ON COMPLEX REFLECTION GROUPS AND THEIR ASSOCIATED BRAID GROUPS

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Abstract. Presentations "à la Coxeter" are given for all (irreducible) finite complex reflection groups. They provide presentations for the corresponding generalized braid groups (still conjectural in some cases) which allow us to generalize some of the known properties of finite Coxeter groups (center of the braid group, construction of Hecke algebras).

1. Background from complex reflection groups

For all the results quoted here, we refer the reader to the classical literature on complex reflections groups, such as [Bou], [Ch], [Co], [ShTo], [Sp], and also to the more recent and fundamental work on the subject by Orlik, Solomon and Terao (see [OrSo], [OrTe]).

Let V be a complex vector space of dimension r. A *pseudo–reflection* of $\mathrm{GL}(V)$ is a non trivial element ρ of $\mathrm{GL}(V)$ which acts trivially on a hyperplane, called the *reflecting hyperplane* of ρ. Let G be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo–reflections. The pair (V, G) is called a "complex reflection group".

A *parabolic subgroup* of G is by definition the subgroup of elements of G which act trivially on a subspace of V. By a theorem of Steinberg ([St], Theorem 1.5), a parabolic subgroup is generated by pseudo–reflections.

We denote by $\mathcal{A}$ the set of reflecting hyperplanes of G, and we set $N := |\mathcal{A}|$. For $H \in \mathcal{A}$, we denote by G_H the fixator (pointwise stabilizer) of H (a minimal parabolic subgroup of G), and we set $e_H := |G_H|$. We denote by N^* the number of pseudo-reflections in G. The centralizer $C_G(G_H)$ of G_H in G is also its normalizer, as well as the normalizer (setwise stabilizer) of H. We set $\omega_H := |G : C_G(G_H)|$.

We denote by S the symmetric algebra of V, by $R = S^G$ the algebra of invariants of G, by R_+ the ideal of R consisting of elements of positive degree, and we set $S_G := S/R_+S$.

The following facts are known.

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This is a summary of work in progress. The final version of this paper will be submitted for publication elsewhere. We thank Jean Michel for useful conversations and for his help in preparing this manuscript.

- There is a family of r integers $d_1, d_2, \ldots, d_r$ called the *degrees* of (V, G), defined by the following condition: the Poincaré polynomial of the graded module $(V \otimes S_G)^G$ is
$$q^{d_1-1} + q^{d_2-1} + \cdots + q^{d_r-1}\,.$$
We have $\sum_{i=1}^{i=r}(d_i - 1) = \sum_{H\in\mathcal{A}}(e_H - 1) = \sum_{H\in\mathcal{A}/G}\omega_H(e_H - 1) = N^*$.
- There is a family of r integers $d_1^*, d_2^*, \ldots, d_r^*$ called the *codegrees* of (V, G), defined by the following condition: the Poincaré polynomial of the graded module $(V^* \otimes S_G)^G$ is
$$q^{d_1^*+1} + q^{d_2^*+1} + \cdots + q^{d_r^*+1}\,.$$
We have $\sum_{i=1}^{i=r}(d_i^* + 1) = \sum_{H\in\mathcal{A}} 1 = \sum_{H\in\mathcal{A}/G}\omega_H = N$.
- We have $N + N^* = \sum_{i=1}^{i=r}(d_i + d_i^*) = \sum_{H\in\mathcal{A}/G}\omega_H e_H$.
- The center Z of G has order $|Z| = \gcd\{d_1, d_2, \ldots, d_r\}$.
- The order of G is $|G| = d_1 d_2 \cdots d_r$.

Remark. The "codegrees" have not been introduced as such in the quoted literature. Nevertheless, the degrees and the codegrees are related to the *exponents* $\{m_1, m_2, \ldots, m_r\}$ and the *coexponents* $\{m_1^*, m_2^*, \ldots, m_r^*\}$ (which are defined in [OrSo]) by the formulae
$$m_j = d_j - 1 \quad \text{and} \quad m_j^* = d_j^* + 1 \quad (j = 1, 2, \ldots, r)\,.$$

2. Presentations

The tables in the Appendix provide a complete list of the irreducible finite pseudo–reflection groups, together with presentations of these groups symbolized by diagrams "à la Coxeter", as well as some of the data attached to these groups.

Here are some definitions, notation, conventions, which will allow the reader to understand the diagrams.

The groups have presentations given by diagrams $\mathcal{D}$ such that

- the nodes correspond to pseudo-reflections in G, the order of which is given inside the circle representing the node (order 2 is omitted),
- they are related by homogeneous relations with the same "support" (of cardinality 2 or 3), which are represented by links beween two or three nodes, or circles between three nodes, weighted with a number representing the degree of the relation (as in the usual case, 3 is omitted, 4 is represented by a double line, 6 is represented by a triple line). These homogeneous relations are called the *braid relations* of $\mathcal{D}$.

More details are provided in §5 below.

Isomorphisms between diagrams.

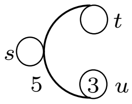
We may notice that the only isomorphisms between the diagrams of our tables are between the diagrams of $G(2, 1, 2)$ and $G(4, 4, 2)$, and between the diagrams of $\mathfrak{S}_3$ and $G(3, 3, 2)$.

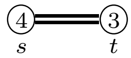
Admissible subdiagrams and parabolic subgroups.

Let $\mathcal{D}$ be one of the diagrams. Let us define an equivalence relation between nodes by

$$s \sim t \quad :\iff \quad s \text{ and } t \text{ are not in a homogeneous relation with support } \{s,t\}\,.$$

Then we see that the equivalences classes have 1 or 3 elements, and that there is at most one class with 3 elements.

If there is no class with 3 elements, the rank r of the group is the number of nodes of the diagram, while it is this number minus 1 in case there is a class with 3 elements. Thus 

has rank 2, as well as 

.

Remark. One must point out that, in the first of the preceding two diagrams, s, t and u must be considered as linked with a line (so t and u *do not* commute).

An **admissible subdiagram** is a full subdiagram of the same type, namely a diagram with 1 or 3 elements per class.

Fact. *Assume G is neither G_{27}, G_{29}, G_{33} nor G_{34}, and let $\mathcal{D}$ be its diagram (see tables).*

(1) *If $\mathcal{D}'$ is an admissible subdiagram of $\mathcal{D}$, it gives a presentation of the corresponding subgroup $G(\mathcal{D}')$ of G. This subgroup is a parabolic subgroup.*
(2) *If $P_1 \subseteq P_2 \subseteq \cdots P_n$ is a chain of parabolic subgroups of G, there exist $g \in G$ and a chain $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \mathcal{D}_n$ of admissible subdiagrams of $\mathcal{D}$ such that*

$$(P_1, P_2, \ldots, P_n) = {}^g(G(\mathcal{D}_1), G(\mathcal{D}_2), \ldots, G(\mathcal{D}_n))\,.$$

Remark.

For groups G_{27} and G_{29}, all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams, but not all *conjugacy classes* of parabolics subgroups are represented by admissible subdiagrams, as noticed by Orlik.

For groups G_{33} and G_{34}, not all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams. In these cases, it seems that a second diagram should be introduced, as suggested by [Hu].

3. Braid groups and diagrams

We set $\mathcal{M} := V - \bigcup_{H \in \mathcal{A}} H$, and we denote by $p\colon \mathcal{M} \to \mathcal{M}/G$ the canonical surjection. Let $x_0 \in \mathcal{M}$. We introduce the following notation for the fundamental groups:

$$P := \Pi_1(\mathcal{M}, x_0) \quad \text{and} \quad B := \Pi_1(\mathcal{M}/G, p(x_0))\,,$$

and we call B and P respectively the *braid group* and the *pure braid group* associated to G.

The projection p induces a short exact sequence between fundamental groups

(br) $$\{1\} \to P \to B \to G \to \{1\}\,.$$

The following statement is conjectured to be true for all complex reflection groups, with the diagrams listed in our tables.

It is well known for Coxeter groups (see for example [BrSa] or [De]). Its first assertion had been noticed by Orlik and Solomon ([OrSo]) for the case of Shephard groups (*i.e.*, groups whose braid diagram – see below – is a Coxeter diagram). It is now proved for all the infinite series but $G(e,e,r)$ for $e \geq 2$, $r > 2$, and for all the exceptional groups but G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}. For the case of non Coxeter-Shephard groups of rank 2, we make use of [Ba].

Theorem–Conjecture.

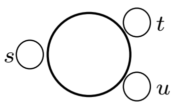
(1) *Let $\mathcal{N}(\mathcal{D})$ be the set of nodes of $\mathcal{D}$, identified with a set of pseudo-reflections in G. For each $s \in \mathcal{N}(\mathcal{D})$, there exists a preimage γ_s of s in B such that the set $\{\gamma_s\}_{s\in\mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}$, is a presentation of B.*

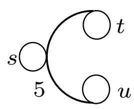
(2) *The center ZB of B is infinite cyclic and it is generated by an element β which belongs to the monoid generated by the $\{\gamma_s\}_{s\in\mathcal{N}(\mathcal{D})}$. The length of β on this set of generators is $\ell(\beta) = (N + N^*)/|ZG|$.*

(3) *The short exact sequence* (br) *induces a short exact sequence between the centers*
$$\{1\} \to ZP \to ZB \to ZG \to \{1\}\,.$$
Hence the cyclic group ZP is generated by an element π which belongs to the monoid generated by the $\{\gamma_s\}_{s\in\mathcal{N}(\mathcal{D})}$. Its length on this set of generators is $\ell(\pi) = N + N^*$.

The element β is given in the last column of our tables. Notice that the knowledge of degrees and codegrees allows then to find the order of ZG, which is not explicitely provided in the tables.

The diagram where the orders of the nodes are "forgotten" and where only the braid relations are kept is called a *braid diagram* for the corresponding group.

Thus the braid diagram 

gives a presentation for the braid groups of both $G(2d,2,2)$ $(d \geq 2)$, G_7, G_{11}, G_{19}, while the diagram 

gives a presentation for the braid groups of both G_{15} and $G(4d,4,2)$.

4. Hecke algebras

Let G be a complex reflection group. We define a set

$$\mathbf{u} = (u_{H,j})_{(H\in\mathcal{A}/G)(0\leq j\leq e_H-1)}$$

of $\sum_{H\in\mathcal{A}/G}(e_H)$ indeterminates. We set

$$\mathbf{u}^{-1} := (u_{H,j}^{-1})_{(H\in\mathcal{A}/G)(0\leq j\leq e_H-1)}\,.$$

Now assume that $\mathcal{D}$ is a diagram for G, and let $s\in\mathcal{N}(\mathcal{D})$ be a node of $\mathcal{D}$. We set $u_{s,j} := u_{H,j}$ for $j=0,1,\ldots,e_H-1$, where H denotes the reflecting hyperplane of s.

Definition. *The Hecke algebra $\mathcal{H}_{\mathbf{u}}$ associated to $\mathcal{D}$ is the algebra over the ring $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ generated by elements $(T_s)_{s\in\mathcal{N}(\mathcal{D})}$ such that*

- *the elements T_s satisfy the braid relations defined by $\mathcal{D}$,*
- *we have $(T_s-u_{s,0})(T_s-u_{s,1})\cdots(T_s-u_{s,e_s-1})=0$.*

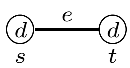
Notice that through the specialization $u_{s,j}\mapsto \det_V(s)^j$ (for $s\in\mathcal{N}(\mathcal{D})$ and $0\leq j\leq e_s-1$), the algebra $\mathcal{H}_{\mathbf{u}}$ becomes the group algebra of G over a suitable cyclotomic extension $\mathbb{Z}[\zeta]$ of $\mathbb{Z}$.

The following statement is well known for Coxeter groups. It is now proved for all infinite series of complex reflection groups, as a consequence of [ArKo], [BrMa], [Ar]. It has been checked for many of the groups of small rank.

Theorem–Conjecture.
$\mathcal{H}_{\mathbf{u}}$ is a free $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$–module of rank $|G|$.

5. Diagrams and tables

Meaning of the diagrams.

- The diagram 

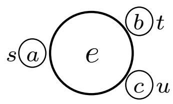
corresponds to the presentation

$$s^d=t^d=1 \text{ and } \underbrace{stst s\cdots}_{e\text{ factors}} = \underbrace{tstst\cdots}_{e\text{ factors}}$$

- The diagram 

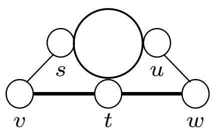
corresponds to the presentation

$$s^5=t^3=1 \text{ and } stst=tsts\,.$$

- The diagram 

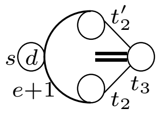
corresponds to the presentation

$$s^a=t^b=u^c=1 \text{ and } \underbrace{stustu\cdots}_{e\text{ factors}} = \underbrace{tustus\cdots}_{e\text{ factors}} = \underbrace{ustust\cdots}_{e\text{ factors}}\,.$$

- The diagram 

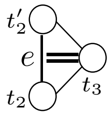
corresponds to the presentation

$$\begin{aligned}
&s^2=t^2=u^2=v^2=w^2=1\,,\\
&uv=vu\,,\ sw=ws\,,\ vw=wv\,,\\
&sut=uts=tsu\,,\\
&svs=vsv\,,\ tvt=vtv\,,\ twt=wtw\,, wuw=uwu\,.
\end{aligned}$$

• The diagram 

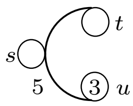
corresponds to the presentation

$$\begin{aligned}
&s^d = {t_2'}^2 = t_2^2 = t_3^2 = 1\,,\ st_3 = t_3 s\,,\\
&st_2't_2 = t_2't_2 s\,,\\
&t_2't_3t_2' = t_3t_2't_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t_2't_2t_3t_2't_2 = t_2't_2t_3t_2't_2t_3\,,\\
&\underbrace{t_2st_2't_2t_2't_2t_2'\cdots}_{e+1 \text{ factors}} = \underbrace{st_2't_2t_2't_2t_2't_2\cdots}_{e+1 \text{ factors}}\,.
\end{aligned}$$

• The diagram 

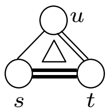
corresponds to the presentation

$$\begin{aligned}
&{t_2'}^2 = t_2^2 = t_3^2 = 1\,,\\
&t_2't_3t_2' = t_3t_2't_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t_2't_2t_3t_2't_2 = t_2't_2t_3t_2't_2t_3\,,\\
&\underbrace{t_2t_2't_2t_2't_2t_2'\cdots}_{e \text{ factors}} = \underbrace{t_2't_2t_2't_2t_2't_2\cdots}_{e \text{ factors}}\,.
\end{aligned}$$

• The diagram 

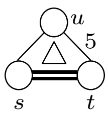
corresponds to the presentation

$$s^2 = t^2 = u^3 = 1\,,\ stu = tus\,,\ ustut = stutu\,.$$

• The diagram 

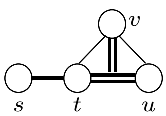
corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutu = utut\,,\ utusut = sutusu\,,\ sus = usu\,.$$

• The diagram 

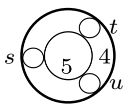
corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutut = ututu\,,\ utusut = sutusu\,,\ sus = usu\,.$$

• The diagram 

corresponds to the presentation

$$\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = 1\,,\ sv = vs\,,\ su = us\,,\\
&sts = tst\,,\ vtv = tvt\,,\ uvu = vuv\,,\ tutu = utut\,,\ vtuvtu = tuvtuv\,.
\end{aligned}$$

• The diagram 

corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ ustus = stust\,,\ tust = ustu\,.$$

Finally, we denote by $a \wedge b$ the greatest common divisor of two integers a and b.

Information provided by the tables: invariants of braid diagrams.

The groups have been ordered by their diagrams, by collecting groups with the same braid diagram. Thus, for example,

- G_{15} has the same braid diagram as the groups $G(4d, 4, 2)$ for all $d \geq 2$,
- G_4, G_8, G_{16}, G_{25}, G_{32} all have the same braid diagrams as groups $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$,
- G_5, G_{10}, G_{18} have the same braid diagram as the groups $G(d, 1, 2)$ for all $d \geq 2$,
- G_7, G_{11}, G_{19} have the same braid diagram as the groups $G(2d, 2, 2)$ for all $d \geq 2$.

Degrees and Codegrees of a braid diagram.

The following property may be noticed on the tables. It generalizes a property already noticed by Orlik and Solomon for the case of Coxeter–Shephard groups (see [OrSo]).

Theorem. *Let $\mathcal{D}$ be a braid diagram of rank r. There exist two families of r integers, $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r)$ and $(\mathbf{d}_1^*, \mathbf{d}_2^*, \ldots, \mathbf{d}_r^*)$, depending only on $\mathcal{D}$, and called respectively the degrees and the codegrees of $\mathcal{D}$, with the following property: whenever G is a complex reflection group with $\mathcal{D}$ as a braid diagram, its degrees and codegrees are given by the formulae*

$$d_j = |Z|\mathbf{d}_j \quad \text{and} \quad d_j^* = |Z|\mathbf{d}_j^* \quad (j = 1, 2, \ldots, r)\,,$$

where $|Z|$ denotes the order of the center of G.

The zeta function of a braid diagram.

In [DeLo], Denef and Loeser compute the *zeta function of local monodromy of the discriminant* of a complex reflection group G, which is the element of $\mathbb{Q}[q]$ defined by the formula

$$Z(q, G) := \prod_i \det(1 - q\mu, H^i(F_0, \mathbb{C}))^{(-1)^{i+1}}\,,$$

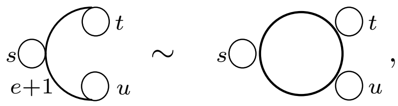
where F_0 denotes the Milnor fiber of the discriminant at 0 and μ denotes the monodromy automorphism (see [DeLo]).

Putting together the tables of [DeLo] and our braid diagrams, one may notice the following fact.

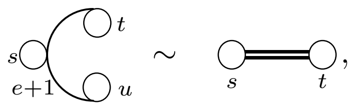
Theorem. *The zeta function of local monodromy of the discriminant of a complex reflection group G depends only on the braid diagram of G.*

Remark. Two different braid diagrams may be associated to isomorphic braid groups. For example, this is the case for the following rank 2 diagrams (where the sign "∼" means that the corresponding groups are isomorphic) :

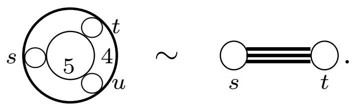
For e even,



for e odd,



and



It should be noticed, however, that the above pairs of diagrams do not have the same degrees and codegrees, nor do they have the same zeta function. Thus, degrees, codegrees and zeta functions are indeed attached to the braid diagrams, not to the braid groups.

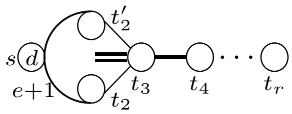
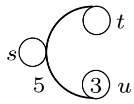
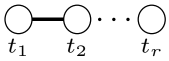
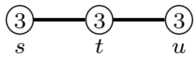
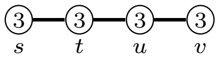
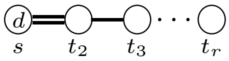
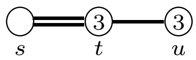
name	diagram	degrees	codegrees	β
$G(de,e,r)$ $e>2, d\geq 2$		$ed, 2ed \ldots$ $(r-1)ed, rd$	$0, ed, \ldots$ $(r-1)ed$	$s^{\frac{r}{e\wedge r}}(t_2' t_2 t_3 \ldots t_r)^{\frac{e(r-1)}{e\wedge r}}$
G_{15}		$12, 24$	$0, 24$	$ustut = s(tu)^2$
$\mathfrak{S}_{r+1}$		$2, 3, \ldots, r+1$	$0, 1, \ldots, r-1$	$(t_1 \ldots t_r)^{r+1}$
G_4		$4, 6$	$0, 2$	$(st)^3$
G_8		$8, 12$	$0, 4$	$(st)^3$
G_{16}		$20, 30$	$0, 10$	$(st)^3$
G_{25}		$6, 9, 12$	$0, 3, 6$	$(stu)^4$
G_{32}		$12, 18, 24, 30$	$0, 6, 12, 18$	$(stuv)^5$
$G(d,1,r)$ $d\geq 2$		$d, 2d, \ldots, rd$	$0, d, \ldots, (r-1)d$	$(st_2 t_3 \ldots t_r)^r$
G_5		$6, 12$	$0, 6$	$(st)^2$
G_{10}		$12, 24$	$0, 12$	$(st)^2$
G_{18}		$30, 60$	$0, 30$	$(st)^2$
G_{26}		$6, 12, 18$	$0, 6, 12$	$(stu)^3$

TABLE 1

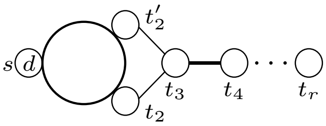
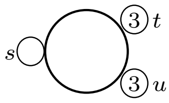
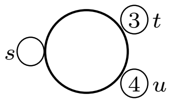
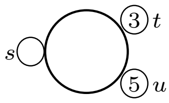
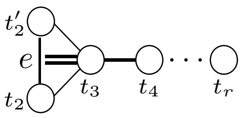
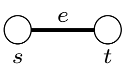
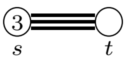
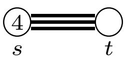
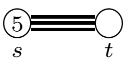
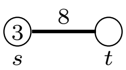
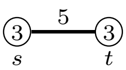
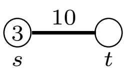
name	diagram	degrees	codegrees	β
$G(2d,2,r)$ $d\geq 2$		$2d, 4d, \dots$ $2(r-1)d, rd$	$0, 2d, \dots$ $2(r-1)d$	$s^{\frac{r}{2\wedge r}}(t'_2t_2t_3\dots t_r)^{\frac{2(r-1)}{2\wedge r}}$
G_7		$12, 12$	$0, 12$	stu
G_{11}		$24, 24$	$0, 24$	stu
G_{19}		$60, 60$	$0, 60$	stu
$G(e,e,r)$ $e\geq 2, r>2$		$e, 2e, \dots$ $(r-1)e, r$	$0, e, \dots, (r-2)e,$ $(r-1)e-r$	$(t'_2t_2t_3\dots t_r)^{\frac{e(r-1)}{e\wedge r}}$
$G(e,e,2)$ $e\geq 3$		$2, e$	$0, e-2$	$(st)^{e/(e\wedge 2)}$
G_6		$4, 12$	$0, 8$	$(st)^3$
G_9		$8, 24$	$0, 16$	$(st)^3$
G_{17}		$20, 60$	$0, 40$	$(st)^3$
G_{14}		$6, 24$	$0, 18$	$(st)^4$
G_{20}		$12, 30$	$0, 18$	$(st)^5$
G_{21}		$12, 60$	$0, 48$	$(st)^5$

TABLE 2

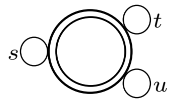
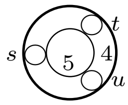
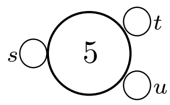
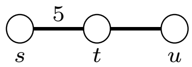
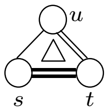
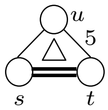
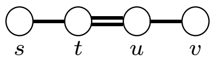
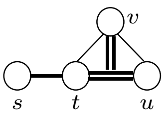
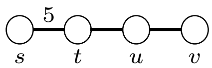
name	diagram	degrees	codegrees	β
G_{12}		$6, 8$	$0, 10$	$(stu)^4$
G_{13}		$8, 12$	$0, 16$	$(stu)^3$
G_{22}		$12, 20$	$0, 28$	$(stu)^5$
G_{23}		$2, 6, 10$	$0, 4, 8$	$(stu)^5$
G_{24}		$4, 6, 14$	$0, 8, 10$	$(stu)^7$
G_{27}		$6, 12, 30$	$0, 18, 24$	$(stu)^5$
G_{28}		$2, 6, 8, 12$	$0, 4, 6, 10$	$(stuv)^6$
G_{29}		$4, 8, 12, 20$	$0, 8, 12, 16$	$(stuv)^5$
G_{30}		$2, 12, 20, 30$	$0, 10, 18, 28$	$(stuv)^{15}$

TABLE 3

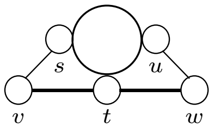
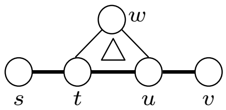
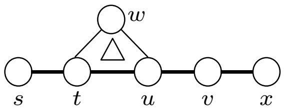
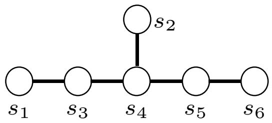
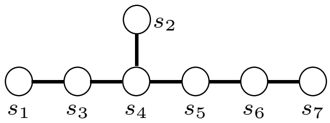
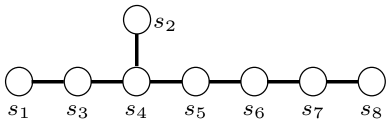
name	diagram	degrees	codegrees	β
G_{31}	 	8,12, 20,24	0,12, 16,28	$(stuwv)^6$
G_{33}	 	4,6,10, 12,18	0,6,8, 12,14	$(ustvw)^9$
G_{34}	 	6,12,18,24, 30,42	0,12,18,24, 30,36	$(stuvwx)^7$
G_{35}	 	2,5,6,8, 9,12	0,3,4,6, 7,10	$(s_1 \dots s_6)^{12}$
G_{36}	 	2,6,8,10,12, 14,18	0,4,6,8,10, 12,16	$(s_1 \dots s_7)^9$
G_{37}	 	2,8,12,14,18,20, 24,30	0,6,10,12,16,18, 22,28	$(s_1 \dots s_8)^{15}$

Table 4

References

[Ar] S. Ariki, *Representation theory of a Hecke algebra of* $G(r,p,n)$, preprint (1993).

[ArKo] S. Ariki and K. Koike, *A Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ *and construction of its irreducible representations*, Advances in Math. **106** (1994), 216–243.

[Ba] E. Bannai, *Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2*, J. Math. Soc. Japan **28** (1976), 447–454.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie, chap. 4, 5 et 6*, Hermann, Paris, 1968.

[BrSa] E. Brieskorn and K. Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. **17** (1972), 245–271.

[BrMa] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189.

[Ch] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math **77** (1955), 778–782.

[Co] A. M. Cohen, *Finite complex reflection groups*, Ann. scient. Éc. Norm. Sup. **9** (1976), 379–436.

[De] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.

[DeLo] J. Denef and F. Loeser, *Regular elements and monodromy of discriminants of finite reflection groups*, Preprint (1994).

[Hu] M. C. Hughes, *Complex reflection groups*, Comm. Algebra **18** (1990), 3999-4029.

[OrSo] P. Orlik and L. Solomon, *Unitary reflection groups and cohomology*, Invent. Math. **59** (1980), 77–94.

[OrTe] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer–Verlag, Berlin–Heidelberg, 1992.

[ShTo] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[Sp] T. A. Springer, *Regular elements of finite reflection groups*, Invent. math. **25** (1974), 159–198.

[St] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Am. Math. Soc. **112** (1964), 392–400.

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