x_*.$$

But, by Theorem 5.18, $\mathrm{R}_{\mathbf{n}''} = \mathrm{R}_{w'} = \mathrm{R}_{\mathbf{n}'}$ and $\mathrm{R}_{\mathbf{n}\bullet(F(\dot{x}),F(\dot{x})^{-1})} = \mathrm{R}_w = \mathrm{R}_{\mathbf{n}}$. □

The following result is a reformulation of Corollary 5.19 as in [BR03, §11.1].

COROLLARY 5.20. *Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$, and let $\mathbf{B}'$ and $\mathbf{B}''$ be two Borel subgroups of $\mathbf{G}$ containing $\mathbf{T}'$. Then $\mathrm{R}^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}'}(\theta') = \mathrm{R}^{\mathbf{G}}_{\mathbf{T}'\subset\mathbf{B}''}(\theta')$ for all $\theta' \in \mathrm{Irr}(\mathbf{T}'^F)$.*

Remark 5.21. Corollary 5.20 is well known. In [DL76, Cor. 4.3], this result is first proved "geometrically" for $\theta' = 1$ [DL76, Th. 1.6] by relating the varieties $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}'}$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}''}$, and extended to the general case using the *character formula* [DL76, Th. 4.2]. Note that this result is then used in [DL76, Th. 6.8] to deduce the *Mackey formula* for Deligne-Lusztig induction functors.

In [Lus78], Lusztig proposed another argument: the Mackey formula is proved "geometrically" and *a priori* [Lus78, Th. 2.3], and Corollary 5.20 follows [Lus78, Cor. 2.4].

Our argument relies neither on the Mackey formula nor on the character formula: we lift Deligne-Lusztig's comparison of $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}'}$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{B}''}$ to a relation between the varieties $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}$ and $\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}''}$. (Here $\mathbf{U}'$ and $\mathbf{U}''$ are the unipotent radicals of $\mathbf{B}'$ and $\mathbf{B}''$ respectively.)

Remark 5.22. Some of the results in [BR06] (Lemma 4.3, Proposition 4.5 and Theorem 4.6) rely on a disjointness result used in [BR06, p. 30, line 16]. This disjointness result was "proved" using the isomorphism in [BR06, p. 30, line 18]: it has been pointed out to the attention of the authors by H. Wang that this equality is false. However, Wang provided a complete proof of this disjointness result [Wan14, Prop. 3.4.3], so [BR06, Lemma 4.3, Prop. 4.5, Th. 4.6] remain valid.

Another proof of this disjointness result has been obtained independently by Nguyen [Ngu] (with slightly different methods). Using a version of Remark 5.17, Wang and Nguyen have been able to keep track of the Frobenius eigenvalues.

6. Independence with respect to the parabolic subgroup

We assume in this section that $\mathbf{G}$ is connected. We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, and we denote by $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ a triple dual to $(\mathbf{G}, \mathbf{T}, F)$.

We fix a family of parabolic subgroups $\mathbf{P}_1$, $\mathbf{P}_2, \ldots,$ $\mathbf{P}_r$ admitting $\mathbf{L}$ as a Levi complement.

The identification of the root system of $\mathbf{G}$ with the coroot system of $\mathbf{G}^*$ allows us to define parabolic subgroups $\mathbf{P}_1^*$, $\mathbf{P}_2^*$,..., $\mathbf{P}_r^*$, admitting a common F^*-stable Levi complement $\mathbf{L}^*$ and such that $\mathbf{L}^*$ and $\mathbf{P}_j^*$ and are dual to $\mathbf{L}$ and $\mathbf{P}_j$ respectively. We denote by $\mathbf{V}_j$ and $\mathbf{V}_j^*$ the unipotent radicals of $\mathbf{P}_j$ and $\mathbf{P}_j^*$ respectively. We denote by $\mathbf{V}_\bullet$ the sequence $(\mathbf{V}_1,\ldots,\mathbf{V}_r)$.

Finally, we fix a semisimple element $s\in\mathbf{L}^{*F^*}$ whose order is invertible in Λ.

6.A. *Isomorphisms.* As announced in the introduction, the isomorphism of functors described in Theorem 1.3 is *canonical*. So, before giving the proof, we will explain how it is realized. For this, let us define

$$\begin{aligned}\mathbf{Y}_{\mathbf{V}_\bullet} = \{(&g_1\mathbf{V}_1,\ldots,g_r\mathbf{V}_r)\\ &\in \mathbf{G}/\mathbf{V}_1\times\cdots\times\mathbf{G}/\mathbf{V}_r \mid \forall\, j\in\{1,2,\ldots,r\},\ g_j^{-1}g_{j+1}\in\mathbf{V}_j\cdot\mathbf{V}_{j+1}\},\end{aligned}$$

where $\mathbf{V}_{r+1}=F(\mathbf{V}_1)$ and $g_{r+1}=F(g_1)$. Given $2\leqslant j\leqslant r$, we set

$$\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j} = \{(g_1\mathbf{V}_1,g_2\mathbf{V}_2,\ldots,g_r\mathbf{V}_r)\in\mathbf{Y}_{\mathbf{V}_\bullet}\mid g_{j-1}^{-1}g_{j+1}\in\mathbf{V}_{j-1}\cdot\mathbf{V}_{j+1}\}.$$

It is a closed subvariety of $\mathbf{Y}_{\mathbf{V}_\bullet}$, and we denote by $i_{\mathbf{V}_\bullet,j}:\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}\hookrightarrow\mathbf{Y}_{\mathbf{V}_\bullet}$ the closed immersion. Let $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j}$ denote its open complement. We define the sequence $c_j(\mathbf{V}_\bullet)$ as obtained from the sequence $\mathbf{V}_\bullet$ by removing the j-th component. We then define

$$\pi_{\mathbf{V}_\bullet,j}:\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}\longrightarrow\mathbf{Y}_{c_j(\mathbf{V}_\bullet)}$$

as the map that forgets the j-th component, and we set

$$d_j(\mathbf{V}_\bullet)=\dim(\mathbf{V}_{j-1}\cap\mathbf{V}_{j+1})-\dim(\mathbf{V}_{j-1}\cap\mathbf{V}_j\cap\mathbf{V}_{j+1}).$$

Note that $\mathbf{G}^F$ acts diagonally on $\mathbf{Y}_{\mathbf{V}_\bullet}$ by left translation, that $\mathbf{L}^F$ acts diagonally by right translation, and that this endows $\mathbf{Y}_{\mathbf{V}_\bullet}$ with a structure of $\mathbf{G}^F$-variety-$\mathbf{L}^F$. The varieties $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet}$ and $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet}$ are stable under these actions, and the morphisms $i_{\mathbf{V}_\bullet,j}$ and $\pi_{\mathbf{V}_\bullet,j}$ are equivariant. As for their analogues $i_{\mathbf{n},j}$ and $\pi_{\mathbf{n},j}$ defined in Section 5.B, we have the following properties, which follow from Corollary 5.3 by base change.

Lemma 6.1. *The map $\pi_{\mathbf{V}_\bullet,j}$ is smooth with fibers isomorphic to an affine space of dimension $d_j(\mathbf{V}_\bullet)$. The codimension of $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet}$ in $\mathbf{Y}_{\mathbf{V}_\bullet}$ is also equal to $d_j(\mathbf{V}_\bullet)$.*

We deduce that $\pi_{\mathbf{V}_\bullet,j}$ induces a quasi-isomorphism between complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j},\Lambda)\simeq\mathrm{R}\Gamma_c(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)[-2d_j(\mathbf{V}_\bullet)](-d_j(\mathbf{V}_\bullet)).$$

The closed immersion $i_{\mathbf{V}_\bullet,j}:\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}\hookrightarrow\mathbf{Y}_{\mathbf{V}_\bullet}$ induces a morphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$i^*_{\mathbf{V}_\bullet,j}:\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_\bullet},\Lambda)\longrightarrow\mathrm{R}\Gamma_c(\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j},\Lambda)$$

which, composed with the previous isomorphism, induces a morphism

$$\Psi_{\mathbf{V}_\bullet,j} : \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet},\Lambda) \longrightarrow \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda).$$

The main result of this section is the following theorem. We put $e_s^{\mathbf{L}^F} = e_{\mathcal{Y}}$, where $\mathcal{Y} \in \nabla_\Lambda(\mathbf{L},F)/\equiv$ is the rational series corresponding to the $\mathbf{L}^{*F^*}$-conjugacy class of s.

THEOREM 6.2. *Let $j \in \{2,3,\dots,r\}$ such that $C_{\mathbf{V}_{j-1}^*\cap\mathbf{V}_{j+1}^*}(s) \subset C_{\mathbf{V}_j^*}(s)$. We have*

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet,j},\Lambda)e_s^{\mathbf{L}^F} = 0,$$

and hence $\Psi_{\mathbf{V}_\bullet,j}$ induces a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$\Psi_{\mathbf{V}_\bullet,j,s} : \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet,j},\Lambda)e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)},\Lambda)e_s^{\mathbf{L}^F}.$$

Proof. The proof will proceed in two steps. We first prove the theorem when $\mathbf{L}$ is a maximal torus: in fact, it will be shown that it is a consequence of Theorem 5.16. We then use [BR03, Th. A′] to deduce the general case from this particular one.

First step: *Assume here that $\mathbf{L}$ is a maximal torus.* Let $a_1,\dots,\ a_r$ be elements of $\mathbf{G}$ such that $(\mathbf{L},\mathbf{P}_i) = {}^{a_i}(\mathbf{T},\mathbf{B})$ for all $i \in \{1,2,\dots,r\}$). As usual, we set $a_{r+1} = F(a_r)$. Now, let $n_i = a_i^{-1}a_{i+1}$. It follows from the definition of the a_i's that $n_i \in N_{\mathbf{G}}(\mathbf{T})$. We set $\mathbf{n} = (n_1,\dots,n_r)$. Note that $n_1n_2\cdots n_r = a_1^{-1}F(a_1)$. We denote by w_i the image of n_i in W, and we set $w = w_1w_2\cdots w_r$. It is then easily checked that the map

$$(g_1\mathbf{V}_1,\dots,g_r\mathbf{V}_r) \longmapsto (g_1\mathbf{V}_1a_1,\dots,g_r\mathbf{V}_ra_r)$$

induces an isomorphism of varieties

$$\mathbf{Y}_{\mathbf{V}_\bullet} \xrightarrow{\sim} \mathbf{Y}(\mathbf{n})$$

that sends $\mathbf{Y}^{\mathrm{cl}}_{\mathbf{V}_\bullet,j}$ to $\mathbf{Y}^{\mathrm{cl}}_j(\mathbf{n})$. Moreover, conjugacy by a_1 induces an isomorphism $\mathbf{T}^{wF} \simeq \mathbf{L}^F$ and it is easily checked that the above isomorphism is $(\mathbf{G}^F,\mathbf{L}^F)$-equivariant through this identification. Now, to s is associated a linear character of $\mathbf{L}^F$ that, through the identification $\mathbf{T}^{wF} \simeq \mathbf{L}^F$, defines a linear character $\theta : \mathbf{T}^{wF} \to \Lambda^\times$.

By Theorem 5.16, we only need to prove that Condition $C_{\mathbf{V}_{j-1}^*\cap\mathbf{V}_{j+1}^*}(s) \subset C_{\mathbf{V}_j^*}(s)$ is equivalent to $\mathcal{P}(\mathbf{n},j,\theta)$. So let us prove this last fact. The property $\mathcal{P}(\mathbf{n},j,\theta)$ can be rewritten as follows:

PROPERTY $\mathcal{P}(\mathbf{n},j,\theta)$. *If $\alpha \in \Phi^+$ is such that $\theta(N_w(w_1\cdots w_{j-2}(\alpha^\vee)))$ $= 1$ and $(w_{j-1}w_j)^{-1}(\alpha) \in \Phi^+$, then $w_{j-1}^{-1}(\alpha) \in \Phi^+$.*

Let $s' = a_1^{-1} s a_1 \in \mathbf{T}^{*wF^*}$. Note that $\mathcal{P}(\mathbf{n}, j, \theta)$ is equivalent to

$$C_{{}^{w_1\cdots w_{j-2}}\mathbf{U}^*}(s') \cap {}^{w_1\cdots w_j}\mathbf{U}^* \subset {}^{w_1\cdots w_{j-1}}\mathbf{U}^*.$$

By conjugating by a_1, and since ${}^{a_1 n_1 \cdots n_i}\mathbf{U}^* = \mathbf{V}_i^*$, we get that $\mathcal{P}(\mathbf{n}, j, \theta)$ is equivalent to $C_{\mathbf{V}_{j-1}^*}(s) \cap \mathbf{V}_{j+1}^* \subset \mathbf{V}_j^*$, as desired.

Second step: *The general case.* Let us now come back to the general case: we no longer assume that $\mathbf{L}$ is a maximal torus. Since $\mathrm{R\Gamma_c}(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet, j}, \Lambda) e_s^{\mathbf{L}^F} = \mathrm{R\Gamma_c}(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet, j}, \Lambda) \otimes_{\Lambda \mathbf{L}^F} \Lambda \mathbf{L}^F e_s^{\mathbf{L}^F}$, and since $\Lambda \mathbf{L}^F e_s^{\mathbf{L}^F}$ lives in the category generated by the complexes $\mathcal{R}^{\mathbf{L}}_{\mathbf{T}' \subset \mathbf{B}'}(\Lambda \mathbf{T}'^F e_s^{\mathbf{T}'^F})$, where $\mathbf{B}'$ runs over the set of Borel subgroups of $\mathbf{L}$ admitting an F-stable maximal torus $\mathbf{T}'$ whose dual torus contains s (see [BR03, Th. A']), it is sufficient to prove that

$$\mathrm{R\Gamma_c}(\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet, j}, \Lambda) \otimes_{\Lambda \mathbf{L}^F} \mathcal{R}^{\mathbf{L}}_{\mathbf{T}' \subset \mathbf{B}'}(\Lambda \mathbf{T}'^F e_s^{\mathbf{T}'^F}) = 0.$$

So let $(\mathbf{T}', \mathbf{B}')$ be a pair as above. Let $\mathbf{U}'$ denote the unipotent radical of $\mathbf{B}'$, let $\mathbf{T}'^*$ be an F^*-stable maximal torus of $\mathbf{L}^*$, containing s and dual to $\mathbf{T}'$, and let $\mathbf{B}'^*$ be a Borel subgroup of $\mathbf{L}^*$ containing $\mathbf{T}'^*$ and dual to $\mathbf{B}'$. Then [DM91, 11.5]

$$\mathbf{Y}_{\mathbf{V}_\bullet} \times_{\mathbf{L}^F} \mathbf{Y}^{\mathbf{L}}_{\mathbf{U}'} \simeq \mathbf{Y}_{\mathbf{U}'\mathbf{V}_\bullet}$$

(as $\mathbf{G}^F$-varieties-$\mathbf{T}'^F$). Here, we have set $\mathbf{U}'\mathbf{V}_\bullet = (\mathbf{U}'\mathbf{V}_1, \dots, \mathbf{U}'\mathbf{V}_r)$. Moreover, through this isomorphism, $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_\bullet, j} \times_{\mathbf{L}^F} \mathbf{Y}^{\mathbf{L}}_{\mathbf{U}'}$ is sent to $\mathbf{Y}^{\mathrm{op}}_{\mathbf{U}'\mathbf{V}_\bullet, j}$ hence, by applying the first step of this proof, we only need to prove that

$$C_{\mathbf{U}'^*\mathbf{V}_{j-1}^* \cap \mathbf{U}'^*\mathbf{V}_{j+1}^*}(s) \subset C_{\mathbf{U}'^*\mathbf{V}_j^*}(s).$$

Since $\mathbf{V}_{j-1}^*$ and $\mathbf{V}_{j+1}^*$ both admit $\mathbf{L}^*$ as a Levi complement and $\mathbf{U}'^* \subset \mathbf{L}^*$, it follows that $\mathbf{U}'^*\mathbf{V}_{j-1}^* \cap \mathbf{U}'^*\mathbf{V}_{j+1}^* \subset \mathbf{U}'^*(\mathbf{V}_{j-1}^* \cap \mathbf{V}_{j+1}^*)$. On the other hand, $C_{\mathbf{U}'^*(\mathbf{V}_{j-1}^* \cap \mathbf{V}_{j+1}^*)}(s) = C_{\mathbf{U}'^*}(s) C_{\mathbf{V}_{j-1}^* \cap \mathbf{V}_{j+1}^*}(s) \subset C_{\mathbf{U}'^*}(s) C_{\mathbf{V}_j^*}(s)$ by assumption, and this completes the proof. □

Remark 6.3. Theorem 6.2 provides a comparison of modules, together with the Frobenius action. We have an isomorphism of $(\Lambda \mathbf{G}^F, \Lambda \mathbf{L}^F)$-bimodules compatible with the Frobenius action

$$H_c^i(\mathbf{Y}_{\mathbf{V}, j}, \Lambda) e_s^{\mathbf{L}^F} \simeq H_c^{i-2r}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)}, \Lambda) e_s^{\mathbf{L}^F}(-r),$$

where $r = d_j(\mathbf{V}_\bullet)$.

Let $\mathrm{sh}(\mathbf{V}_\bullet) = (\mathbf{V}_2, \dots, \mathbf{V}_r, {}^F\mathbf{V}_1)$. The map

$$\begin{array}{rccc} \mathrm{sh}_{\mathbf{V}_\bullet} : & \mathbf{Y}_{\mathbf{V}_\bullet} & \longrightarrow & \mathbf{Y}_{\mathrm{sh}(\mathbf{V}_\bullet)} \\ & (g_1\mathbf{V}_1, \dots, g_r\mathbf{V}_r) & \longmapsto & (g_2\mathbf{V}_2, \dots, g_r\mathbf{V}_r, F(g_1\mathbf{V}_1)) \end{array}$$

is $(\mathbf{G}^F, \mathbf{L}^F)$-equivariant and induces an equivalence of étale sites. Therefore, it induces a quasi-isomorphism of complexes of bimodules

$$\mathrm{sh}^*_{\mathbf{V}_\bullet} : \mathrm{R\Gamma_c}(\mathbf{Y}_{\mathrm{sh}(\mathbf{V}_\bullet)}, \Lambda) \xrightarrow{\sim} \mathrm{R\Gamma_c}(\mathbf{Y}_{\mathbf{V}_\bullet}, \Lambda).$$

Applying twice Theorem 6.2, we obtain the following result.

COROLLARY 6.4. *Let $j \in \{2, \ldots, r\}$, and assume that*

$$C_{\mathbf{V}^*_{j-1} \cap \mathbf{V}^*_{j+1}}(s) \subset C_{\mathbf{V}^*_j}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_j \cap \mathbf{V}^*_{j+2}}(s) \subset C_{\mathbf{V}^*_{j+1}}(s).$$

The map $\Psi_{\mathbf{V}_\bullet, j, s} \circ \mathrm{sh}^*_{\mathbf{V}_\bullet} \circ \Psi^{-1}_{\mathrm{sh}(\mathbf{V}_\bullet), j, s}$ is a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{R\Gamma}^{\dim}_{\mathrm{c}}(\mathbf{Y}_{c_j(\mathrm{sh}(\mathbf{V}_\bullet))}, \Lambda) e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R\Gamma}^{\dim}_{\mathrm{c}}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)}, \Lambda) e_s^{\mathbf{L}^F}.$$

In the case $r = 2$, Corollary 6.4 becomes the following result.

COROLLARY 6.5. *Assume*

$$C_{\mathbf{V}^*_1 \cap F^*\mathbf{V}^*_1}(s) \subset C_{\mathbf{V}^*_2}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_2 \cap F^*\mathbf{V}^*_2}(s) \subset C_{F^*\mathbf{V}^*_1}(s).$$

The map $\Psi_{\mathbf{V}_1, \mathbf{V}_2, 2, s} \circ \mathrm{sh}^*_{\mathbf{V}_1, \mathbf{V}_2} \circ \Psi^{-1}_{\mathbf{V}_2, F(\mathbf{V}_1), 2, s}$ is a quasi-isomorphism of complexes of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{R\Gamma}^{\dim}_{\mathrm{c}}(\mathbf{Y}_{\mathbf{V}_2}, \Lambda) e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R\Gamma}^{\dim}_{\mathrm{c}}(\mathbf{Y}_{\mathbf{V}_1}, \Lambda) e_s^{\mathbf{L}^F}.$$

As a consequence, we obtain a quasi-isomorphism of functors between

$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}_1}[\dim(\mathbf{Y}_{\mathbf{V}_1})] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$$

and
$$\mathcal{R}^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}_2}[\dim(\mathbf{Y}_{\mathbf{V}_2})] : \mathrm{D}^b(\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}) \longrightarrow \mathrm{D}^b(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}).$$

Remark 6.6. The isomorphism of functors of Corollary 6.5 comes with a Tate twist. Keeping track of this twist has important applications [Wan14], [Ngu].

Remark 6.7. Let us make some comments here about the condition

$$(\mathcal{C}_{\mathbf{V}_1, \mathbf{V}_2}) \qquad C_{\mathbf{V}^*_1 \cap F^*\mathbf{V}^*_1}(s) \subset C_{\mathbf{V}^*_2}(s) \quad \textit{and} \quad C_{\mathbf{V}^*_2 \cap F^*\mathbf{V}^*_2}(s) \subset C_{F^*\mathbf{V}^*_1}(s).$$

Note that if $C_{\mathbf{V}^*_1}(s) = C_{\mathbf{V}^*_2}(s)$, then Condition $(\mathcal{C}_{\mathbf{V}_1, \mathbf{V}_2})$ is satisfied. Since $C_{\mathbf{V}^*_i}(s)$ is connected, it follows that if $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, then Condition $(\mathcal{C}_{\mathbf{V}_1, \mathbf{V}_2})$ is satisfied.

Example 6.8. Of course, Condition $(\mathcal{C}_{\mathbf{V}_1, \mathbf{V}_1})$ is fulfilled for all s. Gluing the quasi-isomorphisms obtained from Corollary 6.5, we get a quasi-isomorphism of complexes of bimodules

$$\Theta_{\mathbf{V}_1, \mathbf{V}_1} : \mathrm{R\Gamma}_{\mathrm{c}}(\mathbf{Y}_{\mathbf{V}_1}, \Lambda) \xrightarrow{\sim} \mathrm{R\Gamma}_{\mathrm{c}}(\mathbf{Y}_{\mathbf{V}_1}, \Lambda).$$

But, since $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_1, \mathbf{V}_1} = \varnothing$, it is readily checked that $\Theta_{\mathbf{V}_1, \mathbf{V}_1} = \mathrm{Id}_{\mathrm{R\Gamma}_{\mathrm{c}}(\mathbf{Y}_{\mathbf{V}_1}, \Lambda)}$.

Example 6.9. Similarly, Condition $(\mathcal{C}_{\mathbf{V}_1,F(\mathbf{V}_1)})$ is fulfilled for all s. Gluing the quasi-isomorphisms obtained from Corollary 6.5, we obtain a quasi-isomorphism of complexes of bimodules

$$\Theta_{\mathbf{V}_1,F(\mathbf{V}_1)} : \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{V}_1},\Lambda) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathbf{Y}_{F(\mathbf{V}_1)},\Lambda).$$

But, since $\mathbf{Y}^{\mathrm{op}}_{\mathbf{V}_1,F(\mathbf{V}_1)} = \varnothing$, it is readily checked that $\Theta_{\mathbf{V}_1,F(\mathbf{V}_1)} = F$.

Remark 6.10. If $(\mathcal{C}_{\mathbf{V}_1,\mathbf{V}_2})$ holds, we denote by

$$\Theta_{\mathbf{V}_1,\mathbf{V}_2,s} : \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_2},\Lambda)e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}_1},\Lambda)e_s^{\mathbf{L}^F}$$

the quasi-isomorphism defined by $\Theta_{\mathbf{V}_1,\mathbf{V}_2,s} = \Psi_{\mathbf{V}_1,\mathbf{V}_2,2,s}\circ\mathrm{sh}^*_{\mathbf{V}_1,\mathbf{V}_2}\circ\Psi^{-1}_{\mathbf{V}_2,F(\mathbf{V}_1),2,s}$. Assume moreover that $(\mathcal{C}_{\mathbf{V}_1,\mathbf{V}_3})$ and $(\mathcal{C}_{\mathbf{V}_2,\mathbf{V}_3})$ hold, so that the quasi-isomorphisms of complexes $\Theta_{\mathbf{V}_1,\mathbf{V}_3,s}$ and $\Theta_{\mathbf{V}_2,\mathbf{V}_3,s}$ are also well defined. It is natural to ask the following:

QUESTION. *When does the equality* $\Theta_{\mathbf{V}_1,\mathbf{V}_3,s} = \Theta_{\mathbf{V}_1,\mathbf{V}_2,s}\circ\Theta_{\mathbf{V}_2,\mathbf{V}_3,s}$ *hold*?

For instance, taking Example 6.8 into account, when does the equality $\Theta^{-1}_{\mathbf{V}_1,\mathbf{V}_2,s} = \Theta_{\mathbf{V}_2,\mathbf{V}_1,s}$ hold?

We do not know the answer to this question, but we can just say that the equality does not always hold. Indeed, if m is minimal such that $F^m(\mathbf{V}_1) = \mathbf{V}_1$, then the isomorphisms $\Theta_{\mathbf{V}_1,F(\mathbf{V}_1),s}$, $\Theta_{F(\mathbf{V}_1),F^2(\mathbf{V}_1),s},\ldots,\ \Theta_{F^{m-1}(\mathbf{V}_1),\mathbf{V}_1}$ are well defined and all coincide with the Frobenius endomorphism F (see Example 6.9), and so

$$\Theta_{\mathbf{V}_1,F(\mathbf{V}_1),s}\circ\Theta_{F(\mathbf{V}_1),F^2(\mathbf{V}_1),s}\circ\cdots\circ\Theta_{F^{m-1}(\mathbf{V}_1),\mathbf{V}_1,s} = F^m \neq \mathrm{Id} = \Theta_{\mathbf{V}_1,\mathbf{V}_1,s}$$

(see Example 6.8).

Example 6.11. Let $\mathbf{P}_0$ be a parabolic subgroup admitting an F-stable Levi subgroup $\mathbf{L}_0$ containing $\mathbf{L}$. We denote by $\mathbf{V}_0$ the unipotent radical of $\mathbf{P}_0$ and $\mathbf{L}_0^*$ the corresponding Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ containing $\mathbf{L}^*$, which is dual to $\mathbf{L}_0$. We assume in this example that $C^\circ_{\mathbf{G}^*}(s)\subset\mathbf{L}_0^*$. Then it follows from [BR03, Th. 11.7], Corollary 6.5 and Remark 6.7 that we have an isomorphism of $(\Lambda\mathbf{G}^F,\Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{H}_c^{d_0}(\mathbf{Y}_{\mathbf{V}_0},\Lambda)\otimes_{\Lambda\mathbf{L}_0^F}\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}^{\mathbf{L}_0}_{\mathbf{V}\cap\mathbf{L}_0},\Lambda)e_s^{\mathbf{L}^F}\simeq\mathrm{R}\Gamma_c^{\dim}(\mathbf{Y}_{\mathbf{V}},\Lambda)e_s^{\mathbf{L}^F},$$

where $d_0 = \dim(\mathbf{Y}_{\mathbf{V}_0})$.

Remark 6.12. Let us consider the Harish-Chandra case: assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are F-stable. The functors $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_1}$ and $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}_2}$ are isomorphic without truncating by any series [DD93], [HL94]. Such isomorphisms are given by explicit isomorphisms of bimodules, which do not rely on any algebraic geometry. We do not know if after truncation by a series satisfying $(\mathcal{C}_{\mathbf{V}_1,\mathbf{V}_2})$, that they coincide with our isomorphisms.

6.B. *Transitivity.* We will provide here an analogue to Lemma 5.13 in the more general context of this section. Assume in this subsection, and only in this subsection, that $3 \leqslant j \leqslant r$ (in particular, $r \geqslant 3$). Since $c_{j-1}(c_j(\mathbf{V}_\bullet)) = c_{j-1}(c_{j-1}(\mathbf{V}_\bullet))$, we can build a diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{\mathbf{V}_\bullet}, \Lambda) \arrow[r, "\Psi_{\mathbf{V}_\bullet, j}"] \arrow[d, "\Psi_{\mathbf{V}_\bullet, j-1}"'] & \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_j(\mathbf{V}_\bullet)}, \Lambda) \arrow[d, "\Psi_{c_j(\mathbf{V}_\bullet), j-1}"'] \\
\mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_{j-1}(\mathbf{V}_\bullet)}, \Lambda) \arrow[r, "\Psi_{c_{j-1}(\mathbf{V}_\bullet), j-1}"] & \mathrm{R}\Gamma_{\mathrm{c}}^{\dim}(\mathbf{Y}_{c_{j-1}(c_j(\mathbf{V}_\bullet))}, \Lambda).
\end{tikzcd} \tag{6.13}$$

It does not seem reasonable to expect that the diagram (6.13) is commutative in general. However, we have the following result, obtained from the results of Section 5.A below by copying the proof of Lemma 5.13.

LEMMA 6.14. *Assume that one of the following holds*:

(1) $\mathbf{V}_{j-2} \subset \mathbf{V}_{j+1} \cdot \mathbf{V}_{j-1}$;
(2) $\mathbf{V}_{j-1} \subset \mathbf{V}_{j-2} \cdot \mathbf{V}_{j}$;
(3) $\mathbf{V}_{j} \subset \mathbf{V}_{j-1} \cdot \mathbf{V}_{j+1}$;
(4) $\mathbf{V}_{j+1} \subset \mathbf{V}_{j} \cdot \mathbf{V}_{j-2}$.

Then the diagram (6.13) *is commutative.*

7. Jordan decomposition and quasi-isolated blocks

In this section, we assume $\mathbf{G}$ is connected. We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, and we denote by $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ a triple dual to $(\mathbf{G}, \mathbf{T}, F)$.

We start in Section 7.A with a recollection of some of the results of [BR03] on the vanishing of the truncated cohomology of certain Deligne-Lusztig varieties outside the middle degree. We fix an F-stable Levi subgroup $\mathbf{L}$ and consider $s \in \mathbf{G}^{*F^*}$ of order invertible in Λ such that $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$ — and we take $\mathbf{L}$ minimal with that property. We show that the corresponding middle degree $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodule $\mathrm{H}_c^{\dim(\mathbf{Y_P})}(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ does not depend on the choice of the parabolic subgroup $\mathbf{P}$, up to isomorphism, thanks to the results of Section 6. In particular, it is stable under the action of the stabilizer N of $e_s^{\mathbf{L}^F}$ in $N_{\mathbf{G}^F}(\mathbf{L})$.

Section 7.B develops some Clifford theory tools in order to extend the action of $\mathbf{L}^F$ on $\mathrm{H}_c^{\dim(\mathbf{Y_P})}(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ to an action of N. We apply this in Section 7.C by embedding $\mathbf{G}$ in a group $\tilde{\mathbf{G}}$ with connected center. This provides a Morita equivalence, extending the main result of [BR03] to the quasi-isolated case.

In section Section 7.D, we show that the action of $\mathbf{L}^F$ on the complex of cohomology $C = \mathrm{G}\Gamma_c(\mathbf{Y_P}, \Lambda)e_s^{\mathbf{L}^F}$ also extends to N, and the resulting complex provides a splendid Rickard equivalence. This relies on checking that given Q an ℓ-subgroup of $\mathbf{L}^F$, the complex $\mathrm{Br}_{\Delta Q}(C)$ arises in a Jordan decomposition setting for $C_{\mathbf{G}}(Q)$, and then applying the results of the appendix. The main difficulty is to prove that $\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ is a sum of idempotents associated to a Jordan decomposition setting for $C_{\mathbf{G}}(Q)$. An added difficulty is that the group $C_{\mathbf{G}}(Q)$ need not be connected.

7.A. *Quasi-isolated setting.* We fix a semisimple element $s \in \mathbf{G}^{*F^*}$ whose order is invertible in Λ. Let $\mathbf{L}^* = C_{\mathbf{G}^*}\big(\mathrm{Z}(C^\circ_{\mathbf{G}^*}(s))^\circ\big)$, an F^*-stable Levi complement of some parabolic subgroup $\mathbf{P}^*$ of $\mathbf{G}^*$. Note that $\mathbf{L}^*$ is a minimal Levi subgroup with respect to the property of containing $C^\circ_{\mathbf{G}^*}(s)$ and $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is an abelian ℓ'-group [Bon05, Cor. 2.8(b)]. In particular, the series corresponding to s is $(\mathbf{G}, \mathbf{L})$-regular.

We denote by $(\mathbf{L}, \mathbf{P})$ a pair dual to $(\mathbf{L}^*, \mathbf{P}^*)$. Note that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ admitting $\mathbf{L}$ as an F-stable Levi complement. The unipotent radical of $\mathbf{P}$ will be denoted by $\mathbf{V}$. We put $d = \dim(\mathbf{Y_V})$.

The group $C_{\mathbf{G}^*}(s)$ normalizes $\mathbf{L}^*$, and we set $\mathbf{N}^* = C_{\mathbf{G}^*}(s)^{F^*} \cdot \mathbf{L}^*$: it is a subgroup of $N_{\mathbf{G}^*}(\mathbf{L}^*)$ containing $\mathbf{L}^*$. Via the canonical isomorphism between $N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ and $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, we define the subgroup $\mathbf{N}$ of $N_{\mathbf{G}}(\mathbf{L})$ containing $\mathbf{L}$ such that $\mathbf{N}/\mathbf{L}$ corresponds to $\mathbf{N}^*/\mathbf{L}^*$. Note that $\mathbf{N}^*$ is F^*-stable and so $\mathbf{N}$ is F-stable, and that $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$ and $\mathbf{N}^F/\mathbf{L}^F$ are abelian ℓ'-groups.

Let us first derive some consequences of these assumptions. Note that $\mathbf{N}^*/\mathbf{L}^* = (\mathbf{N}^*/\mathbf{L}^*)^{F^*} = \mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$, so that $\mathbf{N}/\mathbf{L} = (\mathbf{N}/\mathbf{L})^F = \mathbf{N}^F/\mathbf{L}^F$. Also, $\mathbf{N}^{*F^*}$ is the stabilizer, in $N_{\mathbf{G}^{*F^*}}(\mathbf{L}^*)$, of the $\mathbf{L}^{*F^*}$-conjugacy class of s. Therefore,

$$\mathbf{N}^F \textit{ is the stabilizer of } e_s^{\mathbf{L}^F} \textit{ in } N_{\mathbf{G}^F}(\mathbf{L}). \tag{7.1}$$

It follows that $e_s^{\mathbf{L}^F}$ is a central idempotent of $\Lambda\mathbf{N}^F$. By [BR03, Th. 11.7], we have

$$\mathrm{H}^i_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} = 0 \text{ for } i \neq d.$$

Our first result on the Jordan decomposition is the independence of the choice of parabolic subgroups.

THEOREM 7.2. *Given $\mathbf{P}'$ a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L}$ and unipotent radical $\mathbf{V}'$, then $\mathrm{H}_c^{\dim(\mathbf{Y_V})}(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq \mathrm{H}_c^{\dim(\mathbf{Y_{V'}})}(\mathbf{Y_{V'}}, \Lambda)e_s^{\mathbf{L}^F}$ as $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules.*

The $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodule $\mathrm{H}^d_c(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ is $\mathbf{N}^F$-stable.

Proof. The first result follows from Remark 6.7 and Corollary 6.5.

Let $n \in \mathbf{N}^F$. The isomorphism of varieties $\mathbf{G}/\mathbf{V} \xrightarrow{\sim} \mathbf{G}/{}^n\mathbf{V}$, $g\mathbf{V} \mapsto g\mathbf{V}n^{-1}$ induces an isomorphism of varieties $\mathbf{Y_V} \xrightarrow{\sim} \mathbf{Y}_{{}^n\mathbf{V}}$. As a consequence, we have an isomorphism of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules

$$\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda) \simeq n_*\big(\mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big),$$

where $n_*\big(\mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big) = \mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)$ as a left $\Lambda\mathbf{G}^F$-module and the right action of $a \in \Lambda\mathbf{L}^F$ on $n_*\big(\mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)\big)$ is given by the right action of nan^{-1} on $\mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)$.

Since n fixes $e_s^{\mathbf{L}^F}$, we deduce that

$$\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq n_*\big(\mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)e_s^{\mathbf{L}^F}\big).$$

On the other hand, the first part of the theorem shows that

$$\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq \mathrm{H}_c^d(\mathbf{Y}_{{}^n\mathbf{V}}, \Lambda)e_s^{\mathbf{L}^F}.$$

It follows that $\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F} \simeq n_*\big(\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}\big)$. □

Recall that if $\mathbf{N}^F = \mathbf{L}^F$ (that is, if $C_{\mathbf{G}^*}(s)^{F^*} \subset \mathbf{L}^*$), then $\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ induces a Morita equivalence between $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\Lambda\mathbf{L}^F e_s^{\mathbf{L}^F}$ by [BR03, Th. B′]. Note that the assumption in [BR03, Th. B′] is $C_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, but it can easily be seen that the proof requires only the assumption $C_{\mathbf{G}^*}(s)^{F^*} \subset \mathbf{L}^*$. Theorem 7.2 shows that this Morita equivalence does not depend on the choice of a parabolic subgroup.

We will generalize the Morita equivalence to our situation. The main difficulty is to extend the action of $\mathbf{L}^F$ on $\mathrm{H}_c^d(\mathbf{Y_V}, \Lambda)e_s^{\mathbf{L}^F}$ to $\mathbf{N}^F$.

7.B. *Clifford theory.* Let us recall some basic facts of Clifford theory. Let $\mathbf{k}$ be a field. Let Y be a finite group and X a normal subgroup of Y. Let M be a finitely generated $\mathbf{k}X$-module that is Y-stable, and let $A = \mathrm{End}_{kX}(M)$.

Given $y \in Y$, let N_y be the set of $\phi \in \mathrm{End}_{\mathbf{k}}(M)^\times$ such that $\phi(xm) = yxy^{-1}\phi(m)$ for all $x \in X$ and $m \in M$. Note that $N_yN_{y'} = N_{yy'}$ for all $y, y' \in Y$.

Let $N = \bigcup_{y\in Y} N_y$, a subgroup of $\mathrm{End}_{\mathbf{k}}(M)^\times$ containing $N_1 = A^\times$ as a normal subgroup. The action of $x \in X$ on M defines an element of N_x, and this gives a morphism $X \to N$. The Y-stability of M gives a surjective morphism of groups $Y \to N/N_1$, $y \mapsto N_y$.

Let $\hat{Y} = Y \times_{N/N_1} N$. There is a diagonal embedding of X as a normal subgroup of $\hat{Y}$. There is a commutative diagram whose horizontal and vertical sequences are exact:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
1 \arrow[r] & A^\times \arrow[r] \arrow[d, equal] & \hat{Y} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A^\times \arrow[r] & \hat{Y}/X \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1. &
\end{tikzcd}$$

The action of X on M extends to an action of Y if and only if the canonical morphism of groups $\hat{Y} \to Y$ has a splitting that is the identity on X. This is equivalent to the fact that the canonical morphism of groups $\hat{Y}/X \to Y/X$ is a split surjection.

The extension of groups

$$1 \to 1 + J(A) \to A^\times \to A^\times/(1 + J(A)) \to 1$$

splits. Indeed, since A is a finite-dimensional $\mathbf{k}$-algebra, there exists a $\mathbf{k}$-subalgebra S of A such that the composition $S \hookrightarrow A \twoheadrightarrow A/J(A)$ is an isomorphism. Since $A = S \oplus J(A)$, we have $A^\times = (1 + J(A)) \rtimes S^\times$.

If $[Y : X] \in \mathbf{k}^\times$, then every group extension $1 \to 1 + J(A) \to Z \to Y/X \to 1$ splits, since $1 + J(A)$ is the finite extension of abelian groups

$$(1 + J(A)^i)/(1 + J(A)^{i+1}) \simeq J(A)^i/J(A)^{i+1},$$

and those are $\mathbf{k}(Y/X)$-modules. Consequently, if $[Y : X] \in \mathbf{k}^\times$, then the action of X on M extends to an action of Y if and only if the extension

$$1 \to A^\times/(1 + J(A)) \to \hat{Y}/X(1 + J(A)) \to Y/X \to 1$$

splits.

Consider now $\tilde{Y}$ a finite group with Y and $\tilde{X}$ two normal subgroups such that $X = Y \cap \tilde{X}$ and $\tilde{Y} = Y\tilde{X}$. Let $\tilde{M} = \operatorname{Ind}_X^{\tilde{X}}(M)$, a $\tilde{Y}$-stable $\mathbf{k}\tilde{X}$-module. We define $\tilde{N}_y$, $\tilde{N}$ and $\hat{\tilde{Y}}$ as above, replacing M by $\tilde{M}$.

Given $y \in Y$, we define a map $\rho : N_y \to \tilde{N}_y$, $\phi \mapsto (a \otimes m \mapsto yay^{-1} \otimes \phi(m))$ for $a \in \mathbf{k}\tilde{X}$ and $m \in M$. This gives a morphism of groups $N \to \tilde{N}$ extending the canonical morphism $A \to \operatorname{End}_{\mathbf{k}\tilde{X}}(\tilde{M})$ and a morphism of groups $\hat{Y}/X \to$

$\hat{\tilde{Y}}/\tilde{X}$ giving a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & A^\times \arrow[r] \arrow[d, hook] & \hat{Y}/X \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d, "\sim"] & 1 \\ 1 \arrow[r] & \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times \arrow[r] & \hat{\tilde{Y}}/\tilde{X} \arrow[r] & \tilde{Y}/\tilde{X} \arrow[r] & 1. \end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & A^\times/(1+J(A)) \arrow[r] \arrow[d, hook] & \hat{Y}/X(1+J(A)) \arrow[r] \arrow[d] & Y/X \arrow[r] \arrow[d, "\sim"] & 1 \\ 1 \arrow[r] & \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times/(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \arrow[r] & \hat{\tilde{Y}}/\tilde{X}(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \arrow[r] & \tilde{Y}/\tilde{X} \arrow[r] & 1 \end{tikzcd}$$

Assume the inclusion

$$\mathrm{End}_{\mathbf{k}X}(M)^\times/(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \hookrightarrow \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})^\times/(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})))$$

splits; this happens, for example, if $\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})/J\big(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})\big) \simeq \mathbf{k}^n$ for some n, for in that case the algebra embedding

$$\mathrm{End}_{\mathbf{k}X}(M)/J(\mathrm{End}_{\mathbf{k}X}(M)) \hookrightarrow \mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M})/J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))$$

has a section. If the surjection $\hat{\tilde{Y}}/\tilde{X}(1+J(\mathrm{End}_{\mathbf{k}\tilde{X}}(\tilde{M}))) \to \tilde{Y}/\tilde{X}$ splits, then the surjection $\hat{Y}/X(1+J(\mathrm{End}_{\mathbf{k}X}(M))) \to Y/X$ splits.

As a consequence, we have the following proposition.

PROPOSITION 7.3. *Let $\tilde{Y}$ be a finite group and Y, $\tilde{X}$ be two normal subgroups of $\tilde{Y}$. Let $X = Y \cap \tilde{X}$. We assume $\tilde{Y} = Y\tilde{X}$. Let $\mathbf{k}$ be a field with $[Y : X] \in \mathbf{k}^\times$.*

Let M be a finitely generated $\mathbf{k}X$-module that is Y-stable. We assume that

$$\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))/J\big(\mathrm{End}_{\mathbf{k}\tilde{X}}(\mathrm{Ind}_X^{\tilde{X}}(M))\big) \simeq \mathbf{k}^n \text{ for some } n.$$

If $\mathrm{Ind}_X^{\tilde{X}}(M)$ extends to $\tilde{Y}$, then M extends to Y.

7.C. *Embedding in a group with connected center and Morita equivalence.* We fix a connected reductive algebraic group $\tilde{\mathbf{G}}$ containing $\mathbf{G}$ as a closed subgroup, with an extension of F to an endomorphism of $\tilde{\mathbf{G}}$ such that F^δ is a Frobenius endomorphism of $\tilde{\mathbf{G}}$ defining an $\mathbb{F}_q$-structure, and such that $\tilde{\mathbf{G}} = \mathbf{G}\cdot Z(\tilde{\mathbf{G}})$ and $Z(\tilde{\mathbf{G}})$ is connected [DL76, proof of Cor. 5.18]. The inclusion $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ is called a *regular embedding.*

Let $\tilde{\mathbf{T}} = \mathbf{T}\cdot Z(\tilde{\mathbf{G}})$, an F-stable maximal torus of $\tilde{\mathbf{G}}$. Fix a triple $(\tilde{\mathbf{G}}^*, \tilde{\mathbf{T}}^*, F^*)$ dual to $(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}, F)$. The inclusion $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ induces a surjection $i^* : \tilde{\mathbf{G}}^* \twoheadrightarrow \mathbf{G}^*$. Let $\tilde{\mathbf{L}} = \mathbf{L}\cdot Z(\tilde{\mathbf{G}})$, so that $\tilde{\mathbf{L}}^* = (i^*)^{-1}(\mathbf{L}^*)$. Let $\tilde{\mathbf{N}} = \mathbf{N}\tilde{\mathbf{L}}$.

Let J be a set of representatives of conjugacy classes of ℓ'-elements $\tilde{t} \in \tilde{\mathbf{G}}^{*F^*}$ such that $i^*(\tilde{t}) = s$. (Recall that $C_{\tilde{\mathbf{G}}^*}(\tilde{t})$ is connected because $Z(\tilde{\mathbf{G}})$ is connected.) Note that $J \subset \tilde{\mathbf{L}}^{*F^*}$.

Lemma 7.4. *We have idempotent decompositions* $e_s^{\mathbf{G}^F} = \sum_{\tilde{t}\in J} e_{\tilde{t}}^{\tilde{\mathbf{G}}^F}$ *and* $e_s^{\mathbf{L}^F} = \sum_{n\in \mathbf{N}^F/\mathbf{L}^F} \sum_{\tilde{t}\in J} n e_{\tilde{t}}^{\tilde{\mathbf{L}}^F} n^{-1}$.

Proof. The first statement is a classical translation from $\mathbf{G}^*$ to $\mathbf{G}$; cf., for instance, [Bon06, Prop. 11.7].

Let $\tilde{s}$ be a semisimple element of $\tilde{\mathbf{G}}^{*F^*}$ such that $i^*(\tilde{s}) = s$. If $\Lambda \neq K$, we will assume that $\tilde{s}$ has order prime to ℓ. (This is always possible as we may replace $\tilde{s}$ by its ℓ'-part if necessary.) Note that $\tilde{s} \in \tilde{\mathbf{L}}^{*F^*}$.

Let $n \in \tilde{\mathbf{N}}^{*F^*}$ such that $n\tilde{s}n^{-1}$ is $\tilde{\mathbf{L}}^{*F^*}$-conjugate to $\tilde{s}$. Then $n \in \tilde{\mathbf{L}}^{*F^*} \cdot C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Since $i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) \subset C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, it follows that $i^*(n) \in \mathbf{L}^{*F^*}$. We have $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*} = \tilde{\mathbf{N}}^{*F^*}/\tilde{\mathbf{L}}^{*F^*}$, hence $n \in \tilde{\mathbf{L}}^{*F^*}$.

It follows that $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$ acts freely on the set of conjugacy classes of $\tilde{\mathbf{L}}^{*F^*}$ whose image under i^* is the $\mathbf{L}^{*F^*}$-conjugacy class of s. Through the identification of $\mathbf{N}^{*F^*}/\mathbf{L}^{*F^*}$ with $\mathbf{N}^F/\mathbf{L}^F$, this shows that given $\tilde{t} \in J$, the stabilizer in $\mathbf{N}^F$ of $e_{\tilde{t}}^{\tilde{\mathbf{L}}^F}$ is $\mathbf{L}^F$. $\square$

Theorem 7.5. *The action of* $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{L}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$ *on* $\mathrm{H}^d_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$ *extends to an action of* $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$. *The resulting bimodule induces a Morita equivalence between* $k\mathbf{G}^F e_s^{\mathbf{G}^F}$ *and* $k\mathbf{N}^F e_s^{\mathbf{L}^F}$.

Proof. Let $\tilde{\mathbf{P}} = \mathbf{P} \cdot Z(\tilde{\mathbf{G}})$ and let $\tilde{\mathbf{P}}^* = i^{*-1}(\mathbf{P}^*)$. Note that $\tilde{\mathbf{L}}$ (resp. $\tilde{\mathbf{L}}^*$) is a Levi complement of $\tilde{\mathbf{P}}$ (resp. $\tilde{\mathbf{P}}^*$) and it is F-stable (resp. F^*-stable) and the pair $(\tilde{\mathbf{L}}^*, \tilde{\mathbf{P}}^*)$ is dual to $(\tilde{\mathbf{L}}, \tilde{\mathbf{P}})$.

We put

$$X = (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}) \cdot \Delta\tilde{\mathbf{L}}^F, \quad Y = (\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}) \cdot \Delta\tilde{\mathbf{N}}^F,$$

$$\tilde{X} = \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}} \quad \text{and} \quad \tilde{Y} = \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{N}}^F)^{\mathrm{opp}}.$$

Let $\tilde{\mathbf{Y}}_\mathbf{V} = \mathbf{Y}_\mathbf{V}^{\tilde{\mathbf{G}}}$. Through the embedding $\mathbf{G}/\mathbf{V} \hookrightarrow \tilde{\mathbf{G}}/\mathbf{V}$, we identify $\mathbf{Y_V}$ with a subvariety of $\tilde{\mathbf{Y}}_\mathbf{V}$. The stabilizer in $\tilde{X}$ of the subvariety $\mathbf{Y_V}$ of $\tilde{\mathbf{Y}}_\mathbf{V}$ is X, hence we have an isomorphism of $\tilde{X}$-varieties $\mathrm{Ind}_X^{\tilde{X}} \mathbf{Y_V} \xrightarrow{\sim} \tilde{\mathbf{Y}}_\mathbf{V}$.

Let $M = \mathrm{H}^d_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$, a $(kX(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-module. Let $\tilde{M} = \mathrm{Ind}_X^{\tilde{X}} M$, a $(k\tilde{X}(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-module. We have an isomorphism of $(k\tilde{X}(e_s^{\mathbf{G}^F} \otimes e_s^{\mathbf{L}^F}))$-modules $\tilde{M} \xrightarrow{\sim} \mathrm{H}^d_c(\tilde{\mathbf{Y}}_\mathbf{V}, k)e_s^{\mathbf{L}^F}$.

We put $e = \sum_{\tilde{t}\in J} e_{\tilde{t}}^{\tilde{\mathbf{L}}^F}$. We have $e_s^{\mathbf{L}^F} = \sum_{n\in\tilde{\mathbf{N}}^F/\tilde{\mathbf{L}}^F} nen^{-1}$, and e is a central idempotent of $k\tilde{\mathbf{L}}^F$ (Lemma 7.4).

The kX-module M is $\mathbf{N}^F$-stable (Theorem 7.2), hence the $k\tilde{X}$-module $\tilde{M}$ is $\mathbf{N}^F$-stable as well. It follows that given $\tilde{t} \in J$ and $n \in \mathbf{N}^F$, we have

$n_*(\mathrm{H}^d_c(\tilde{\mathbf{Y}}_{\mathbf{V}})e^{\tilde{\mathbf{L}}^F}_{\tilde{t}}) \simeq \mathrm{H}^d_c(\tilde{\mathbf{Y}}_{\mathbf{V}})e^{\tilde{\mathbf{L}}^F}_{n\tilde{t}n^{-1}}$ as $k\tilde{X}$-modules. The classical Mackey formula for induction and restriction in finite groups shows now that

$$\mathrm{H}^d_c(\tilde{\mathbf{Y}}_{\mathbf{V}}, k)\Big(\sum_{n\in\mathbf{N}^F/\mathbf{L}^F} e^{\tilde{\mathbf{L}}^F}_{n\tilde{t}n^{-1}}\Big) \simeq \mathrm{Res}^{\tilde{Y}}_{\tilde{X}}\,\mathrm{Ind}^{\tilde{Y}}_{\tilde{X}}\big(\mathrm{H}^d_c(\tilde{\mathbf{Y}}_{\mathbf{V}})e^{\tilde{\mathbf{L}}^F}_{\tilde{t}}\big),$$

hence

$$\tilde{M} \simeq \mathrm{Res}^{\tilde{Y}}_{\tilde{X}}\,\mathrm{Ind}^{\tilde{Y}}_{\tilde{X}}\big(\tilde{M}e\big).$$

Lemma 7.4 shows that $\tilde{M}e$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e^{\mathbf{G}^F}_s$ and $k\tilde{\mathbf{L}}^F e$ (cf. [BR03, Th. B′]). In particular, it is a direct sum of indecomposable modules, no two of which are isomorphic.

Since $ek\tilde{\mathbf{N}}^F$ induces a Morita equivalence between $k\tilde{\mathbf{L}}^F e$ and $k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s$, we deduce that the right action of $\tilde{\mathbf{L}}^F$ on $\tilde{M} \simeq \tilde{M}e\otimes_{k\tilde{\mathbf{L}}e} ek\tilde{\mathbf{N}}^F$ extends to an action of $\tilde{\mathbf{N}}^F$ commuting with the left action of $\tilde{\mathbf{G}}^F$ and the extended bimodule $\tilde{M}'$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e^{\mathbf{G}^F}_s$ and $k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s$. It follows that

$$\mathrm{End}_{k\tilde{X}}(\tilde{M}) \simeq \mathrm{End}_{k(\tilde{\mathbf{N}}^F\times(\tilde{\mathbf{L}}^F)^{\mathrm{opp}})}(k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s).$$

Given $n_1, n_2 \in \tilde{\mathbf{N}}^F$ with $n_1 \notin n_2\tilde{\mathbf{L}}^F$, the central idempotents $n_1en_1^{-1}$ and $n_2en_2^{-1}$ of $k\tilde{\mathbf{L}}^F$ are orthogonal. It follows that

$$\begin{aligned}\mathrm{End}_{k(\tilde{\mathbf{N}}^F\times(\tilde{\mathbf{L}}^F)^{\mathrm{opp}}}(k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s) &\simeq \prod_{n\in\mathbf{N}^F/\mathbf{L}^F} \mathrm{End}_{k(\tilde{\mathbf{N}}^F\times(\tilde{\mathbf{L}}^F)^{\mathrm{opp}})}(k\tilde{\mathbf{N}}^F nen^{-1})\\ &\simeq \big(Z(k\tilde{\mathbf{L}}^F e)\big)^{[\mathbf{N}^F/\mathbf{L}^F]},\end{aligned}$$

the last isomorphism following from the fact that $k\tilde{\mathbf{N}}^F nen^{-1}$ induces a Morita equivalence between $k\tilde{\mathbf{N}}e^{\mathbf{L}^F}_s$ and $k\tilde{\mathbf{L}}^F nen^{-1} \simeq k\tilde{\mathbf{L}}^F e$.

We deduce that $\mathrm{End}_{k\tilde{X}}(\tilde{M})^\times/\big(1 + J(\mathrm{End}_{k\tilde{X}}(\tilde{M}))\big) \simeq (k^\times)^r$ for some r. Since $[Y : X] = [\mathbf{N} : \mathbf{L}]$ is prime to ℓ, it follows from Proposition 7.3 that the action of X on M extends to an action of Y. Denote by M' the extended module. We have $\mathrm{Res}^{\tilde{Y}}_{\tilde{X}}\,\mathrm{Ind}^{\tilde{Y}}_{Y}(M')e \simeq \tilde{M}e \simeq \mathrm{Res}^{\tilde{Y}}_{\tilde{X}}(\tilde{M}')e$, hence $\mathrm{Ind}^{\tilde{Y}}_{Y}(M') \simeq \tilde{M}'$. It follows that $\mathrm{Ind}^{\tilde{Y}}_{Y}(M')$ induces a Morita equivalence between $k\tilde{\mathbf{G}}^F e^{\mathbf{G}^F}_s$ and $k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s$. We have

$$\begin{aligned}\mathrm{End}_{k\tilde{\mathbf{G}}^F}(\mathrm{Ind}^{\tilde{Y}}_{Y}(M')) &\simeq \mathrm{End}_{k\tilde{\mathbf{G}}^F}(k\tilde{\mathbf{G}}^F \otimes_{k\mathbf{G}^F} M')\\ &\simeq \mathrm{Hom}_{k\mathbf{G}^F}(M', M' \otimes_{k\mathbf{N}^F} k\tilde{\mathbf{N}}^F)\\ &\simeq \mathrm{End}_{k\mathbf{G}^F}(M') \otimes_{k\mathbf{N}^F} k\tilde{\mathbf{N}}^F.\end{aligned}$$

The canonical map $k\tilde{\mathbf{N}}^F e^{\mathbf{L}^F}_s \to \mathrm{End}_{k\tilde{\mathbf{G}}^F}(\mathrm{Ind}^{\tilde{Y}}_{Y} M')$ is an isomorphism, hence the canonical map $k\mathbf{N}^F e^{\mathbf{L}^F}_s \to \mathrm{End}_{k\mathbf{G}^F}(M')$ is an isomorphism as well. Also, M is a faithful $k\mathbf{G}^F e^{\mathbf{G}^F}_s$-module, since $\tilde{M} = \mathrm{Ind}^{\tilde{\mathbf{G}}^F}_{\mathbf{G}^F} M$ is a faithful $k\tilde{\mathbf{G}}^F e^{\mathbf{G}^F}_s$-module. We deduce that M' induces a Morita equivalence between $k\mathbf{G}^F e^{\mathbf{G}^F}_s$ and $k\mathbf{N}^F e^{\mathbf{L}^F}_s$. ∎

7.D. *Splendid Rickard equivalence and local structure.* Recall that $\mathbf{L}$ is the minimal F-stable Levi subgroup of $\mathbf{G}$ such that $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$.

THEOREM 7.6. *The action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{L}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$ on $\mathrm{G\Gamma}_c(\mathbf{Y}_\mathbf{V}, k)e_s^{\mathbf{L}^F}$ extends to an action of $k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}}$. The resulting complex induces a splendid Rickard equivalence between $k\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $k\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

Proof. Step 1: *Identification of* $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathrm{G\Gamma}_c(\mathbf{Y}_\mathbf{V}, k)e_s^{\mathbf{L}^F})$ *in* $\mathrm{Ho}^b(k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}))$. Let $C = (\mathrm{G\Gamma}_c(\mathbf{Y}_\mathbf{V}, k)e_s^{\mathbf{L}^F})^{\mathrm{red}}$. The vertices of the indecomposable direct summands of components of C are contained in $\Delta\mathbf{L}^F$ by Corollary 3.8. Let Q be an ℓ-subgroup of $\mathbf{L}^F$. We have $\mathrm{Br}_{\Delta Q}(C) \simeq \mathrm{G\Gamma}_c(\mathbf{Y}^{C_\mathbf{G}(Q)}_{C_\mathbf{V}(Q)}, k)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ in $\mathrm{Ho}^b(k(C_{\mathbf{G}^F}(Q) \times C_{\mathbf{L}^F}(Q)^{\mathrm{opp}}))$ by Proposition 3.4. Let $\mathcal{X}$ be the rational series of $(\mathbf{L}, F)$ corresponding to s, so that $e_s^{\mathbf{L}^F} = e_\mathcal{X}$. Theorem 4.14 shows that

$$\mathrm{br}_Q(e_\mathcal{X}) = \sum_{\mathcal{Y} \in (i_Q^\mathbf{L})^{-1}(\mathcal{X})} e_\mathcal{Y}.$$

Let $\mathcal{Y} \in (i_Q^\mathbf{L})^{-1}(\mathcal{X})$. Proposition 4.11 shows that $\mathcal{Y}$ is $(C^\circ_\mathbf{G}(Q), C^\circ_\mathbf{L}(Q))$-regular. It follows from [BR03, Theorem 11.7] that $\mathrm{H}^i_c(\mathbf{Y}^{C^\circ_\mathbf{G}(Q)}_{C_\mathbf{V}(Q)}, k)e_\mathcal{Y} = 0$ for $i \neq \dim \mathbf{Y}^{C^\circ_\mathbf{G}(Q)}_{C_\mathbf{V}(Q)}$, hence $\mathrm{H}^i_c(\mathbf{Y}^{C_\mathbf{G}(Q)}_{C_\mathbf{V}(Q)}, k)e_\mathcal{Y} = 0$ for $i \neq \dim \mathbf{Y}^{C_\mathbf{G}(Q)}_{C_\mathbf{V}(Q)}$. We have shown that the cohomology of $\mathrm{Br}_{\Delta Q}(C)$ is concentrated in a single degree. Note that $\mathrm{Res}_{kC_{\mathbf{G}^F}(Q)}(\mathrm{Br}_{\Delta Q}(C))$ is a perfect complex, hence its homology is projective as a $kC_{\mathbf{G}^F}(Q)$-module. We deduce from Theorem A.4 that

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C) \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(C) \quad \text{in } \mathrm{Ho}^b(k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}})).$$

Step 2: *Study of* $\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))}(\mathrm{Ind}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}} \mathrm{G\Gamma}_c(\mathbf{Y}_\mathbf{V}, k)e_s^{\mathbf{L}^F})$. Let $C' = \mathrm{Ind}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}} C$. Let P be a projective resolution of $k\mathbf{N}^F$, i.e., a complex of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})$-proj with $P^i = 0$ for $i > 0$, together with a quasi-isomorphism $P \to k\mathbf{N}^F$ of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})$-modules. As the terms of C' are projective $k\mathbf{G}^F$-modules, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))}(C') & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))}(k\mathbf{N}^F, \mathrm{End}^\bullet_{k\mathbf{G}^F}(C')) \\
\downarrow & & \downarrow \\
\mathrm{End}_{D^b(k(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))}(C') & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))}(P, \mathrm{End}^\bullet_{k\mathbf{G}^F}(C'))
\end{array}$$

Using the isomorphisms of complexes in $\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$,

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(C') \simeq \mathrm{Ind}^{\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(\mathrm{End}^\bullet_{k\mathbf{G}^F}(C))$$

and

$$\mathrm{End}_{D^b(k\mathbf{G}^F)}(C') \simeq \mathrm{Ind}^{\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(\mathrm{End}_{D^b(k\mathbf{G}^F)}(C)),$$

we deduce that

$$\mathrm{End}^{\bullet}_{k\mathbf{G}^F}(C') \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(C') \text{ in } \mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}})).$$

Now, the canonical map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(k\mathbf{N}^F, \mathrm{End}_{D^b(k\mathbf{G}^F)}(C'))&\\ \to \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}&(P, \mathrm{End}_{D^b(k\mathbf{G}^F)}(C'))\end{aligned}$$

is an isomorphism, since $\mathrm{End}_{D^b(k\mathbf{G}^F)}(C')$ is a complex concentrated in degree 0. It follows that the top horizontal map in the commutative diagram above is an isomorphism, hence we have canonical isomorphisms

$$\begin{aligned}\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') &\xrightarrow{\sim} \mathrm{End}_{D^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C')\\ &\xrightarrow{\sim} \mathrm{End}_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}})}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}^d_c(\mathbf{Y_V}, k)).\end{aligned}$$

Step 3: *Construction of a summand* $\tilde{C}$ *of* $\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\mathrm{G}\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F})$. We have shown (Theorem 7.5) that there is a direct summand M' of the module $\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}^d_c(\mathbf{Y_V}, k)$ whose restriction to $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$ is isomorphic to $\mathrm{H}^d_c(\mathbf{Y_V}, k)$. Let i be the corresponding idempotent of

$$\mathrm{End}_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}})}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}} \mathrm{H}_c(\mathbf{Y_V}, k))$$

and j its inverse image in $\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C')$ via the isomorphisms above. We have a surjective homomorphism of finite-dimensional k-algebras

$$\mathrm{End}_{\mathrm{Comp}(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C') \twoheadrightarrow \mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C').$$

Consequently, j lifts to an idempotent j' of $\mathrm{End}_{\mathrm{Comp}(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(C')$ [Thé95, Th. 3.2]. It corresponds to a direct summand $\tilde{C}$ of C' quasi-isomorphic to M' and $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\tilde{C})$ is a direct summand of $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(C') \simeq C^{\oplus[\mathbf{N}^F:\mathbf{L}^F]}$.

Step 4: $\tilde{C}$ *lifts* $\mathrm{G}\Gamma_c(\mathbf{Y_V}, k)e_s^{\mathbf{L}^F}$. Let $C = \bigoplus_{1 \leqslant r \leqslant n} C_r$ be a decomposition into a direct sum of indecomposable objects of $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$. This induces a decomposition $M = \bigoplus_{1 \leqslant r \leqslant n} M_r$, where $M_r = H^d(C_r)$ and M_r and $M_{r'}$ have no isomorphic indecomposable summands for $r \neq r'$ (cf. proof of Theorem 7.5). We have $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\tilde{C}) \simeq \bigoplus_{1 \leqslant r \leqslant n} C_r^{\oplus a_r}$ in $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$ for some integers $0 \leqslant a_r \leqslant [\mathbf{N}^F : \mathbf{L}^F]$ and $\bigoplus_{1 \leqslant r \leqslant n} H^d(C_r)^{\oplus a_r} \simeq M$. It follows that $a_r = 1$ for all r, hence $\mathrm{Res}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}}(\tilde{C}) \simeq C$ in $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$. This shows the first statement.

Step 5: *Rickard equivalence.* We have shown above that $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\tilde{C}) \simeq \mathrm{End}_{D^b(k\mathbf{G}^F)}(\tilde{C})$ in $\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$. On the other hand, $\mathrm{End}_{D^b(k\mathbf{G}^F)}(\tilde{C}) \simeq \mathrm{End}_{k\mathbf{G}^F}(M') \simeq k\mathbf{N}^F e_s^{\mathbf{L}^F}$. It follows from Corollary A.5 that $\tilde{C}$ induces a splendid Rickard equivalence. $\square$

We now summarize and complete the description of the Jordan decomposition of blocks.

THEOREM 7.7. *There is an extension of the complex $G\Gamma_c(\mathbf{Y_V}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}$ of $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}, \mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F})$-bimodules to a complex C of $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}, \mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})$-bimodules. The complex C induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

There is a (unique) bijection $b \mapsto b'$ between blocks of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$ such that $bC \simeq Cb'$.

Given b a block of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$, then

- *the bimodule $H^{\dim \mathbf{Y_V}}(bCb')$ induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F b$ and $\mathcal{O}\mathbf{N}^F b'$;*
- *the complex bCb' induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F b$ and $\mathcal{O}\mathbf{N}^F b'$;*
- *there is a (unique) equivalence $(Q, b'_Q) \mapsto (Q, b_Q)$ from the category of b'-subpairs to the category of b-subpairs such that $b_Q \mathrm{Br}_{\Delta Q}(C) = \mathrm{Br}_{\Delta Q}(C) b'_Q$. In particular, if D is a defect group of b', then D is a defect group of b.*

Proof. Theorem 7.6 provides a complex C' of $(k\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (k\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}})$-modules inducing a splendid Rickard equivalence. By Rickard's Lifting Theorem [Ric96, Th. 5.2], there is a splendid complex C of $(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F} \otimes (\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})^{\mathrm{opp}})$-modules, unique up to isomorphism in $\mathrm{Comp}(\mathcal{O}(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}))$, such that $kC \simeq C'$. Also, [Ric96, proof of Th. 5.2] shows that $G\Gamma_c(\mathbf{Y_V}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}$ is the unique splendid complex that lifts $G\Gamma_c(\mathbf{Y_V}, k)^{\mathrm{red}} e_s^{\mathbf{L}^F}$. As a consequence, there is an isomorphism of complexes

$$\mathrm{Res}^{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}}_{\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}}(C) \simeq G\Gamma_c(\mathbf{Y_V}, \mathcal{O})^{\mathrm{red}} e_s^{\mathbf{L}^F}.$$

By [Ric96, Th. 5.2], the complex C induces a splendid Rickard equivalence.

Since $H^d(bkCb')$ induces a Morita equivalence, it follows that $H^d(bCb')$ induces a Morita equivalence (cf., e.g., [Ric96, proof of Th. 5.2]).

The existence of the bijection between blocks follows from the isomorphism of algebras $Z(\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}) \xrightarrow{\sim} Z(\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F})$ induced by the Morita equivalence, and the blockwise statements on Morita and Rickard equivalence are clear.

By [Pui99, Th. 19.7], it follows that the Brauer categories of $k\mathbf{G}^F b$ and $k\mathbf{N}^F b'$ are equivalent and, in particular, $k\mathbf{G}^F b$ and $k\mathbf{N}^F b'$ have isomorphic defect groups. $\square$

Remark 7.8. If was already known that given b a block of $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$, then b and b' have isomorphic defect groups under one of the following assumptions:

- ℓ does not divide $|Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$ nor $|Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ)^F|$, $\ell \geqslant 5$ and $\ell \geqslant 7$ if $\mathbf{G}$ has a component of type E_8 [CE99, Prop. 5.1];
- $C_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$, and either b or b' has a defect group that is abelian modulo the ℓ-center of $\mathbf{G}^F$ [KM13, Th. 1.3].

Example 7.9. Assume in this example that $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$, and assume that $(C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s))^{F^*}$ is cyclic. The element s defines a linear character $\hat{s} : \mathbf{L}^F \to \mathcal{O}^\times$ that induces an isomorphism of algebra $\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F} \simeq \mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$. The linear character $\hat{s}$ is stable under the action of $\mathbf{N}^F$ so, since $\mathbf{N}^F/\mathbf{L}^F$ is cyclic, it extends to a linear character $\hat{s}^+ : \mathbf{N}^F \to \mathcal{O}^\times$. Again, $\hat{s}^+$ induces an isomorphism of algebra $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F} \simeq \mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$. Combined with this, Theorem 7.5 provides a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.

Example 7.10 (Type A). Assume in this example that all the simple components of $\mathbf{G}$ are of Type A. (No assumption is made on the action of F.) Then $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ and $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is cyclic. Therefore, Example 7.9 can be applied to provide a Morita equivalence between $\mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$ and $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$.

Remark 7.11. This article was announced at the end of the introduction of [BR03]. Unfortunately, we have not been able to settle the problem of finiteness of source algebras. On the other hand, in addition to what was announced in [BR03], we have provided an extension of the Jordan decomposition to the quasi-isolated case.

Appendix A. About ℓ-permutation modules

In this section, we assume $\Lambda = \mathcal{O}$ or $\Lambda = k$. Let us recall here some results of Broué and Puig; cf. [Bro85, §3.6]. Let G be a finite group. Note that an ℓ-permutation $\mathcal{O}G$-module M is indecomposable if and only if kM is an indecomposable kG-module.

Let P be an ℓ-subgroup of G. An indecomposable ℓ-permutation ΛG-module M has a vertex containing P if and only if $\mathrm{Br}_P(M) \neq 0$. Also, given V an indecomposable projective $k[N_G(P)/P]$-module (it is then an ℓ-permutation kG-module), there exists a unique indecomposable ℓ-permutation ΛG-module $\mathrm{M}(P, V)$ such that $\mathrm{Br}_P\mathrm{M}(P, V) \simeq V$. The ΛG-module $\mathrm{M}(P, V)$ has vertex P. Moreover, every indecomposable ℓ-permutation ΛG-module with vertex P is isomorphic to such an $\mathrm{M}(P, V)$.

The following lemma is a variant of [Bou98, Prop. 6.4].

Lemma A.1. *Let M and N be ℓ-permutation ΛG-modules, and let $\psi \in \mathrm{Hom}_{\Lambda G}(M, N)$. Assume that all indecomposable summands of N have a vertex* *equal to a given subgroup P of G and that $\mathrm{Br}_P(\psi)$ is a surjection. Then ψ is a split surjection.*

Proof. Proceeding by induction on the dimension of N, we can assume that N is indecomposable. Fix a decomposition $M = \bigoplus_{i\in I} M_i$ where M_i is indecomposable for all $i \in I$, and let $\psi_i : M_i \to N$ denote the restriction of ψ. Since $\mathrm{Br}_P(\psi)$ is a surjection and $\mathrm{Br}_P(N)$ is an indecomposable projective $k[N_G(P)/P]$-module, we deduce that $\mathrm{Br}_P(\psi_i) : \mathrm{Br}_P(M_i) \to \mathrm{Br}_P(N)$ is a split surjection for some $i \in I$.

By [Bro85, Th. 3.2(4)], it follows that N is isomorphic to a direct summand of M_i. Since M_i is indecomposable, there is an isomorphism $\psi' : N \xrightarrow{\sim} M_i$. The morphism $\mathrm{Br}_P(\psi_i\psi') = \mathrm{Br}_P(\psi_i)\mathrm{Br}_P(\psi')$ is an isomorphism, so it is not nilpotent. Therefore, $\psi_i\psi'$ does not belong to the radical of $\mathrm{End}_{\Lambda G}(N)$, hence it is invertible (because $\mathrm{End}_{\Lambda G}(N)$ is a local ring). So ψ_i is an isomorphism, as desired. □

LEMMA A.2. *Let C be a bounded complex of ℓ-permutation ΛG-modules and P an ℓ-subgroup of G such that $\mathrm{Br}_Q(C)$ is acyclic for all ℓ-subgroups of G that are not conjugate to a subgroup of P. Let D be a bounded complex of finitely generated projective $k[N_G(P)/P]$-modules. We assume that $\mathrm{Br}_P(C) \simeq D$ in $\mathrm{Ho}^b(k[N_G(P)/P])$.*

Then there exists a bounded complex C' of ℓ-permutation ΛG-modules, all of whose indecomposable summands have a vertex contained in P, such that $C' \simeq C$ in $\mathrm{Ho}^b(\Lambda G)$ and $\mathrm{Br}_P(C') \simeq D$ in $\mathrm{Comp}^b(k[N_G(P)/P])$.

Proof. Up to isomorphism in $\mathrm{Ho}^b(\Lambda G)$, we may assume that $C = C^{\mathrm{red}}$. We write $C = (C^\bullet, d^\bullet)$. We will first show by induction on the length of C that $\mathrm{Br}_P(C) = \mathrm{Br}_P(C)^{\mathrm{red}}$ and that the indecomposable summands of C have a vertex contained in P.

Let n be maximal such that $C^{n+1} \neq 0$. We fix a decomposition $C^{n+1} = \bigoplus_{i\in I} M_i$ where M_i is indecomposable for all $i \in I$, and we denote by $p_i : C^{n+1} \to M_i$ the projection.

Let $i \in I$, and let Q be the vertex of M_i. Assume that the composition

$$\mathrm{Br}_Q(C^n) \xrightarrow{\mathrm{Br}_Q(d^n)} \mathrm{Br}_Q(C^{n+1}) \xrightarrow{\mathrm{Br}_Q(p_i)} \mathrm{Br}_Q(M_i)$$

is surjective. It follows from Lemma A.1 that $p_i d^n : C^n \to M_i$ is a split surjection: this contradicts the fact that $C = C^{\mathrm{red}}$. If Q is not conjugate to a subgroup of P, then $\mathrm{Br}_Q(d^n)$ is surjective by assumption, hence a contradiction. We deduce by induction that the indecomposable summands of C have a vertex contained in P.

$\mathrm{Br}_Q(C) = 0$ if Q is not conjugate to a subgroup of P.

We deduce also that the complex

$$0 \to \mathrm{Br}_P(C^n) \xrightarrow{\mathrm{Br}_P(d^n)} \mathrm{Br}_P(C^{n+1}) \to 0$$

has no nonzero direct summand that is homotopy equivalent to 0. By the induction hypothesis, the complex

$$\cdots \to \mathrm{Br}_P(C^{n-1}) \xrightarrow{\mathrm{Br}_P(d^{n-1})} \mathrm{Br}_P(C^n) \to 0$$

has no nonzero direct summand that is homotopy equivalent to 0. It follows that $\mathrm{Br}_P(C) = \mathrm{Br}_P(C)^{\mathrm{red}}$.

We deduce from this that $D \simeq \mathrm{Br}_P(C) \oplus D'$, where D' is homotopy equivalent to 0. So D' is a sum of complexes of the form $0 \to V \xrightarrow{\mathrm{Id}} V \to 0$ with V projective indecomposable (up to a shift), hence there is a bounded complex C' of ℓ-permutation ΛG-modules that is a direct sum of complexes of the form $0 \to \mathrm{M}(P,V) \xrightarrow{\mathrm{Id}} \mathrm{M}(P,V) \to 0$ with V projective indecomposable such that $\mathrm{Br}_P(C') \simeq D'$. We have $\mathrm{Br}_P(C \oplus C') \simeq D$, as desired. □

The following lemma is close to [Bou98, Prop. 7.9].

LEMMA A.3. *Let G be a finite group and C be a bounded complex of ℓ-permutation kG-modules. Assume $H^i(\mathrm{Br}_Q(C)) = 0$ for all $i \neq 0$ and all ℓ-subgroups Q of G. Then $C \simeq H^0(C)$ in $\mathrm{Ho}^b(kG)$.*

Proof. Replacing C by C^{red}, we can and will assume that C has no nonzero direct summands that are homotopy equivalent to 0.

Let $i > 0$ be maximal such that $C^i \neq 0$. The map $d^{i-1}_{\mathrm{Br}_Q(C)} = \mathrm{Br}_Q(d^{i-1}_C)$: $\mathrm{Br}_Q(C^{i-1}) \to \mathrm{Br}_Q(C^i)$ is surjective for all ℓ-subgroups Q. It follows from Lemma A.1 that d^{i-1}_C is a split surjection: this contradicts our assumption on C. So $C^i = 0$ for $i > 0$. Replacing C by C^*, we obtain similarly that $C^i_Q = 0$ for $i < 0$. The lemma follows. □

The following theorem is a variant of [Rou01, Th. 5.6].

THEOREM A.4. *Let G be a finite group and H a subgroup of G. Let C be a bounded complex of ℓ-permutation $k(G \times H^{\mathrm{opp}})$-modules all of whose indecomposable summands have a vertex contained in ΔH.*

Assume $\mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C), \mathrm{Br}_{\Delta Q}(C)[i]) = 0$ for all $i \neq 0$ and all ℓ-subgroups Q of H. Then $\mathrm{End}^\bullet_{kG}(C)$ is isomorphic to $\mathrm{End}_{D^b(kG)}(C)$ in $\mathrm{Ho}^b(k(H \times H^{\mathrm{opp}}))$.

Proof. Let R be an ℓ-subgroup of $H \times H^{\mathrm{opp}}$. By [Ric96, proof of Th. 4.1], we have $\mathrm{Br}_R(\mathrm{End}^\bullet_{kG}(C)) = 0$ if R is not conjugate to a subgroup of ΔH, and given $Q \leqslant H$ an ℓ-subgroup, we have

$$\mathrm{Br}_{\Delta Q}(\mathrm{End}^\bullet_{kG}(C)) \simeq \mathrm{End}^\bullet_{kC_G(Q)}(\mathrm{Br}_{\Delta Q}(C))$$

in $\mathrm{Comp}(k(C_H(Q) \times C_H(Q)^{\mathrm{opp}}))$.

Note that the indecomposable summands of $\mathrm{Br}_{\Delta Q}(C)$ are projective for $kC_G(Q)$ since their vertices are contained in $x(\Delta H)x^{-1} \cap (C_G(Q) \times 1)$ for some $x \in G \times H^{\mathrm{opp}}$, hence

$$H^i(\mathrm{End}^\bullet_{kC_G(Q)}(\mathrm{Br}_{\Delta Q}(C))) \simeq \mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C), \mathrm{Br}_{\Delta Q}(C)[i])$$

and this vanishes for $i \neq 0$. Consequently,

$$\mathrm{Br}_{\Delta Q}(\mathrm{End}^\bullet_{kG}(C)) \simeq \mathrm{End}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C))$$

in $D^b(k(C_H(Q) \times C_H(Q)^{\mathrm{opp}}))$.

The conclusion of the theorem follows now from Lemma A.3 applied to the complex $\mathrm{End}^\bullet_{kG}(C)$. □

The following corollary, used in the proof of Theorem 7.6, might be useful in other settings.

Corollary A.5. *Let G be a finite group, H a subgroup of G, b a block idempotent of $\mathcal{O}G$, and c a block idempotent of ΛH. Let C be a bounded complex of ℓ-permutation $(\Lambda Gb, \Lambda Hc)$-bimodules, all of whose indecomposable summands have a vertex contained in ΔH. Assume*

$$\mathrm{Hom}_{D^b(kC_G(Q))}(\mathrm{Br}_{\Delta Q}(C), \mathrm{Br}_{\Delta Q}(C)[i]) = 0$$

for all $i \neq 0$ and all ℓ-subgroups Q of H and the canonical map $kHc \to \mathrm{End}_{D^b(kG)}(kC)$ is an isomorphism. Then C induces a splendid Rickard equivalence between ΛGb and ΛHc.

Proof. Theorem A.4 shows that the canonical map $kHc \to \mathrm{End}^\bullet_{kG}(kC)$ is an isomorphism in $\mathrm{Ho}^b(k(H \times H^{\mathrm{opp}}))$. It follows from [Ric96, Th. 2.1] that kC induces a Rickard equivalence between kGb and kHc. The result follows now from [Ric96, proof of Th. 5.2]. □

References

[Bon05] C. Bonnafé, Quasi-isolated elements in reductive groups, *Comm. Algebra* **33** (2005), 2315–2337. MR 2153225. Zbl 1096.20037. https://doi.org/10.1081/AGB-200063602.

[Bon06] C. Bonnafé, *Sur les Caractères des Groupes Réductifs Finis à Centre Non Connexe: Applications aux Groupes Spéciaux Linéaires et Unitaires*, *Astérisque* **306**, Math. Soc. France, Paris, 2006. MR 2274998. Zbl 1157.20022.

[BM11] C. Bonnafé and J. Michel, Computational proof of the Mackey formula for $q > 2$, *J. Algebra* **327** (2011), 506–526. MR 2746047. Zbl 1231.20041. https://doi.org/10.1016/j.jalgebra.2010.10.030.

[BR03] C. Bonnafé and R. Rouquier, Catégories dérivées et variétés de Deligne-Lusztig, *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 1–59. MR 2010739. Zbl 1054.20024. https://doi.org/10.1007/s10240-003-0013-3.

[BR06] C. Bonnafé and R. Rouquier, Coxeter orbits and modular representations, *Nagoya Math. J.* **183** (2006), 1–34. MR 2253885. Zbl 1109.20038. https://doi.org/10.1017/S0027763000009259.

[Bou98] S. Bouc, Résolutions de foncteurs de Mackey, in *Group Representations: Cohomology, Group Actions and Topology* (Seattle, WA, 1996), *Proc. Sympos. Pure Math.* **63**, Amer. Math. Soc., Providence, RI, 1998, pp. 31–83. MR 1603131. Zbl 0897.19001. https://doi.org/10.1090/pspum/063/1603131.

[Bro85] M. Broué, On Scott modules and p-permutation modules: an approach through the Brauer morphism, *Proc. Amer. Math. Soc.* **93** (1985), 401–408. MR 0773988. Zbl 0574.20005. https://doi.org/10.2307/2045600.

[Bro90] M. Broué, Isométries de caractères et équivalences de Morita ou dérivées, *Inst. Hautes Études Sci. Publ. Math.* **71** (1990), 45–63. MR 1079643. Zbl 0727.20005. https://doi.org/10.1007/BF02699877.

[BM89] M. Broué and J. Michel, Blocs et séries de Lusztig dans un groupe réductif fini, *J. Reine Angew. Math.* **395** (1989), 56–67. MR 0983059. Zbl 0654.20048. https://doi.org/10.1515/crll.1989.395.56.

[Cab13] M. Cabanes, On Jordan decomposition of characters for $\mathrm{SU}(n, q)$, *J. Algebra* **374** (2013), 216–230. MR 2998804. Zbl 1272.20049. https://doi.org/10.1016/j.jalgebra.2012.10.015.

[CE99] M. Cabanes and M. Enguehard, On blocks of finite reductive groups and twisted induction, *Adv. Math.* **145** (1999), 189–229. MR 1704575. Zbl 0954.20023. https://doi.org/10.1006/aima.1998.1814.

[CE04] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, *New Math. Monogr.* **1**, Cambridge Univ. Press, Cambridge, 2004. MR 2057756. Zbl 1069.20032. https://doi.org/10.1017/CBO9780511542763.

[Dat16] J.-F. Dat, Equivalences of tame blocks for p-adic linear groups, 2016. arXiv 1603.07226.

[DL76] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161. MR 0393266. Zbl 0336.20029. https://doi.org/10.2307/1971021.

[DM91] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*, *London Math. Soc. Stud. Texts* **21**, Cambridge Univ. Press, Cambridge, 1991. MR 1118841. Zbl 0815.20014. https://doi.org/10.1017/CBO9781139172417.

[DM94] F. Digne and J. Michel, Groupes réductifs non connexes, *Ann. Sci. École Norm. Sup.* **27** (1994), 345–406. MR 1272294. Zbl 0846.20040. Available at http://www.numdam.org/item?id=ASENS_1994_4_27_3_345_0.

[DM15] F. Digne and J. Michel, Complements on disconnected reductive groups, *Pacific J. Math.* **279** (2015), 203–228. MR 3437776. Zbl 06526352. https://doi.org/10.2140/pjm.2015.279.203.

[DMR07] F. Digne, J. Michel, and R. Rouquier, Cohomologie des variétés de Deligne-Lusztig, *Adv. Math.* **209** (2007), 749–822. MR 2296313. Zbl 1118.20006. https://doi.org/10.1016/j.aim.2006.06.001.

[DD93] R. Dipper and J. Du, Harish-Chandra vertices, *J. Reine Angew. Math.* **437** (1993), 101–130. MR 1212254. Zbl 0768.20004. https://doi.org/10.1515/crll.1993.437.101.

[DR14] O. Dudas and R. Rouquier, Coxeter orbits and Brauer trees III, *J. Amer. Math. Soc.* **27** (2014), 1117–1145. MR 3230819. Zbl 1307.20010. https://doi.org/10.1090/S0894-0347-2014-00791-8.

[HL94] R. B. Howlett and G. I. Lehrer, On Harish-Chandra induction and restriction for modules of Levi subgroups, *J. Algebra* **165** (1994), 172–183. MR 1272585. Zbl 0802.20036. https://doi.org/10.1006/jabr.1994.1104.

[KM13] R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, *Ann. of Math.* **178** (2013), 321–384. MR 3043583. Zbl 1317.20006. https://doi.org/10.4007/annals.2013.178.1.6.

[KM15] R. Kessar and G. Malle, Brauer's height zero conjecture for quasi-simple groups, 2015. arXiv 1510.07907.

[Lus78] G. Lusztig, *Representations of Finite Chevalley Groups*, *CBMS Reg. Conf. Ser. Mat.* **39**, Amer. Math. Soc., Providence, RI, 1978. MR 0518617. Zbl 0418.20037. https://doi.org/10.1090/cbms/039.

[Ngu] T. H. Nguyen, Cohomologie des variétés de Coxeter pour le groupe linéaire: algèbre d'endomorphismes, compactification, Ph.D. thesis (in preparation).

[Pui99] L. Puig, *On the Local Structure of Morita and Rickard Equivalences Between Brauer Blocks*, *Progr. Math.* no. 178, Birkhäuser, 1999. MR 1707300. Zbl 0929.20012.

[Ric94] J. Rickard, Finite group actions and étale cohomology, *Inst. Hautes Études Sci. Publ. Math.* **80** (1994), 81–94 (1995). MR 1320604. Zbl 0838.14013. https://doi.org/10.1007/BF02698896.

[Ric96] J. Rickard, Splendid equivalences: derived categories and permutation modules, *Proc. London Math. Soc.* **72** (1996), 331–358. MR 1367082. Zbl 0862.20010. https://doi.org/10.1112/plms/s3-72.2.331.

[Rou01] R. Rouquier, Block theory via stable and Rickard equivalences, in *Modular Representation Theory of Finite Groups* (Charlottesville, VA, 1998), de Gruyter, Berlin, 2001, pp. 101–146. MR 1889341. Zbl 0998.20006.

[Rou02] R. Rouquier, Complexes de chaînes étales et courbes de Deligne-Lusztig, *J. Algebra* **257** (2002), 482–508. MR 1947973. Zbl 1030.18008. https://doi.org/10.1016/S0021-8693(02)00530-6.

[Rou14] R. Rouquier, Finite generation of cohomology of finite groups, 2014, preprint. arXiv 1608.03579.

[Spr98] T. A. Springer, *Linear Algebraic Groups*, second ed., *Progr. Math.* **9**, Birkhäuser, Boston, 1998. MR 1642713. Zbl 0927.20024. https://doi.org/10.1007/978-0-8176-4840-4.

[Ste68] R. Steinberg, *Endomorphisms of Linear Algebraic Groups*, *Mem. Amer. Math. Soc.* **80**, Amer. Math. Soc., Providence, RI, 1968. MR 0230728. Zbl 0164.02902.

[Thé95] J. Thévenaz, *G-Algebras and Modular Representation Theory*, *Oxford Math. Monogr.*, The Clarendon Press, Oxford Univ. Press, New York, 1995. MR 1365077. Zbl 0837.20015.

[Tho97] R. W. Thomason, The classification of triangulated subcategories, *Compositio Math.* **105** (1997), 1–27. MR 1436741. Zbl 0873.18003. https://doi.org/10.1023/A:1017932514274.

[Wan14] H. Wang, L'espace symétrique de Drinfeld et correspondance de Langlands locale II, 2014. arXiv 1402.1265.

(Received: November 19, 2015)
(Revised: July 27, 2016)

Institut de Mathématiques et de Modélisation de Montpellier, Montpellier, France
E-mail: cedric.bonnafe@univ-montp2.fr
http://www.math.univ-montp2.fr/~bonnafe/

Institut de Mathématiques de Jussieu, Paris, France
E-mail: jean-francois.dat@imj-prg.fr
https://webusers.imj-prg.fr/~jean-francois.dat/

University of California, Los Angeles, Los Angeles, CA
E-mail: rouquier@math.ucla.edu
http://www.math.ucla.edu/~rouquier/