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ALPERIN'S CONJECTURE FOR ALGEBRAIC GROUPS

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Abstract

We prove analogues for reductive algebraic groups of some results for finite groups due to Knörr and Robinson from 'Some remarks on a conjecture of Alperin', *J. London Math. Soc* (2) **39** (1989), 48–60, which play a central rôle in their reformulation of Alperin's conjecture for finite groups.

1. Introduction

Let G be a finite group, p a prime and k an algebraically closed field of characteristic p. By kG we denote the modular group algebra of G. Alperin's conjecture [**1**] asserts that the number of isomorphism classes of simple kG-modules equals the sum of the number of isomorphism classes of projective simple $k[N_G(P)/P]$-modules, where P is a p-subgroup of G and the sum is taken over all p-subgroups P of G up to G-conjugacy. Knörr and Robinson [**3**, Theorem 3.8] reformulated this conjecture in terms of the vanishing of an alternating sum of the number of simple modules for normalizers of p-subgroups. More precisely, they showed that Alperin's conjecture holds for all finite groups if and only if their alternating sum conjecture holds for all finite groups. For finite groups of Lie type, Alperin's original conjecture was first proved by Cabanes [**2**] (see also [**3**, Theorem 5.3; **4**; **6**]).

The aim of this note is to prove analogues for reductive algebraic groups of some results of Knörr and Robinson from [**3**] that are relevant in their reformulation of Alperin's conjecture.

2. Complexes of nilpotent subalgebras of $\mathfrak{g}$

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k. We denote the Lie algebra of G by $\operatorname{Lie} G$ or by $\mathfrak{g}$; likewise for closed subgroups of G. For a closed subgroup H of G, the normalizer of $\operatorname{Lie} H = \mathfrak{h}$ in G is defined by $N_G(\mathfrak{h}) = \{g \in G \mid \operatorname{Ad} g(\mathfrak{h}) \subseteq \mathfrak{h}\}$, where $\operatorname{Ad} g$ denotes the adjoint action of $g \in G$ on $\mathfrak{g}$.

By $R_u(H)$, we denote the unipotent radical of H and frequently write $\operatorname{nil}(\mathfrak{h})$ for the nilradical $\operatorname{Lie}(R_u(H))$ of $\mathfrak{h}$.

We define several simplicial complexes consisting of various chains of nilpotent subalgebras of $\mathfrak{g}$. They are analogues of the subcomplexes of p-subgroups in finite group theory mentioned above. To our knowledge, they have not been studied yet in the context of reductive algebraic groups.

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Let $\mathcal{N}$ denote the simplicial complex associated to the partially ordered set of all chains of nilpotent subalgebras of $\mathfrak{g}$. We define $\mathcal{I}$ to be the subcomplex of $\mathcal{N}$, where for a fixed chain C in $\mathcal{I}$ there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that each member of C is an ideal of $\mathfrak{b}$; equivalently, there exists a Borel subgroup B of G such that each member of C is a B-submodule of nil(Lie B). Moreover, $\mathcal{A}$ is the subcomplex of $\mathcal{I}$, where each member of a given chain C is an Abelian ideal of a Borel subalgebra associated to C. Finally, by $\mathcal{R}$ we denote the subcomplex of $\mathcal{I}$ of chains C, where each member $\mathfrak{n}$ in C satisfies $\mathfrak{n} = \mathrm{nil}(\mathrm{Lie}\, N_G(\mathfrak{n}))$.

The empty chain is considered to be a (-1)-simplex in each case. We will assume that every non-empty chain C in $\mathcal{N}$ considered is of the form $\mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_n$, where $\mathfrak{n}_0 = \{0\}$. The *chain stabilizer* G_C of C in G is defined to be $G_C := \bigcap_{i=0}^{n} N_G(\mathfrak{n}_i)$. We define the *length* of the chain C in $\mathcal{N}$ by $|C| = n$, so that $|C| = \dim C + 1$, where $\dim C$ is the dimension of C as a simplex.

The adjoint representation of G on $\mathfrak{g}$ induces an action of G on each of the simplicial complexes defined; for C as above and $g \in G$ we define $g \cdot C$ to be the chain $\{0\} = (\mathrm{Ad}\, g)\mathfrak{n}_0 \subset (\mathrm{Ad}\, g)\mathfrak{n}_1 \subset \cdots \subset (\mathrm{Ad}\, g)\mathfrak{n}_n$. Let $\mathcal{N}/G$ denote the set of G-conjugacy classes of chains in $\mathcal{N}$; likewise for the other complexes. Since all the chains we consider consist of nilpotent subalgebras of $\mathfrak{g}$, we may assume that, up to G-conjugacy, any given chain lies in the nilradical $\mathrm{nil}(\mathfrak{b})$ of a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Thus, in particular, each of the sets of G-classes $\mathcal{N}/G$, $\mathcal{I}/G$, $\mathcal{R}/G$ and $\mathcal{A}/G$ is finite.

The following is the analogue of [3, Proposition 3.3] in the context of reductive algebraic groups.

PROPOSITION 2.1 *Let A be an Abelian group and let f be a G-equivariant function from the set of subgroups of G to A (that is, f is constant on conjugacy classes of subgroups of G). Then*

$$\sum_{C\in\mathcal{N}/G} (-1)^{|C|} f(G_C) = \sum_{C\in\mathcal{I}/G} (-1)^{|C|} f(G_C) = \sum_{C\in\mathcal{R}/G} (-1)^{|C|} f(G_C) = \sum_{C\in\mathcal{A}/G} (-1)^{|C|} f(G_C).$$

Proof. Observe that by the above remark, each of the sums is finite. We imitate the proof of [3, Propostion 3.3]. The idea is to pair up chains which lie outside $\mathcal{A}$ (respectively, outside $\mathcal{R}$), so that their contributions in the above alternating sums cancel each other out.

First we show that the G-classes of chains in $\mathcal{I} \setminus \mathcal{A}$ do not contribute to the alternating sum $\sum_{C\in\mathcal{I}/G}(-1)^{|C|} f(G_C)$.

Let C be a chain $\mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_n$ in $\mathcal{I} \setminus \mathcal{A}$. Let B be a Borel subalgebra of G so that $\mathfrak{n}_n \subseteq \mathfrak{u} = \mathrm{nil}(\mathfrak{b})$. We pair C with a chain C' in $\mathcal{I} \setminus \mathcal{A}$ as follows. Since $\mathfrak{n}_n$ is not Abelian, the commutator subalgebra $[\mathfrak{n}_n, \mathfrak{n}_n]$ is non-trivial. Let $j > 0$ be minimal so that $[\mathfrak{n}_n, \mathfrak{n}_n] \subseteq \mathfrak{n}_j$. Observe that $\tilde{\mathfrak{n}}_j := \mathfrak{n}_{j-1} + [\mathfrak{n}_n, \mathfrak{n}_n]$ is again a B-submodule of $\mathfrak{u}$. Now, if $\tilde{\mathfrak{n}}_j \neq \mathfrak{n}_j$, then we insert $\tilde{\mathfrak{n}}_j$ (between $\mathfrak{n}_{j-1}$ and $\mathfrak{n}_j$) into C to obtain C', and if $\tilde{\mathfrak{n}}_j = \mathfrak{n}_j$, then we remove $\mathfrak{n}_j$ from C to obtain C'. In any case, C' again belongs to $\mathcal{I} \setminus \mathcal{A}$, since $\mathfrak{n}_n$ still belongs to C'; if $\mathfrak{n}_{n-1} + [\mathfrak{n}_n, \mathfrak{n}_n] = \mathfrak{n}_n$, then we have $\mathfrak{n}_{n-1} = \mathfrak{n}_n$, a contradiction. One readily checks that $(C')' = C$, $\left|C'\right| = |C| \pm 1$, and that $(g \cdot C)' = g \cdot (C')$. It follows that the chain stabilizers G_C and $G_{C'}$ coincide. We may pair the contributions of the G-orbits of C and C' and this shows that

$$\sum_{C\in\mathcal{I}/G} (-1)^{|C|} f(G_C) = \sum_{C\in\mathcal{A}/G} (-1)^{|C|} f(G_C).$$

The very same argument as one above, with C taken from $\mathcal{N}$ instead of $\mathcal{I}$, shows that in fact the G-classes of chains in $\mathcal{N} \setminus \mathcal{A}$ do not contribute to the alternating sum $\sum_{C\in\mathcal{N}/G}(-1)^{|C|} f(G_C)$. Thus we obtain

$$\sum_{C\in\mathcal{N}/G}(-1)^{|C|}f(G_C)=\sum_{C\in\mathcal{A}/G}(-1)^{|C|}f(G_C).$$

Finally, we show that chains in $\mathcal{I}\setminus\mathcal{R}$ do not make a contribution to $\sum_{C\in\mathcal{I}/G}(-1)^{|C|}f(G_C)$. Let C be a chain $\mathfrak{n}_0\subset\mathfrak{n}_1\subset\cdots\subset\mathfrak{n}_n$ in $\mathcal{I}\setminus\mathcal{R}$. We pair C with a chain C' in $\mathcal{I}\setminus\mathcal{R}$ as follows. Let $i>0$ be minimal so that $\tilde{\mathfrak{n}}_i:=\mathrm{nil}(\mathrm{Lie}\,N_G(\mathfrak{n}_i))\neq\mathfrak{n}_i$. Since $\mathfrak{n}_i\subseteq\tilde{\mathfrak{n}}_i$, we have $\tilde{\mathfrak{n}}_i\not\subseteq\mathfrak{n}_i$, by hypothesis on i. Now let $j\geq i$ be maximal so that $\tilde{\mathfrak{n}}_i\not\subseteq\mathfrak{n}_j$. If $j<n$ and $\tilde{\mathfrak{n}}_i+\mathfrak{n}_j=\mathfrak{n}_{j+1}$ we remove $\mathfrak{n}_{j+1}$ from C and if $\tilde{\mathfrak{n}}_i+\mathfrak{n}_j\neq\mathfrak{n}_{j+1}$ or if $j=n$ we insert $\tilde{\mathfrak{n}}_i+\mathfrak{n}_j$ into C to obtain C'. In any event, C' still belongs to $\mathcal{I}\setminus\mathcal{R}$, since $\mathfrak{n}_i$ is still a member of the resulting chain C'. Moreover, one readily checks that $(C')'=C$, $\left|C'\right|=|C|\pm1$, and $(g\cdot C)'=g\cdot(C')$. Since $N_G(\mathfrak{n}_i)=N_G(\tilde{\mathfrak{n}}_i)$, we obtain $G_C=G_{C'}$ in any case, and we may pair the contributions of the G-orbits of C and C'; this shows that

$$\sum_{C\in\mathcal{I}/G}(-1)^{|C|}f(G_C)=\sum_{C\in\mathcal{R}/G}(-1)^{|C|}f(G_C).$$

The result now follows.

Let $\mathfrak{n}$ be a member of a chain C in $\mathcal{R}$. Then as $\mathfrak{n}$ is normalized by a Borel subgroup of G, its normalizer $N_G(\mathfrak{n})$ is therefore a parabolic subgroup of G. Thus by definition, each member of a chain C in $\mathcal{R}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$. Consequently, the chain stabilizer G_C is simply the parabolic subgroup whose nilpotent radical is the largest member in C.

We define another complex of chains, $\mathcal{P}$, consisting of chains of parabolic subgroups of G. If C is a chain in $\mathcal{R}$, then we can associate to it a chain D in $\mathcal{P}$ of the corresponding parabolic subgroups in G, that are the normalizers of the members of C, and conversely for a chain D in $\mathcal{P}$ we can form a chain C in $\mathcal{R}$ by taking the nilradicals of the parabolic subgroups in D. Note that both operations are maps of complexes and both are order-reversing and preserve the lengths of chains. Further, since parabolic subgroups are self-normalizing, the chain stabilizer G_D of a chain D in $\mathcal{P}$ is simply the smallest parabolic subgroup in D. In particular, if C in $\mathcal{R}$ and D in $\mathcal{P}$ correspond in this way, then $G_C=G_D$. This immediately yields our next result.

PROPOSITION 2.2 *Let A be an Abelian group and let f be a G-equivariant function from the set of subgroups of G to A. Then*

$$\sum_{C\in\mathcal{R}/G}(-1)^{|C|}f(G_C)=\sum_{C\in\mathcal{P}/G}(-1)^{|C|}f(G_C).$$

Let $\mathcal{B}$ denote the spherical Tits building of G. We can view $\mathcal{B}$ as the complex consisting of the parabolic subgroups of G with reversed inclusion giving the poset structure [7]. Then, as complexes, $\mathcal{P}$ is simply the barycentric subdivision of $\mathcal{B}$ and thus both $\mathcal{P}$ and $\mathcal{B}$ are homotopy equivalent (cf. [5, equation (1.4)]). Let T be a maximal torus of G and let S be a set of simple roots of G with respect to T. For a subset I of S let P_I be the (standard) parabolic subgroup of G associated with I and note that any parabolic subgroup of G is conjugate to P_I for some $I\subseteq S$.

Our next result follows from Proposition 2.2, the comments in the previous paragraph and the fact that a parabolic subgroup of G is self-normalizing. For a parabolic subgroup P of G conjugate to P_I let $\mathrm{cr}(P)$ denote its (semisimple) *corank* in G, that is, $\mathrm{cr}(P_I)=|S\setminus I|$.

PROPOSITION 2.3 *Let A be an Abelian group and let f be a G-equivariant function from the set of subgroups of G to A. Then*

$$\sum_{C\in\mathcal{P}/G}(-1)^{|C|}f(G_C)=\sum_{P\in\mathcal{B}/G}(-1)^{\mathrm{cr}(P)}f(P)=\sum_{I\subseteq S}(-1)^{|S\setminus I|}f(P_I).$$

REMARK 2.4 Let $p>0$ be the characteristic of k and let $q=p^a$ for some integer a. For a suitable choice of a G-equivariant function f in Proposition 2.3, the resulting sum gives the number of isomorphism classes of projective simple $kG(q)$-modules, where $G(q)$ is the finite group of Lie type associated to G and q [3, Theorem 5.3].

REMARK 2.5 There are analogues of all the results above, using complexes of chains of unipotent subgroups of G (with G acting by conjugation) in place of nilpotent subalgebras of $\mathfrak{g}$. We leave the details to the reader.

REMARK 2.6 All the above results are independent of the characteristic of the underlying field. In particular, they are valid even if the characteristic of k is a bad prime for G, leading to degeneracies in the commutator relations. Amusingly, all the above results are also true in characteristic zero.

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