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Affineness of Deligne–Lusztig varieties for minimal length elements

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Abstract

We prove that the Deligne–Lusztig varieties associated to elements of the Weyl group which are of minimal length in their twisted class are affine. Our proof differs from the proof of He and Orlik–Rapoport and it is inspired by the case of regular elements, which correspond to the varieties involved in Broué's conjectures.

Keywords: Deligne–Lusztig varieties; Finite groups of Lie type; Weyl groups; Conjugacy classes

1. Introduction

Let p be a prime number, let $\mathbb{F}$ denote an algebraic closure of the finite field with p elements and let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F}$. We assume that $\mathbf{G}$ is endowed with an isogeny $F : \mathbf{G} \to \mathbf{G}$ such that F^δ is a Frobenius endomorphism with respect to some $\mathbb{F}_q$-structure on $\mathbf{G}$ (here, δ is a non-zero natural number, q is a power of p and $\mathbb{F}_q$ denotes the subfield of $\mathbb{F}$ with q elements).

We denote by $\mathcal{B}$ the variety of Borel subgroups of $\mathbf{G}$ and by $\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} \mathcal{O}(w)$ the decomposition into orbits for the diagonal action of $\mathbf{G}$. Here, W is the Weyl group of $\mathbf{G}$, with set of simple reflections S corresponding to the orbits of dimension $1 + \dim \mathcal{B}$, and the first and last

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projections define an isomorphism $\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\sim} \mathcal{O}(ww')$ when $\ell(ww') = \ell(w) + \ell(w')$, where $\ell : W \to \mathbb{Z}_{\geqslant 0}$ is the length function on W associated to S.

Given $w \in W$, we define the *Deligne–Lusztig variety* [4, Definition 1.4] associated to w by

$$\mathbf{X}(w) = \mathbf{X}_{\mathbf{G}}(w) = \{\mathbf{B} \in \mathcal{B} \mid (\mathbf{B}, F(\mathbf{B})) \in \mathcal{O}(w)\}.$$

By studying a class of ample sheaves on $\mathbf{X}(w)$, Deligne and Lusztig proved that these varieties are affine when $q^{1/\delta}$ is larger than the Coxeter number of $\mathbf{G}$ [4, Theorem 9.7].

They proved more generally that the existence of coweights satisfying certain inequalities ensures that $\mathbf{X}(w)$ is affine. Recently, Orlik–Rapoport and He studied this question. Recall that $x, y \in W$ are *F-conjugate* if there exists $a \in W$ such that $y = a^{-1}xF(a)$. By a case-by-case analysis based on Deligne–Lusztig's criterion, they obtained the following result ([13, §5], [10, Theorem 1.3]):

Theorem A *(Orlik–Rapoport, He). If $w \in W$ is an element of minimal length in its F-conjugacy class then $\mathbf{X}(w)$ is affine.*

When w is a Coxeter element, the result is due to Lusztig [12, Corollary 2.8]. In this note we prove a more general affineness result and we deduce Theorem A by applying a combinatorial result on elements of minimal length in their F-conjugacy class.

Before stating our results, we need some further notation. We denote by B^+ the *braid monoid* associated to (W, S). It is the monoid with presentation

$$B^+ = \langle (\underline{x})_{x \in W} \mid \forall x, x' \in W,\ \ell(xx') = \ell(x) + \ell(x') \Rightarrow \underline{xx'} = \underline{x}\,\underline{x'} \rangle.$$

The automorphism F of W extends to an automorphism of B^+ still denoted by F.

Given $I \subset S$, let W_I denote the subgroup of W generated by I and let w_I be the longest element of W_I (the element w_S will be denoted by w_0). The main result of this note is the following:

Theorem B. *Let I be an F-stable subset of S and let $w \in W_I$ be such that there exists a positive integer d and $a \in B^+$ with*

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_I a.$$

Then $\mathbf{X}(w)$ is affine.

The proof of Theorem B is by a general argument while our deduction of Theorem A relies on combinatorial results on finite Coxeter groups (see [7, Theorem 1.1], [6, §6] and [9, Theorem 7.5]) which are proved by a case-by-case analysis.

There is a case where our criterion can be applied easily. Indeed, if d is a regular number for (W, F) (in the sense of Springer) then by [2, Proposition 6.5], there exists a regular element $w \in W$ such that

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_0\underline{w}_0.$$

Therefore, by Theorem B, the variety $\mathbf{X}(w)$ is affine: this variety is of fundamental interest for the geometric version of Broué's abelian defect group conjecture for finite reductive groups

[2, §5.B]. In particular, if $i \neq j$, this conjecture predicts that, as $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules, the cohomology groups $H_c^i(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ and $H_c^j(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ have no common irreducible constituents.

Finally, note that there exist elements satisfying the criterion of Theorem B but which do not satisfy Deligne–Lusztig's criterion. For instance, if W is of type B_5 (and F acts trivially on W), the element $w = s_1 t s_3 s_2 s_1 t s_1 s_4 s_3 s_2 s_1 t s_1 s_2 s_3$ does not satisfy Deligne–Lusztig's criterion (for $q = 2$) but satisfies $(\underline{w})^5 = (\underline{w_0})^3$ (here, $S = \{t, s_1, s_2, s_3, s_4\}$, ts_1 has order 4 and $s_i s_{i+1}$ has order 3 for $i = 1, 2, 3$). However, this element w is F-conjugate by cyclic shift (see Section 2 for the definition) to $s_4 w s_4 = s_1 t s_3 s_2 s_1 t s_1 s_2 s_3 s_4 s_3 s_2 s_1 t s_1$ which satisfies Deligne–Lusztig's criterion, so the affineness of the variety $\mathbf{X}(w)$ can also be obtained from Deligne–Lusztig's criterion (see Proposition 2). These computations have been checked using `GAP3/CHEVIE` programs written by Jean Michel.

2. Preliminaries

Levi subgroup

Let us fix an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ and an F-stable maximal torus $\mathbf{T}$ of $\mathbf{B}_0$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}_0$. We identify $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ with W by requiring that $(\mathbf{B}_0, w\mathbf{B}_0 w^{-1}) \in \mathcal{O}(w)$.

Let I be an F-stable subset of S, let $\mathbf{P}_I = \mathbf{B} W_I \mathbf{P}$, let $\mathbf{V}_I$ denote the unipotent radical of $\mathbf{P}_I$ and let $\mathbf{L}_I$ denote the unique Levi subgroup of $\mathbf{P}_I$ containing $\mathbf{T}$. Given $w \in W_I$, there is an isomorphism [11, Lemma 3]

$$\mathbf{X}_{\mathbf{G}}(w) \xrightarrow{\sim} \mathbf{G}^F / \mathbf{V}_I^F \times_{\mathbf{L}_I^F} \mathbf{X}_{\mathbf{L}_I}(w).$$

In particular,

$$\mathbf{X}_{\mathbf{G}}(w) \text{ is affine if and only if } \mathbf{X}_{\mathbf{L}_I}(w) \text{ is affine.} \tag{1}$$

Cyclic shift

If $w, w' \in W$, we say that w and w' are *F-conjugate by cyclic shift* (and we write $w \overset{F}{\longleftrightarrow} w'$) if there exists three sequences $(x_i)_{1 \leqslant i \leqslant n}$, $(y_i)_{1 \leqslant i \leqslant n}$ and $(w_i)_{1 \leqslant i \leqslant n+1}$ of elements of W such that

(1) $w_1 = w$ and $w_{n+1} = w'$;
(2) for all $i \in \{1, 2, \dots, n\}$, $w_i = x_i y_i$, $w_{i+1} = y_i F(x_i)$ and $\ell(w_i) = \ell(w_{i+1}) = \ell(x_i) + \ell(y_i)$.

The relation $\overset{F}{\longleftrightarrow}$ is an equivalence relation. Two elements which are F-conjugate by cyclic shift have the same length.

Proposition 2. *If $w \overset{F}{\longleftrightarrow} w'$, then $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine.*

Proof. By induction, we may assume that there exists x and y in W such that $w = xy$, $w' = yF(x)$ and $\ell(w) = \ell(w') = \ell(x) + \ell(y)$. The result follows from the existence of a purely inseparable morphism $\mathbf{X}(w) \to \mathbf{X}(w')$ [4, page 108]. □

3. Proof of Theorem B

Let I be an F-stable subset of S, let $w \in W_I$ and assume that there exists $a \in B^+$ and a positive integer d such that

$$\underline{w} F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_I a.$$

The aim of this section is to prove that $\mathbf{X}(w)$ is affine (Theorem B). By (1), we may (and we will) assume that $I = S$.

Sequences of elements of $\mathbf{W}$

Given $(x_1, \dots, x_r)$ a sequence of elements of W, we set

$$\mathcal{O}(x_1, \dots, x_r) = \mathcal{O}(x_1) \times_{\boldsymbol{\mathcal{B}}} \cdots \times_{\boldsymbol{\mathcal{B}}} \mathcal{O}(x_r).$$

If $(y_1, \dots, y_s)$ is a sequence of elements of W such that $\underline{x}_1 \cdots \underline{x}_r = \underline{y}_1 \cdots \underline{y}_s$ in B^+, then $\mathcal{O}(x_1, \dots, x_r) \simeq \mathcal{O}(y_1, \dots, y_s)$ (the varieties are actually canonically isomorphic [3, Application 2]). For a general treatment of these varieties $\mathcal{O}(x_1, \dots, x_r)$ (and the corresponding Deligne–Lusztig varieties), the reader may refer to [5].

Proposition 3. *If there exists $v \in B^+$ such that $\underline{x}_1 \cdots \underline{x}_r = \underline{w}_0 v$, then the variety $\mathcal{O}(x_1, \dots, x_r)$ is affine.*

Proof. Let $v_1, \dots, v_n \in W$ be such that $\underline{v}_1 \cdots \underline{v}_n = v$. We have $\mathcal{O}(x_1, \dots, x_r) \simeq \mathcal{O}(w_0, v_1, \dots, v_n)$, so it remains to prove that $\mathcal{O}(w_0, v_1, \dots, v_n)$ is affine.

For each $x \in W$, we fix a representative $\dot{x}$ of x in $N_{\mathbf{G}}(\mathbf{T})$. We set

$$\tilde{\mathcal{O}}(x_1, \dots, x_r) = \bigl\{(g_0\mathbf{U}, g_1\mathbf{U}, \dots, g_r\mathbf{U}) \in (\mathbf{G}/\mathbf{U})^{r+1} \bigm| \forall 1 \leqslant i \leqslant r,\ g_{i-1}^{-1} g_i \in \mathbf{U}\dot{x}_i\mathbf{U}\bigr\}.$$

The group $\mathbf{T}$ acts on the right on $\tilde{\mathcal{O}}(x_1, \dots, x_r)$ as follows:

$$(g_0\mathbf{U}, g_1\mathbf{U}, \dots, g_r\mathbf{U}) * t = \bigl(g_0 t\mathbf{U}, g_1^{x_1^{-1}} t\mathbf{U}, \dots, g_r^{x_r^{-1} \cdots x_1^{-1}} t\mathbf{U}\bigr).$$

The canonical map

$$\begin{array}{ccc} \tilde{\mathcal{O}}(x_1, \dots, x_r) & \longrightarrow & \mathcal{O}(x_1, \dots, x_r), \\ (g_0\mathbf{U}, g_1\mathbf{U}, \dots, g_r\mathbf{U}) & \longmapsto & \bigl(g_0\mathbf{B}_0 g_0^{-1}, g_1\mathbf{B}_0 g_1^{-1}, \dots, g_r\mathbf{B}_0 g_r^{-1}\bigr) \end{array}$$

identifies $\mathcal{O}(x_1, \dots, x_r)$ with the quotient of $\tilde{\mathcal{O}}(x_1, \dots, x_r)$ by $\mathbf{T}$: indeed, both varieties are smooth (hence normal), the above map is smooth (hence separable) and it is easily checked that its fibres are precisely the $\mathbf{T}$-orbits. Since $\mathbf{T}$ acts freely on $\tilde{\mathcal{O}}(x_1, \dots, x_r)$, and since the quotient of an affine variety by a free action of a torus is affine, [1, Corollary 8.21], the result will follow if we are able to prove that $\tilde{\mathcal{O}}(w_0, v_1, \dots, v_n)$ is affine. Therefore, it is sufficient to show that the map

$$\begin{aligned}\varphi : \mathbf{G} \times \prod_{i=1}^{n}(\mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^-) &\longrightarrow \tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n),\\ (g; h_1, \ldots, h_n) &\longmapsto (g\mathbf{U}, g\dot{w}_0\mathbf{U}, g\dot{w}_0h_1\mathbf{U}, g\dot{w}_0h_1h_2\mathbf{U}, \ldots, g\dot{w}_0h_1\cdots h_n\mathbf{U})\end{aligned}$$

is an isomorphism of varieties (here, $\mathbf{U}^- = {}^{w_0}\mathbf{U}$).

In order to prove that φ is an isomorphism, we shall construct its inverse. For this, we shall need some notation. First, the map $\mathbf{U} \times \mathbf{U} \to \mathbf{U}\dot{w}_0\mathbf{U}$, $(x, y) \mapsto x\dot{w}_0y$ is an isomorphism of varieties: we shall denote by $\mathbf{U}\dot{w}_0\mathbf{U} \to \mathbf{U} \times \mathbf{U}$, $g \mapsto (\eta(g), \eta'(g))$ its inverse. Also, the map $\mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^- \times \mathbf{U} \to \mathbf{U}\dot{v}_i\mathbf{U}$, $(x, y) \mapsto xy$ is an isomorphism of varieties ($i = 1, 2, \ldots, n$), and we shall denote by $\eta_i : \mathbf{U}\dot{v}_i\mathbf{U} \to \mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^-$ the composition of its inverse with the first projection. Note that, if $g \in \mathbf{U}\dot{w}_0\mathbf{U}$, $h \in \mathbf{U}\dot{v}_i\mathbf{U}$ and $u, v \in \mathbf{U}$, then

$$\eta(ugv) = u\eta(g), \quad \eta(g)\dot{w}_0\mathbf{U} = g\mathbf{U}, \quad \eta_i(hv) = \eta_i(h) \quad \text{and} \quad \eta_i(h)\mathbf{U} = h\mathbf{U}. \qquad (*)$$

Now, if $x = (g\mathbf{U}, g_0\mathbf{U}, g_1\mathbf{U}, \ldots, g_n\mathbf{U}) \in \tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n)$, we set

$$\begin{aligned}\psi(x) &= g\eta\big(g^{-1}g_0\big),\\ \psi_0(x) &= \psi(x)\dot{w}_0,\\ \psi_i(x) &= \eta_i\big(\big(\psi_0(x)\psi_1(x)\cdots\psi_{i-1}(x)\big)^{-1}g_i\big),\end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. By $(*)$, the maps ψ and ψ_j are well-defined morphisms of varieties and it is easily checked that the morphism of varieties

$$\begin{aligned}\tilde{\mathcal{O}}(w_0, v_1, \ldots, v_n) &\longrightarrow \mathbf{G} \times \prod_{i=1}^{n}(\mathbf{U}\dot{v}_i \cap \dot{v}_i\mathbf{U}^-),\\ x &\longmapsto \big(\psi(x); \psi_1(x), \ldots, \psi_n(x)\big)\end{aligned}$$

is well defined and is an inverse of φ. □

Introducing Frobenius

The morphism

$$\mathbf{X}(w) \to \mathcal{B}^d, \quad \mathbf{B} \mapsto \big(\mathbf{B}, F(\mathbf{B}), \ldots, F^{d-1}(\mathbf{B})\big)$$

indentifies $\mathbf{X}(w)$ with the closed subvariety $\Delta_d \cap \mathcal{O}(w, F(w), \ldots, F^{d-1}(w))$, where $\Delta_d = \{(\mathbf{B}, F(\mathbf{B}), \ldots, F^{d-1}(\mathbf{B})) \mid \mathbf{B} \in \mathcal{B}\}$ is a closed subvariety of $\mathcal{B}^d$. By Proposition 3, the variety $\mathcal{O}(w, F(w), \ldots, F^{d-1}(w))$ is affine, hence $\mathbf{X}(w)$ is affine as well. The proof of Theorem B is complete.

4. Proof of Theorem A

Let C be an F-conjugacy class in W and $C_{\min}$ its subset of elements of minimal length. Let d be the smallest positive integer k such that $wF(w)\ldots F^{k-1}(w) = 1$ and F^k acts as the identity on W for $w \in C_{\min}$. Following [7, Theorem 1.1] (in the split case) and [6, Definition 5.3] (in the general case), we say that an element $w \in C_{\min}$ is *good* if there exists a sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r$ of subsets of S such that

$$\underline{w}F(\underline{w})\dots F^{d-1}(\underline{w}) = \underline{w}_{I_1}^2\underline{w}_{I_2}^2\cdots\underline{w}_{I_r}^2 \tag{$*$}$$

in B^+.

Proposition 4. *If w is a good element of $C_{\min}$, then $\mathbf{X}(w)$ is affine.*

Proof. By Theorem B, it remains to show that the subset I_1 of the identity $(*)$ is F-stable. Let I be the set of simple reflections occurring in a reduced expression of w (note that I does not depend on the choice of the reduced expression [8, Corollary 1.2.3]). Then the set of $s \in S$ such that $\underline{s}$ occurs in a reduced expression of $\underline{w}F(\underline{w})\dots F^{d-1}(\underline{w})$ is equal to $I \cup F(I) \cup \cdots \cup F^{d-1}(I)$ (by looking at the left-hand side of $(*)$) and is also equal to I_1 (by looking at the right-hand side). Since F^d acts as the identity on W, we get that I_1 is F-stable. □

Let us now come back to the proof of Theorem A. Let $w \in C_{\min}$. Let I be the minimal F-stable subset of S such that $w \in W_I$. By (1), we may assume that $I = S$. Now, if $w' \in C_{\min}$, then $w \xleftrightarrow{F} w'$ (see [8, Theorem 3.2.7] for the split case, [6, §6] for twisted exceptional groups and [9, Theorem 7.5] for twisted classical groups), so $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine by Proposition 2. Therefore, the result follows from Proposition 4 and the next theorem:

Theorem 6 *(Geck–Michel, Geck–Kim–Pfeiffer, He). There exists a good element in $C_{\min}$.*

Proof. By standard arguments (see [6, §5.5]), we may assume that W is irreducible. If F acts trivially on W, the Theorem is [7, Theorem 1.1]. If F does not act trivially and W is not of type A, this is [6, §5.5]. When W is of type A and F acts non-trivially on W, this follows from [9, Corollary 7.25]. □

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