CACTUS GROUPS AND LUSZTIG'S ASYMPTOTIC ALGEBRA

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Abstract. We construct a morphism from the cactus group associated with a Coxeter group to the group of invertible elements of Lusztig's asymptotic algebra. This relates to the cactus group action on elements of Coxeter groups defined in [Lo, Bo2] and we propose a conjecture on how to fully recover those actions.

1. Introduction

Cactus groups are "crystal limits" of braid groups, originally introduced implicitely in type A by Drinfeld [Dr] and explicitely in [HeKa]. The braid group action on tensor powers of representations of quantum groups becomes an action of the cactus group on tensor powers of a crystal.

Cactus groups are (orbifold) fundamental groups of the Deligne-Mumford compactification of the moduli space of genus 0 real curves with marked points [De, DaJaSc].

Cactus groups have been generalized to other Coxeter groups. They can be defined by generators and relations and, for finite Coxeter groups, are orbifold fundamental groups of real points of the wonderful compactification of projectivized hyerplane complements [DaJaSc].

In [Lo], Losev constructed an action of the cactus group associated to a Weyl group on the set of elements of the Weyl group in terms of the combinatorics of perverse self-equivalences of the category $\mathcal{O}$ of a complex semi-simple Lie algebra. Bonnafé [Bo2] generalized this construction to all Coxeter groups and unequal parameters, with a direct algebraic approach using the Hecke algebra. The cactus group orbits are proven to be unions of Kazhdan-Lusztig cells (actual Kazhdan-Lusztig cells in type A).

Cactus groups are expected to play a role in the geometry of ramification of Calogero-Moser spaces, which conjecturally provides another construction of Kazhdan-Lusztig cells [BoRou1, BoRou2].

In this article, following a suggestion of Etingof, we start from Drinfeld's unitarization trick, providing a morphism from the cactus group to the (completed) braid group. This enables us to obtain a direct connection from the cactus group to the Hecke algebra and to Lusztig's asymptotic algebra. Our work follows Etingof's proposal and answers some of his questions.

A key point in our approach is a characterization of the element of the Hecke algebra that corresponds to the longest element of a finite Coxeter group, given a suitable isomorphism of the Hecke algebra with the group algebra of the Coxeter group (§3).

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2. Braid groups and cactus groups

2.1. **Coxeter groups.** Let (W, S) be a Coxeter group with S finite. Let $(m_{st})_{s,t\in S}$ be the associated Coxeter matrix.

Given I a subset of S, let W_I be the subgroup of W generated by I. When W_I is finite, we say that I is spherical and we denote by w_I the longest element of W_I. We put $w_0 = w_S$ when W is finite.

2.2. **Braid groups.** We denote by B_W the braid group of W. It is generated by $(\beta_s)_{s\in S}$ and relations

$$\underbrace{\beta_s\beta_t\beta_s\cdots}_{m_{st}\text{ terms}} = \underbrace{\beta_t\beta_s\beta_t\cdots}_{m_{st}\text{ terms}}.$$

There is a surjective morphism of groups

$$p_W : B_W \to W,\ \beta_s \mapsto s.$$

Its kernel P_W is the pure braid group.

Let $w \in W$. Given $w = s_1 \cdots s_n$ a reduced decomposition, the element $\beta_w = \beta_{s_1} \cdots \beta_{s_n}$ of B_W is independent of the reduced decomposition.

Given I a subset of S such that W_I is finite, we put $\beta_I = \beta_{w_I}$. Note that $\beta_I^2 \in P_W$.

2.3. **Completions.** Let $\mathcal{I}$ be the augmentation ideal of the group algebra $\mathbf{Q}[P_W]$ (the kernel of the algebra morphism $\mathbf{Q}[P_W] \to \mathbf{Q},\ P_W \ni g \mapsto 1$). Let $\widehat{\mathbf{Q}[P_W]}$ be the completion of $\mathbf{Q}[P_W]$ at $\mathcal{I}$. This is a complete cocommutative Hopf algebra and we denote by $\widehat{\mathbf{Q}[P_W]}^*$ its topological dual. The prounipotent completion of P_W is $\hat{P}_W = \mathrm{Spec}(\widehat{\mathbf{Q}[P_W]}^*)$. Note that given $g \in P_W$ and $\alpha \in \mathbf{Q}$, we have an element $g^\alpha \in \hat{P}_W$ corresponding to $\sum_{n\geq 0} \binom{\alpha}{n}(g-1)^n \in \widehat{\mathbf{Q}[P_W]}$.

Let $\mathcal{I}'$ be the kernel of $\mathbf{Q}[p_W] : \mathbf{Q}[B_W] \to \mathbf{Q}[W],\ B_W \ni g \mapsto p_W(g)$. This is the two-sided ideal of $\mathbf{Q}[B_W]$ generated by $\mathcal{I}$. We denote by $\widehat{\mathbf{Q}[B_W]}$ be the completion of $\mathbf{Q}[B_W]$ at $\mathcal{I}'$ and we put $\hat{B}_W = \mathrm{Spec}(\widehat{\mathbf{Q}[B_W]}^*)$. This is a proalgebraic group, the connected component of the identity is $\hat{P}_W$ and p_W extends to a surjective morphism of groups $\hat{B}_W \to W$ with kernel $\hat{P}_W$.

2.4. **Cactus group.** The cactus group C_W is the group generated by $(\gamma_I)_{I\subset S\text{ spherical}}$ with relations

$$\gamma_I^2 = 1$$

$$\gamma_I\gamma_J = \gamma_{I\cup J} \text{ if } W_{I\cup J} = W_I \times W_J$$

$$\gamma_I\gamma_J = \gamma_J\gamma_{w_J(I)} \text{ if } I \subset J.$$

There is a surjective morphism of groups

$$\pi_W : C_W \to W,\ \gamma_I \mapsto w_I.$$

Note that C_W is generated by those elements γ_I such that (W_I, I) is a finite irreducible Coxeter group.

2.5. **Cactus to completed braids.** In type A, the following result is in [EtHeKaRa, proof of Theorem 3.14], based on Drinfeld's unitarization trick [Dr].

Proposition 2.1. *The assignment $\gamma_I \mapsto \beta_I(\beta_I^2)^{-1/2}$ for $I \subset S$ spherical defines a morphism of groups $\phi : C_W \to \hat{B}_W$. We have $p_W \circ \phi = \pi_W$.*

Proof. Since β_I commutes with β_I^2, it commutes with $(\beta_I^2)^{-1/2}$, hence $(\beta_I(\beta_I^2)^{-1/2})^2 = 1$.

Similarly, if $W_{I\cup J} = W_I \times W_I$, then β_I and β_J commute, hence $\beta_I(\beta_I^2)^{-1/2}$ and $\beta_J(\beta_J^2)^{-1/2}$ commute.

Finally, when $I \subset J$, we have $\beta_J^{-1}\beta_I\beta_J = \beta_{w_J(I)}$, hence $\beta_J^{-1}(\beta_I^2)^{-1/2}\beta_J = (\beta_{w_J(I)}^2)^{-1/2}$. It follows that $\beta_I(\beta_I^2)^{-1/2} \cdot \beta_J(\beta_J^2)^{-1/2} = \beta_J(\beta_J^2)^{-1/2} \cdot \beta_{w_J(I)}(\beta_{w_J(I)}^2)^{-1/2}$ since $(\beta_J^2)^{-1/2}$ commutes with $\beta_{w_J(I)}$ and with $(\beta_{w_J(I)}^2)^{-1/2}$. ☐

Remark 2.2. The map of [EtHeKaRa, proof of Theorem 3.14] and [Dr] is defined using a different set of generators for C_W. That new set of generators can be defined for an arbitrary W as follows.

Let F be the set of pairs (I, s) where $I \subset S$, $s \in I$ and where W_I is finite and irreducible (i.e. its Coxeter diagram is connected). Let $(I, s) \in F$. We put $\gamma_{I,s} = \gamma_I\gamma_{I\setminus\{s\}}$, $\beta'_{I,s} = \beta_I\beta_{I\setminus\{s\}}^{-1}$ and $\beta''_{I,s} = \beta_{I\setminus\{s\}}^{-1}\beta_I$, where $\gamma_\emptyset = 1$ and $\beta_\emptyset = 1$. The set $\{\gamma_{I,s}\}_{(I,s)\in F}$ generates C_W and we have $\phi(\gamma_{I,s}) = \beta'_{I,s}(\beta''_{I,s}\beta'_{I,s})^{-1/2}$.

3. Hecke algebra

Let H_W be the Hecke algebra of W. This is the quotient of the group ring $\mathbf{Z}[v^{\pm 1}]B_W$ by the ideal generated by $(\beta_s - v)(\beta_s + v^{-1})$ for $s \in S$. We denote by $\kappa : \mathbf{Z}[v^{\pm 1}]B_W \to H_W$ the quotient map and we put $T_s = \kappa(\beta_s)$. Note that the composition

$$\mathbf{Z}[v^{\pm 1}]B_W \xrightarrow{\kappa} H_W \xrightarrow{v\to 1} \mathbf{Z}W$$

is given by $\beta \mapsto p_W(\beta)$.

Let R be the completion of $\mathbf{Q}[v^{\pm 1}]$ at the ideal $(v-1)$. We have $\kappa(\mathcal{I}) \subset (v-1)\mathbf{Q}[v^{\pm 1}]H_W$. It follows that κ induces a morphism of proalgebraic groups (still denoted by κ) $\hat{B}_W \to (RH_W)^\times$ and we have a commutative diagram

$$\begin{array}{ccccc} C_W & \xrightarrow{\phi} & \hat{B}_W & \xrightarrow{\kappa} & (RH_W)^\times \\ {\scriptstyle \pi_W}\downarrow & & & & \downarrow{\scriptstyle v\to 1} \\ W & & \xhookrightarrow[\text{can}]{\qquad\qquad} & & (\mathbf{Q}W)^\times \end{array} \tag{3.1}$$

Given $I \subset S$, we have a corresponding commutative diagram with W replaced by W_I. It is a subdiagram of the commutative diagram (3.1).

Lemma 3.1. *If $S = \{s\}$, then $\kappa \circ \phi(\gamma_s) = \frac{1-v^2}{1+v^2} + \frac{2v}{1+v^2}T_s$.*

Proof. Let $a = \kappa \circ \phi(\gamma_s)$. We have $a_{|v=1} = s$ and $a^2 = 1$. This shows that $a = \frac{1-v^2}{1+v^2} + \frac{2v}{1+v^2}T_s$. ☐

Proposition 3.2. *Assume W is finite. There exists a unique element $\tilde{w}_0$ of RH_W such that*

- $\tilde{w}_0^2 = 1$

- $(\tilde{w}_0)_{|v=1} = w_0$
- $\tilde{w}_0 T_s \tilde{w}_0^{-1} = T_{w_0 s w_0}$ *for all* $s \in S$.

Proof. Given $s \in S$, we have $\gamma_S \gamma_s \gamma_S = \gamma_{w_0 s w_0}$, hence

$$\big(\kappa \circ \phi(\gamma_S)\big) T_s \big(\kappa \circ \phi(\gamma_S)\big)^{-1} = T_{w_0 s w_0}$$

by Lemma 3.1. The commutativity of the diagram (3.1) shows that $\tilde{w}_0 = \kappa \circ \phi(\gamma_S)$ satisfies the properties of the proposition.

Consider now an element $h \in RH_W$ satisfying the properties of $\tilde{w}_0$. Let $z = h\tilde{w}_0 \in Z(RH_W)$. We have $z^2 = 1$ and $z_{|v=1} = 1$. There is an isomorphism of R-algebras $i : Z(RH_W) \xrightarrow{\sim} R^n$ for some n. Since $i(z)^2 = 1$, it follows that $i(z) \in \mathbf{Q}^n$. In addition, $i(z)_{|z=1} = 1$, hence $i(z) = 1$. So, we have shown that $z = 1$. This proves the unicity of the element $\tilde{w}_0$. □

Given $I \subset S$ spherical, we denote by $\tilde{w}_I$ the element of $RH_I \subset RH_W$ defined for W_I as in Proposition 3.2. The proof of Proposition 3.2 shows the following.

Proposition 3.3. *Given* $I \subset S$ *with* W_I *finite, we have* $\kappa \circ \phi(\gamma_I) = \tilde{w}_I$.

The next proposition follows from Proposition 3.3 and Lemma 3.1.

Proposition 3.4. *Given* $s \in S$, *we have* $\tilde{w}_s = \frac{1-v^2}{1+v^2} + \frac{2v}{1+v^2} T_s$.

The next result is immediate.

Proposition 3.5. *Assume* W *is finite. Let* $\lambda : RW \xrightarrow{\sim} RH_W$ *be an isomorphism of* R*-algebras such that*

- $\lambda_{|v=1}$ *is the identity*
- $\lambda(w_0 s w_0) = T_{w_0} \lambda(s) T_{w_0}^{-1}$ *for all* $s \in S$.

We have $\lambda(w_0) = \tilde{w}_0$.

Note that Lusztig has constructed an explicit isomorphism with these properties which is already defined over $\mathbf{Q}[v^{\pm 1}]_{(v-1)}$ [Lu1, Theorem 3.1]. As a consequence, we have the following result answering a question of Etingof.

Corollary 3.6. *Given* $I \subset S$ *with* W_I *finite, we have* $\tilde{w}_I \in \mathbf{Q}[v^{\pm 1}]_{(v-1)} H_W$.

4. Asymptotic algebra

Let $h \mapsto \bar{h}$ the $\mathbf{Z}$-algebra involution of H_W given by $\bar{v} = v^{-1}$ and $\overline{T_s} = T_s^{-1}$. There is a unique family $(C_w)_{w \in W}$ of elements of H_W such that $\overline{C_w} = C_w$ and $C_w - T_w \in \bigoplus_{w' < w} \mathbf{Z}[v^{\pm 1}] T_{w'}$. This is the Kazhdan-Lusztig basis of H_W. We refer to [Bo1] for basics of Kazhdan-Lusztig and Lusztig theory.

Given $w, w', w'' \in W$, we define $h_{w,w',w''} \in \mathbf{Z}[v^{\pm 1}]$ so that $C_w C_{w'} = \sum_{w'' \in W} h_{w,w',w''} C_{w''}$.

We put $a(w) = -\min_{w',w'' \in W} \deg(h_{w',w'',w})$ and we put $\gamma_{w,w',w''} = (v^{a(w'')} h_{w,w',(w'')^{-1}})_{|v=0}$.

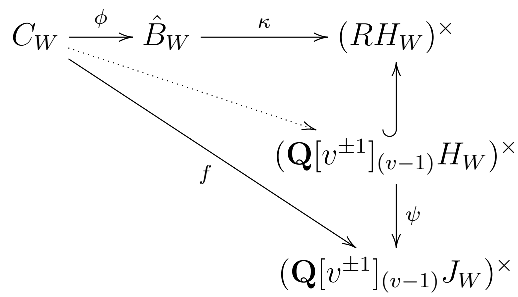
Lusztig's asymptotic J-ring is the $\mathbf{Z}$-algebra with basis $\{t_w\}_{w \in W}$ and multiplication given by $t_w t_{w'} = \sum_{w'' \in W} \gamma_{w,w',w''} t_{w''}$. There is a subset $\mathcal{D}$ of W such that $1 = \sum_{d \in \mathcal{D}} t_d$.

There is a morphism of $\mathbf{Z}[v^{\pm 1}]$-algebras

$$\psi : H_W \to \mathbf{Z}[v^{\pm 1}] J_W,\ C_w \mapsto \sum_{y \in W, d \in \mathcal{D}, a(d) = a(y)} h_{w,d,y} t_y.$$

By Corollary 3.6, the map $\kappa \circ \phi$ takes values in $\mathbf{Q}[v^{\pm 1}]_{(v-1)}$ and we put $f = \psi \circ \kappa \circ \phi : C_W \to (\mathbf{Q}[v^{\pm 1}]J_W)^\times$.

We have a commutative diagram where all the maps are equivariant for the automorphisms of (W, S).



The following result describing the image of $\gamma_S \in C_W$ in the asymptotic algebra is our main result.

Theorem 4.1. *Assume W is finite. We have $f(\gamma_S) = \sum_{d \in \mathcal{D}} (-1)^{\ell(w_0)+a(w_0 d)} t_{w_0 d}$.*

Proof. Let $t = \sum_{d \in \mathcal{D}} (-1)^{\ell(w_0)+a(w_0 d)} t_{w_0 d}$. We have $t^2 = 1$ and $t t_w t = t_{w_0 w w_0}$ [Lu2, §2.9] (cf also [Bo1, Example 19.3.3]), hence $\psi^{-1}(t)^2 = 1$ and $\psi^{-1}(t) T_w \psi^{-1}(t) = T_{w_0 w w_0}$ for all $w \in W$ since ψ is equivariant for the diagram automorphism induced by w_0.

Since $\psi^{-1}(t) - T_{w_0} \in (v-1)H_W$ [Lu2, Corollary 2.8], we have $\psi^{-1}(t) - \tilde{w}_0 \in (v-1)H_W$, hence $\psi^{-1}(t) = \tilde{w}_0$ by Proposition 3.2.

Proposition 3.3 completes the proof of the Theorem. □

It follows from [Ma, Theorem 3.1] (cf [Bo1, Example 19.3.3]) that given $w \in W$, there is $\sigma_W(w) \in W$ such that

(4.1) $$f(\gamma_S) t_w = (-1)^{a(w_0 w)} t_{\sigma_W(w)} \text{ and } t_w f(\gamma_S) = (-1)^{a(w_0 w)} t_{w_0 \sigma_W(w) w_0}.$$

The map σ_W is an involution of the set W.

In general, the image of $f(\gamma_I)$ in J_W is difficult to describe explictly. When $I = \{s\}$, this is equivalent to the description of $\psi(C_s)$ (cf Proposition 3.4). We propose that $f(\gamma_I)$ has an explicit description modulo $v\mathbf{Z}[v]J_W$ and that left and right multiplication by $f(\gamma_I)$ give the cactus group actions on W of Losev and Bonnafé.

Conjecture 4.2. (1) *Given $I \subset S$ spherical, we have $f(\gamma_I) \in \mathbf{Z}[v]_{(v(v-1))} J_W$.*

(2) *Denote by $\bar{f} : C_W \to J_W^\times$ the composition of f with specialization of v to 0. Given $I \subset S$ spherical, we have*
$$\bar{f}(\gamma_I) = \sum_{d \in \mathcal{D}} (-1)^{a(w_I d_1)} t_{\sigma_{W_I}(d_1) d_2}$$
where $d = d_1 d_2$ with $d_1 \in W_I$ and d_2 of minimal length in $W_I w$.

(3) *Consider $I \subset S$ spherical and $w \in W$. Write $w = xy$ with $x \in W_I$ and y of minimal length in wW_I (resp. $w = yx$ with $x \in W_I$ and y of minimal length in $W_I w$). We have*
$$\bar{f}(\gamma_I) t_w = (-1)^{a(w_I x)} t_{\sigma_{W_I}(x) y} \text{ and } t_w \bar{f}(\gamma_I) = (-1)^{a(w_I x)} t_{y w_I \sigma_{W_I}(x) w_I}.$$

Proposition 4.3. *Conjecture 4.2 holds when $I = S$ or $|I| = 1$.*

Proof. Assume $I = \{s\}$. We have $\tilde{w}_s = -1 + \frac{2v}{1+v^2} C_s$, hence

$$f(\gamma_S) t_w = -t_w + \frac{2v}{1+v^2} \sum_{w' \in W, a(w')=a(w)} h_{s,w,w'} t_{w'}.$$

We have

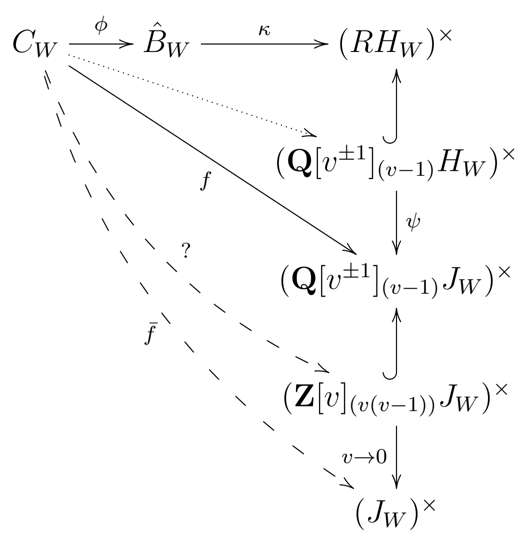
$$C_s C_w = \begin{cases} (v+v^{-1}) C_w & \text{if } sw < w \\ \sum_{w' \in W, sw' < w'} \mu_{s,w,w'} C_{w'} & \text{otherwise,} \end{cases}$$

where $\mu_{s,w,w'} = h_{s,w,w'} \in \mathbf{Z}$ when $sw > w$. It follows that

$$f(\gamma_S) t_w = \begin{cases} t_w & \text{if } sw < w \\ -t_w + \frac{2v}{1+v^2} \sum_{w' \in W, a(w')=a(w), sw' < w'} \mu_{s,w,w'} t_{w'} & \text{otherwise.} \end{cases}$$

This shows that $f(\gamma_s) = \sum_{d \in \mathcal{D}} f(\gamma_S) t_d \in \mathbf{Z}[v]_{(v(v-1))} J_W$ and this also shows the first statement of (3). The second statement of (3) is proven similarly by considering $t_w f(\gamma_S)$. Also we obtain $\bar{f}(\gamma_s) = \sum_{d \in \mathcal{D}} (-1)^{\delta_{sd<d}} t_d$, which shows (2).

Assume now $I = S$. Statements (1) and (2) are given by Theorem 4.1 while (3) is statement (4.1). $\square$



Remark 4.4. We do not know if the map $f : C_W \to (\mathbf{Q}[v^{\pm 1}]_{(v-1)} J_W)^\times$ is faithful or not, and the faithfulness of $\phi : C_W \to \hat{B}_W$ does not seem to be known either. On the other hand, cactus groups are known to be linear when W is finite [Yu].

Example 4.5. Fix $m \geq 3$ and consider the dihedral Coxeter group

$$W = I_2(m) = \langle s_1, s_2 \mid s_i^2 = 1,\ \underbrace{s_1 s_2 s_1 \cdots}_{m \text{ terms}} = \underbrace{s_2 s_1 s_2 \cdots}_{m \text{ terms}} \rangle.$$

We have

$$C_W = \langle \gamma_{s_1}, \gamma_{s_2}, \gamma_S \mid \gamma_{s_i}^2 = \gamma_S^2 = 1,\ \gamma_S \gamma_{s_i} \gamma_S = \begin{cases} \gamma_{3-i} & \text{if } m \text{ is odd} \\ \gamma_i & \text{otherwise} \end{cases} \rangle$$

and

$$f(\gamma_{s_i}) = -t_1 + t_{s_i} - t_{s_{3-i}} + \frac{2v}{1+v^2} t_{s_i s_{3-i}} + t_{w_0}$$
$$f(\gamma_S) = (-1)^m t_1 - t_{w_0 s_1} - t_{w_0 s_2} + t_{w_0}.$$

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