KHOVANOV-ROZANSKY HOMOLOGY AND 2-BRAID GROUPS

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1. Introduction

Khovanov [Kh] has given a construction of the Khovanov-Rozansky link invariants (categorifying the HOMFLYPT invariant) using Hochschild cohomology of 2-braid groups. We give a direct proof that his construction does give link invariants. We show more generally that, for any finite Coxeter group, his construction provides a Markov "2-trace", and we actually show that the invariant takes value in suitable derived categories. This makes more precise a result of Trafim Lasy who has shown that, after taking the class in K_0, this provides a Markov trace [La1, La2]. It coincides with Gomi's trace [Go] for Weyl groups (Webster and Williamson [WeWi]) as well as for dihedral groups [La1].

In the first section, we recall the construction of 2-braid groups [Rou1], based on complexes of Soergel bimodules. The second section is devoted to Markov traces, and a category-valued version, 2-Markov traces. We provide a construction using Hochschild cohomology. The third section is devoted to the proof of the Markov property for Hochschild cohomology.

2. Notations

Let k be a commutative ring. We write $\otimes$ for $\otimes_k$. Let A be a k-algebra. We denote by A^{opp} the opposite algebra to A and we put $A^{\mathrm{en}} = A \otimes A^{\mathrm{opp}}$.

We denote by A-Mod the category of A-modules, by A-mod the category of finitely generated A-modules, by A-Proj the category of projective A-modules and by A-proj the category of finitely generated projective A-modules. Assume A is graded. We denote by A-modgr (resp. A-projgr) the category of finitely generated (resp. and projective) graded A-modules.

Given M a graded k-module and $n \in \mathbf{Z}$, we denote by $M\langle n\rangle$ the graded k-module given by $M\langle n\rangle_i = M_{n+i}$.

Given $\mathcal{A}$ an additive category, we denote by $\mathrm{Comp}(\mathcal{A})$ (resp. $\mathrm{Ho}(\mathcal{A})$) the category (resp. the homotopy category) of complexes of objects of $\mathcal{A}$. If $\mathcal{A}$ is an abelian category, we denote by $D(\mathcal{A})$ its derived category.

Given $\mathcal{C}$ a category, we denote by $\{1\}$ the self equivalence of $\mathcal{C}^{\mathbf{Z}}$ given by $(M\{1\})_i = M_{i+1}$.

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Let $\mathcal{T}$ be a triangulated category equipped with an automorphism $M \mapsto M\langle 1\rangle$. We denote by q the automorphism of $K_0(\mathcal{T})$ given by $[M] \mapsto [M\langle 1\rangle]$. This endows $K_0(\mathcal{T})$ with a structure of $\mathbf{Z}[q,q^{-1}]$-module.

3. 2-braid groups

3.1. **Braid groups.** Let (W,S) be a finite Coxeter group. Let V be the geometric representation of W over $k=\mathbf{C}$: it comes with a basis $\{e_s\}_{s\in S}$. Given $s\in S$, we denote by α_s the linear form on V such that $s(x)-x=\alpha_s(x)e_s$ for all $x\in V$. The set $\{\alpha_s\}_{s\in S}$ is a basis of V^*. Let $P=P_S=P_{(W,S)}=k[V]$ (we will denote by X_S or $X_{(W,S)}$ a given object constructed from (W,S)).

The braid group $B_S=B_{(W,S)}$ associated to (W,S) is the group generated by $\{\sigma_s\}_{s\in S}$ with relations

$$\underbrace{\sigma_s\sigma_t\sigma_s\cdots}_{m_{st}\text{ terms}}=\underbrace{\sigma_t\sigma_s\sigma_t\cdots}_{m_{st}\text{ terms}}$$

for any $s,t\in S$ such that the order m_{st} of st is finite.

We denote by $l: B_S\to\mathbf{Z}$ the length function. It is the morphism of groups defined by $l(\sigma_s)=1$ for $s\in S$.

3.2. **Lift.** Let us recall, following [Rou1], how to lift in a non-trivial way the action of W on the derived category $D(P)$ to an action of B_S on the homotopy category $\mathrm{Ho}(P)$.

Let $s\in S$. We put

$$\theta_s=P\otimes_{P^s}P \text{ and } F_s=0\to\theta_s\langle 1\rangle\xrightarrow{m}P\langle 1\rangle\to 0.$$

The latter is a complex of graded P^{en}-modules, where $P\langle 1\rangle$ is in cohomological degree 1 and m denotes the multiplication map. We put

$$F_s^{-1}=0\to P\langle -1\rangle\xrightarrow{a\mapsto a\alpha_s\otimes 1+a\otimes\alpha_s}\theta_s\to 0.$$

This is a complex of graded P^{en}-modules, where $P\langle -1\rangle$ is in cohomological degree -1.

Let us recall a result of [Rou1, §9]. Given $i_1,\ldots,i_r,\ j_1,\ldots,j_{r'}\in S$ and $\delta_1,\ldots,\delta_r$, $\varepsilon_1,\ldots,\varepsilon_{r'}\in\{\pm 1\}$ such that $\sigma_{i_1}^{\delta_1}\cdots\sigma_{i_r}^{\delta_r}=\sigma_{j_1}^{\varepsilon_1}\cdots\sigma_{j_{r'}}^{\varepsilon_{r'}}$, there is a canonical isomorphism in $\mathrm{Ho}(P^{\mathrm{en}}\text{-modgr})$

$$F_{i_1}^{\delta_1}\otimes_{P_n}\cdots\otimes_{P_n}F_{i_r}^{\delta_r}\xrightarrow{\sim}F_{j_1}^{\varepsilon_1}\otimes_{P_n}\cdots\otimes_{P_n}F_{j_{r'}}^{\varepsilon_{r'}}$$

and these isomorphisms form a transitive system of isomorphisms.

Given $b\in B_S$, we put

$$F_b=\lim_{\substack{i_1,\ldots,i_r\\ \varepsilon_1,\ldots,\varepsilon_r\\ b=\sigma_{i_1}^{\varepsilon_1}\cdots\sigma_{i_r}^{\varepsilon_r}}}F_{i_1}^{\varepsilon_1}\otimes_P\cdots\otimes_P F_{i_r}^{\varepsilon_r}\in\mathrm{Ho}(P^{\mathrm{en}}\text{-modgr}).$$

The 2-braid group $\mathcal{B}_{(W,S)}$ is the full monoidal subcategory of $\mathrm{Ho}(P^{\mathrm{en}}\text{-modgr})$ with objects the F_b's, with $b\in B_S$.

3.3. **Parabolic subgroups.** Let $I \subset S$ and let W_I be the subgroup of W generated by I. Let $V_I = \bigoplus_{s\in I} ke_s$ and $P_I = k[V_I]$. We have $V = V_I \oplus V^I$, hence $P = P_I \otimes k[V^I]$. We deduce also that $V^* = (V^I)^\perp \oplus (V_I)^\perp$, hence the composition of canonical maps $(V^I)^\perp \hookrightarrow V^* \twoheadrightarrow (V_I)^*$ is an isomorphism. We identify $(V^I)^\perp = \bigoplus_{s\in I} k\alpha_s$ and $(V_I)^*$ via this isomorphism.

The compositions of canonical maps $\bigcap_{s\notin I} \ker \alpha_s \to V \to V/V^I$ and $V_I \to V \to V/V^I$ are isomorphisms: this provides an isomorphism $V_I \xrightarrow{\sim} \bigcap_{s\notin I} \ker \alpha_s$. We denote by $\rho_I : P \twoheadrightarrow P_I$ the morphism given by the composition $V_I \xrightarrow{\sim} \bigcap_{s\notin I} \ker \alpha_s \hookrightarrow V$.

We have a functor $\gamma_I : P_I^{\mathrm{en}}\text{-Mod} \to P^{\mathrm{en}}\text{-Mod}$ sending M to $k[V^I] \otimes M$, where $k[V^I]$ is the regular $k[V^I]^{\mathrm{en}}$-module and P^{en} is decomposed as $P^{\mathrm{en}} = k[V^I]^{\mathrm{en}} \otimes P_I^{\mathrm{en}}$. We obtain a fully faithful monoidal functor

$$\mathcal{B}_{W_I} \to \mathcal{B}_W,\ F \mapsto \gamma_I(F) = k[V^I] \otimes F.$$

4. Hochschild cohomology and traces

4.1. Markov traces and 2-traces.

4.1.1. *Markov traces.* Let $\mathcal{C}ox$ be the poset of finite Coxeter groups, viewed as a category. The objects are Coxeter groups (W, S) and $\mathrm{Hom}((W, S), (W', S'))$ is the set of injective maps $f : S \to S'$ such that $m_{f(s),f(t)} = m_{st}$ for all $s, t \in S$. Given $s \in S$, we denote by $i_s : (W_{S\setminus s}, S \setminus s) \to (W, S)$ the inclusion.

Let $\mathcal{F}$ be a full subposet of $\mathcal{C}ox$ closed below.

Let $\mathcal{H}_{(W,S)} = \mathbf{Z}[q^{\pm1}]B_{(W,S)}/((T_s - 1)(T_s + q))_{s\in S}$ be the Hecke algebra of (W, S).

Definition 4.1. *Let R be a $\mathbf{Z}[t_-, t_+, q^{\pm1}]$-module. A Markov trace on $\mathcal{F}$ is the data of a family of $\mathbf{Z}[q^{\pm1}]$-linear maps $\tau_{(W,S)} : \mathcal{H}_{(W,S)} \to R$ for $(W, S) \in \mathcal{F}$ such that*

- $\tau_S(hh') = \tau_S(h'h)$ *for* $h, h' \in \mathcal{H}_S$
- $\tau_S(hT_s^{\pm1}) = t_\pm \tau_{S\setminus s}(h)$ *for all* $s \in S$ *and* $h \in \mathcal{H}_{S\setminus s}$.

Markov traces, with a possibly more general definition, have been studied by Jones and Ocneanu in type A [Jo], Geck-Lambropoulou in type B [GeLa], Geck in type D [Ge], and Kihara in type $I_2(n)$ [Ki]. Gomi has provided a general construction of Markov traces for Weyl groups, using Lusztig's Fourier transform [Go]. More recently, Lasy has studied Markov traces in relation with Gomi's definition and Soergel bimodules [La1].

4.1.2. *Markov 2-traces.*

Definition 4.2. *Let $\mathcal{C} : \mathcal{F} \to \mathcal{C}at$ be a functor.*

A Markov 2-trace on $\mathcal{F}$ (relative to $\mathcal{C}$) is the data of functors $M_{(W,S)} : \mathcal{B}_{(W,S)} \to \mathcal{C}_{(W,S)}$ such that the following holds

- $M_S(?_1 \cdot ?_2) \simeq M_S(?_2 \cdot ?_1)$ *as functors* $\mathcal{B}_S \times \mathcal{B}_S \to \mathcal{C}_S$
- $M_S(\gamma_{S\setminus s}(?) \cdot F_s^{\pm1}) \simeq T_{S,s,\pm}\mathcal{C}(i_s)M_{S\setminus s}(?)$ *as functors* $\mathcal{B}_{S\setminus s} \to \mathcal{C}_S$*, for some endofunctors* $T_{S,s,\pm}$ *of* $\mathcal{C}_S$*, for all* $s \in S$.

One can ask in addition that the functors $T_{S,s,\pm}$ are invertible. On the other hand, one can get a more general definition by dropping the functoriality of $\mathcal{C}$ and by requiring the existence of functors $D_{S,s,\pm} : \mathcal{C}_{S\setminus s} \to \mathcal{C}_S$ such that $M_S(\gamma_{S\setminus s}(?) \cdot F_s^{\pm 1}) \simeq D_{S,s,\pm} M_{S\setminus s}(?)$.

Remark 4.3. Let $\bar{\mathcal{C}} = \operatorname{colim} \mathcal{C}$ and assume there are endofunctors $T_\pm$ of $\bar{\mathcal{C}}$ which restrict to $T_{S,s,\pm}$ for any S and $s \in S$. Replacing $\mathcal{C}_{(W,S)}$ by $\bar{\mathcal{C}}$, one can construct from a Markov 2-trace another one taking value in the constant category $\bar{\mathcal{C}}$, and with fixed endofunctors $T_\pm$.

The first "trace" condition, once formulated in the appropriate homotopical setting, leads to a universal solution ("abelianization") that can be described explicitly [BeNa]. It would be interesting to find a suitable formulation of the second condition in this setting leading to a universal solution. It would also be interesting to study functoriality with respect to cobordism in type A, along the lines of [KhTh] and [ElKr].

4.1.3. *From Markov 2-traces to Markov traces.* Let $\mathcal{S}oe$ be the category of Soergel bimodules: this is the full subcategory of P^{en}-modgr whose objects are direct summands of direct sums of objects of the form $\theta_{s_1} \cdots \theta_{s_n}\langle r\rangle$, for some $s_1, \ldots, s_n \in S$ and $r \in \mathbf{Z}$.

There is a $\mathbf{Z}[q^{\pm 1}]$-algebra morphism $\mathcal{H} \to K_0(\mathcal{S}oe)$ given by $T_s \mapsto [F_s]$, and that morphism is actually an isomorphism (cf [Soe, Theorem 1] and [Li, Théorème 2.4]).

We consider now a Markov 2-trace in the following setting. Assume the functor $\mathcal{C}$ takes values in graded triangulated categories and $M_{(W,S)}$ is the restriction of a graded triangulated functor $M_{(W,S)} : \mathrm{Ho}^b(\mathcal{S}oe_{(W,S)}) \to \mathcal{C}_{(W,S)}$. In particular, it induces a $\mathbf{Z}[q^{\pm 1}]$-linear map $\mathcal{H}_{(W,S)} \to K_0(\mathcal{C}_{(W,S)})$. Let $R = \operatorname{colim}_{(W,S)\in\mathcal{F}} K_0(\mathcal{C}_{(W,S)})$, a $\mathbf{Z}[q^{\pm 1}]$-module. Assume there are commuting endomorphisms $t_\pm$ of R compatible with the action of $[T_{S,s,\pm}]$ on $K_0(\mathcal{C}_{(W,S)})$, for all $(W,S) \in \mathcal{F}$ and $s \in S$, via the canonical maps $\iota_{(W,S)} : K_0(\mathcal{C}_{(W,S)}) \to R$.

Define $\tau_{(W,S)} : B_{(W,S)} \to R$ by $\tau(b) = \iota_S([M_S(F_b)])$. We have the following immediate proposition.

Proposition 4.4. *The maps $\tau_{(W,S)}$ come uniquely from $\mathbf{Z}[q^{\pm 1}]$-linear maps $\mathcal{H}_{(W,S)} \to R$. They define a Markov trace on $\mathcal{F}$.*

4.2. **Hochschild homology.**

4.2.1. *Main Theorem.* We put $\mathrm{HH}_i = \mathrm{HH}_i^{(W,S)} = \mathrm{Tor}_i^{P^{\mathrm{en}}}(P, -) : P^{\mathrm{en}}\text{-modgr} \to P\text{-modgr}$. This gives rise to functors $\mathrm{HH}_i : \mathrm{Ho}^b(P^{\mathrm{en}}\text{-modgr}) \to \mathrm{Ho}^b(P\text{-modgr})$ and to a functor $\mathrm{HH}_* : \mathrm{Ho}^b(P^{\mathrm{en}}\text{-modgr}) \to \mathrm{Ho}^b(P\text{-modgr})^{\mathbf{Z}}$.

Given $I \subset S$, we have a functor $\rho_I^* : D^b(P_I\text{-modgr})^{\mathbf{Z}} \to D^b(P\text{-modgr})^{\mathbf{Z}}$. This defines a functor from $\mathcal{C}ox$ to graded triangulated categories $(W,S) \mapsto D^b(P_S\text{-modgr})^{\mathbf{Z}}$. Our grading here is the one coming from P_S-modgr.

The following theorem is a consequence of Theorem 5.1 below.

Theorem 4.5. *The functors HH_*^S define a Markov 2-trace on finite Coxeter groups with value in $\mathcal{C}_S = D^b(P_S\text{-modgr})^{\mathbf{Z}}$ and with $T_{S,s,+} = [-1]\{-1\}$ and $T_{S,s,-} = \mathrm{Id}$.*

Passing to homology, we obtain the following result.

Corollary 4.6. *The functors $H^* \mathrm{HH}_*^S$ define a Markov 2-trace on finite Coxeter groups with value in $\mathcal{C}_S = \left((P_S\text{-modgr})^{\mathbf{Z}}\right)^{\mathbf{Z}}$ and with $T_{S,s,+} = [-1]\{-1\}$ and $T_{S,s,-} = \mathrm{Id}$.*

The construction of §4.1.3 provides a Markov trace, recovering a result of Lasy [La1, La2]. Note that Webster and Williamson [WeWi] have shown that for finite Weyl groups, this is actually Gomi's trace, as conjectured by J. Michel. That has been shown to hold also in type $I_2(n)$ by Lasy [La1].

Corollary 4.7 (Lasy)**.** *The following defines a Markov trace on finite Coxeter groups:*

$$\mathcal{B}_{(W,S)} \ni b \mapsto \sum_{d,i,j} (-1)^j \dim H^j(\mathrm{HH}_i^S(F_b))_d q^{-d} t^{-i} \in \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$$

corresponding to $t_+ = -t$ and $t_- = 1$.

4.2.2. *Shift adjustment.* By shifting suitably the invariants, we can get rid of the automorphisms $T_{S,s,\pm}$, but we lose functoriality (it would be interesting to see if functoriality with respect to an appropriate notion of cobordisms can be implemented). In order to do this, we need to use $\frac{1}{2}\mathbf{Z}$-complexes.

Given $\mathcal{A}$ an additive category, the category of $\frac{1}{2}$-complexes in $\mathcal{A}$ has objects $(C^i, d^i)_{i \in \frac{1}{2}\mathbf{Z}}$ where the differential has degree 1, and morphisms are $\frac{1}{2}\mathbf{Z}$-graded maps commuting with the differential. Its homotopy category is denoted by $\mathrm{Ho}_{\frac{1}{2}}(\mathcal{A})$ and, when $\mathcal{A}$ is an abelian category, its derived category by $D_{\frac{1}{2}}(\mathcal{A})$.

Corollary 4.8. *Given (W, S) a finite Coxeter group and $b \in B_S$, let*

$$N_S(F_b) = \mathrm{HH}^S_{*+\frac{|S|+l(b)}{2}}(F_b)\left[\frac{|S| + l(b)}{2}\right] \in D^b_{\frac{1}{2}}(P_S\text{-modgr})^{\frac{1}{2}\mathbf{Z}}.$$

- *We have $N_S(F_b F_{b'}) \simeq N_S(F_{b'} F_b)$ for all $b, b' \in B_S$.*
- *Given $s \in S$ and $b \in B_{S \setminus s}$, we have $N_S(\gamma_{S\setminus s}(F_b)F_s^{\pm 1}) \simeq \rho^*_{S\setminus s} N_{S\setminus s}(F_b)$.*

4.3. **Khovanov-Rozansky homology of links.** We specialize now to the case of the classical Artin braid groups considered by Khovanov in [Kh]. Note that Khovanov conjectured ten years ago that the F_b's should give rise to interesting link invariants.

We take here $V = (\bigoplus_{i=1}^n ke_i)/k(e_1 + \cdots e_n)$, the reflection representation of $W = \mathfrak{S}_n$, with $S = \{(1,2), \ldots, (n-1,n)\}$. Let $P_n = k[V] = k[\alpha_1, \ldots, \alpha_{n-1}]$, where $\alpha_i = X_{i+1} - X_i$. We put $B_n = B_{(W,S)}$.

Let $P_\infty = \lim_n P_n$, where the limit is taken over the morphisms of P_n-algebras $\rho_n : P_{n+1} \to P_n,\ \alpha_n \mapsto 0$. This provides functors between derived categories

$$\cdots \to D^b(P_n\text{-modgr}) \to D^b(P_{n+1}\text{-modgr}) \to \cdots \to D^b(P_\infty\text{-modgr}).$$

Theorem 4.9. *The assignment to $b \in B_{n+1}$ of the isomorphism class of $\mathrm{HH}_{*+\frac{n+l(b)}{2}}(F_b)[\frac{n+l(b)}{2}]$ in $D^b_{\frac{1}{2}}(P_\infty\text{-modgr})^{\frac{1}{2}\mathbf{Z}}$ defines an invariant of oriented links.*

Passing to homology, we recover the following result of Khovanov [Kh]. Khovanov identifies the invariant as the Khovanov-Rozansky homology, a categorification of the HOMFLYPT polynomial.

Theorem 4.10 (Khovanov). *The assignment to $b \in B_{n+1}$ of*

$$X_b = (t_2t_3)^{-(n+l(b))/2} \sum_{d,i,j} \dim H^j(\mathrm{HH}_i(F_b))_d t_1^d t_2^i t_3^j \in \mathbf{N}[t_2^{\pm 1/2}, t_3^{\pm 1/2}][[t_1]][t_1^{-1}]$$

defines an invariant of oriented links.

Note that $X_1 = 1$, where $1 \in B_1$ corresponds to the trivial knot.

We define now a two variables invariant $Y_b = (X_b)_{|t_3^{1/2}=\sqrt{-1}}$. The following corollary shows that Y_b is the HOMFLYPT polynomial, as expected.

Corollary 4.11 (Khovanov). *Given $b, b' \in B_n$ and $r \in \{1, \ldots, n-1\}$, we have*

$$(t_1t_2)^{-1/2} Y_{b\sigma_r^{-1}b'} + (t_1t_2)^{1/2} Y_{b\sigma_r b'} = \sqrt{-1}(t_1^{-1/2} - t_1^{1/2}) Y_{bb'}.$$

5. Proofs

5.1. Multiple complexes.

5.1.1. *Total objects.* For a more intrinsic approach to this section, cf [De, §1.1].

Let $\mathcal{A}$ be an additive category and $n \geq 0$. We denote by $\mathrm{Comp}(\mathcal{A})$ the category of complexes of objects of $\mathcal{A}$. The category n-$\mathrm{Comp}(\mathcal{A})$ of n-fold complexes is defined inductively by n-$\mathrm{Comp}(\mathcal{A}) = \mathrm{Comp}((n-1)\text{-}\mathrm{Comp}(\mathcal{A}))$ and 0-$\mathrm{Comp}(\mathcal{A}) = \mathcal{A}$. Its objects are families $(X, d_1, \ldots, d_n)$ where X is an object of $\mathcal{A}$ graded by $\mathbf{Z}^n = \bigoplus_{i=1}^n \mathbf{Z}e_i^*$, d_i is a graded map of degree e_i and $d_i^2 = [d_i, d_j] = 0$ for all i, j.

Given X an n-complex and $i \in \{1, \ldots, n\}$, we define $Y = X[e_i]$ as the n-complex given by $Y^b = X^{e_i+b}$ and differentials $\partial_i^b = (-1)^{\delta_{ij}} d_j^{e_i+b}$.

Let $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ be a map. It induces a map $\sigma : \mathbf{Z}^n \to \mathbf{Z}^m$ and gives by duality a map $\mathbf{Z}^m \to \mathbf{Z}^n$. This provides a functor from $\mathbf{Z}^n$-graded objects to $\mathbf{Z}^m$-graded objects of $\mathcal{A}$. Let X be an n-complex. We have a corresponding $\mathbf{Z}^m$-graded object X'. We define a structure of m-complex by

$$d_i'^a = \sum_{\substack{b \in \sigma^{-1}(a) \\ j \in f^{-1}(i)}} (-1)^{\sum_{k \in f^{-1}(i), k<j} b_k} d_j^b$$

where $b = \sum_i b_i e_i$.

Note that when f is an injection, the sum above has only positive signs. When f is a bijection, then Tot^f is a self-equivalence of n-$\mathrm{Comp}(\mathcal{A})$. We write $\mathrm{Tot} = \mathrm{Tot}^f$ when $m = 1$.

We have defined an additive functor

$$\mathrm{Tot}^f : n\text{-}\mathrm{Comp}(\mathcal{A}) \to m\text{-}\mathrm{Comp}(\mathcal{A}).$$

Let $g : \{1,\ldots,m\} \to \{1,\ldots,p\}$ be a map and $\tau : \mathbf{Z}^m \to \mathbf{Z}^p$ the associated morphism. Let $X \in n\text{-Comp}(\mathcal{A})$. The $\mathbf{Z}^p$-graded objects underlying $\mathrm{Tot}^{gf}(X)$ and $\mathrm{Tot}^g(\mathrm{Tot}^f(X))$ have their component of degree a equal to $\bigoplus_{c\in(\tau\sigma)^{-1}(a)} X^c$. We define an isomorphism between these p-complexes by multiplication by $(-1)^{\varepsilon(c)}$ on X^c, where

$$\varepsilon(c) = \sum_{\substack{l<l' \\ f(l)>f(l') \\ gf(l)=gf(l')}} c_l c_{l'}.$$

This gives an isomorphism of functors

$$\mathrm{Tot}^{gf} \xrightarrow{\sim} \mathrm{Tot}^g \circ \mathrm{Tot}^f .$$

Let k be a commutative ring and A, B and C be three k-algebras. Let $X \in n\text{-Comp}((A\otimes B)\text{-Mod})$ and $Y \in m\text{-Comp}((B \otimes C)\text{-Mod})$. Then $X \otimes_B Y$ defines an object of $(n+m)\text{-Comp}((A\otimes C)\text{-Mod})$. We have $(X\otimes_B Y)^{(a_1,\ldots,a_{n+m})} = X^{(a_1,\ldots,a_n)} \otimes_B Y^{(a_{n+1},\ldots,a_{n+m})}$.

5.1.2. *Cohomology.* Assume $\mathcal{A}$ is an abelian category. Let $X \in n\text{-Comp}(\mathcal{A})$. Let $r \in \{1,\ldots,n\}$ and $Y = \ker d_r / \operatorname{im} d_r$. This is an n-complex with $d_{r,Y} = 0$. Let $i \in \mathbf{Z}$. Consider the map $f : \mathbf{Z}^{n-1} \to \mathbf{Z}^n$, $(a_1,\ldots,a_{n-1}) \mapsto (a_1,\ldots,a_{r-1},i,a_r,\ldots,a_{n-1})$. We put $H^i_{d_r}(X) = \bigoplus_{a\in\mathbf{Z}^{n-1}} Y^{f(a)}$. This defines a functor

$$H^i_r : n\text{-Comp}(\mathcal{A}) \to (n-1)\text{-Comp}(\mathcal{A}).$$

Let $g : \{1,\ldots,n\} \to \{1,\ldots,m\}$ be a map and let $r \in \{1,\ldots,m\}$ such that $f^{-1}(r) = \{s\}$ for some $s \in \{1,\ldots,n\}$. Define maps

$$\alpha : \{1,\ldots,n-1\} \to \{1,\ldots,n\},\ i \mapsto \begin{cases} i & \text{if } i < s \\ i+1 & \text{if } i \geq s \end{cases}$$

and

$$\beta : \{1,\ldots,m\} \to \{1,\ldots,m-1\},\ i \mapsto \begin{cases} i & \text{if } i < r \\ i-1 & \text{if } i \geq r \end{cases}$$

Then, we have a canonical isomorphism

$$H^i_{d_r}(\mathrm{Tot}^f(M)) \xrightarrow{\sim} \mathrm{Tot}^{\beta f \alpha}(H^i_{d_s}(M)). \tag{1}$$

5.1.3. *Resolutions.* Let k be a commutative ring and A a k-algebra. The k-linear functor $H^0 : \mathrm{Ho}^-(A\text{-Proj}) \to A\text{-Mod}$ restricted to the full subcategory of complexes M with $H^i(M) = 0$ for $i \neq 0$ is an equivalence. Let $C = C_A : A\text{-Mod} \to \mathrm{Ho}^-(A\text{-Proj})$ be an inverse, composed with the inclusion functor. By construction the resolution functor C is fully faithful. It induces a functor, still denoted by C,

$$C : \mathrm{Ho}^b(A\text{-Mod}) \to \mathrm{Ho}^b\big(\mathrm{Ho}^-(A\text{-Proj})\big).$$

We view $\mathrm{Ho}^b\big(\mathrm{Ho}^-(A\text{-Proj})\big)$ as a triangulated category with the canonical structure on $\mathrm{Ho}^b(\mathcal{C})$, where $\mathcal{C}$ is the additive category $\mathrm{Ho}^-(A\text{-Proj})$. The functor C is a fully faithful triangulated functor.

Assume A is projective as a k-module. Let $X = C_{A^{\mathrm{en}}}(A)$ be a projective resolution of A as an A^{en}-module. The functor

$$- \otimes_{A^{\mathrm{opp}}} X : \mathrm{Comp}^b(A\text{-Mod}) \to 2\text{-Comp}(A\text{-Proj})$$

composed with the canonical functor $2\text{-Comp}(A\text{-Proj}) \to \mathrm{Ho}\big(\mathrm{Ho}(A\text{-Proj})\big)$ is isomorphic to C: we have $M \otimes_{A^{\mathrm{opp}}} X \xrightarrow{\sim} C_A(M)$ for $M \in \mathrm{Ho}^b(A\text{-Mod})$.

Let $i \in \mathbf{Z}$. The functor $H^i : \mathrm{Ho}^-(A\text{-Proj}) \to A\text{-Mod}$ induces a functor

$$H^i_{d_2} : \mathrm{Ho}^b\big(\mathrm{Ho}^-(A\text{-Proj})\big) \to \mathrm{Ho}^b(A\text{-Mod}).$$

It extends the functor $H^i_{d_2} : 2\text{-Comp}(A\text{-Mod}) \to \mathrm{Comp}(A\text{-Mod})$.

Let B, B' be two k-algebras, projective as k-modules. Let $L \in \mathrm{Ho}^b\big((A \otimes B^{\mathrm{opp}})\text{-Mod}\big)$ and $M \in \mathrm{Ho}^b((B \otimes B'^{\mathrm{opp}})\text{-Mod})$. Assume the components of M are projective right B'-modules. We deduce from (1) an isomorphism

$$\mathrm{Tot}^{13\to 1, 2\to 2}\big(C_{A\otimes B^{\mathrm{opp}}}(L) \otimes_B M\big) \xrightarrow{\sim} C_{A\otimes B'^{\mathrm{opp}}}\big(\mathrm{Tot}(L \otimes_B M)\big). \tag{2}$$

Note that when A is coherent, then A-Mod can be replaced by the abelian category A-mod and A-Proj by A-proj. If A is graded, we can replace these categories by the corresponding categories of graded modules.

5.2. **Markov moves.** Let (W, S) be a finite Coxeter group. Let P-exact be the category of finitely generated graded P^{en}-modules whose restrictions to P and P^{opp} are projective. Let $M \in \mathrm{Ho}^b(P\text{-exact})$ and $i \in \mathbf{Z}$. We put

$$K^i(M) = K^i_S(M) = H^i_{d_2}\big(P \otimes_{P^{\mathrm{en}}} C_{P^{\mathrm{en}}}(M)\big) \in \mathrm{Ho}^b(P\text{-modgr}).$$

This defines a triangulated functor

$$K^i : \mathrm{Ho}^b(P\text{-exact}) \to \mathrm{Ho}^b(P\text{-modgr}).$$

Theorem 5.1. *Given $N, N' \in \mathrm{Ho}^b(P\text{-exact})$, we have functorial isomorphisms $K^i(N \otimes_P N') \simeq K^i(N' \otimes_P N)$ in $\mathrm{Ho}(P\text{-modgr})$.*

Let $s \in S$ and let z_s be a non-zero element of $(V^*)^{S\setminus s}$. Let $M \in \mathrm{Ho}^b(P_{S\setminus s}\text{-exact})$. We have functorial isomorphisms

- $K^i_S\big(\gamma_{S\setminus s}(M) \otimes_P F_s\big) \simeq \rho^*_{S\setminus s} K^{i+1}_{S\setminus s}(M)[-1]$ *in* $D(P\text{-modgr})$
- $K^i_S\big(\gamma_{S\setminus s}(M) \otimes_P F_s^{-1}\big) \simeq \rho^*_{S\setminus s} K^i_{S\setminus s}(M)$ *in* $D(P\text{-modgr})$
- $K^i_S\big(\gamma_{S\setminus s}(M)) \simeq P \otimes_{P_{S\setminus s}} K^{i+1}_{S\setminus s}(M)\langle -1\rangle \oplus P \otimes_{P_{S\setminus s}} K^i_{S\setminus s}(M)$ *in* $\mathrm{Ho}(P\text{-modgr})$.

Proof. Thanks to (2), we have

$$C_{P^{\mathrm{en}}}\big(\mathrm{Tot}(N\otimes_P N')\big)\simeq \mathrm{Tot}^{12\to 1,3\to 2}(C_{P^{\mathrm{en}}}(N)\otimes_P N')$$

hence

$$\begin{aligned}P\otimes_{P^{\mathrm{en}}} C_{P^{\mathrm{en}}}\big(\mathrm{Tot}(N\otimes_P N')\big)&\simeq \mathrm{Tot}^{12\to 1,3\to 2}\big(C_{P^{\mathrm{en}}}(N)\otimes_{P^{\mathrm{en}}} N'\big)\\ &\simeq \mathrm{Tot}^{12\to 1,3\to 2}\big(N'\otimes_{P^{\mathrm{en}}} C_{P^{\mathrm{en}}}(N)\big)\\ &\simeq P\otimes_{P^{\mathrm{en}}} C_{P^{\mathrm{en}}}\big(\mathrm{Tot}(N'\otimes_P N)\big)\end{aligned}$$

and the first assertion follows.

We have

$$C_{P^{\mathrm{en}}}(M\otimes k[z_s])\simeq \mathrm{Tot}^{1\to 1,23\to 2}\big(C_{P^{\mathrm{en}}_{S\backslash s}}(M)\otimes X\big)$$

where $X=0\to k[z_s]^{\mathrm{en}}\langle -1\rangle \xrightarrow{z_s\otimes 1-1\otimes z_s} k[z_s]^{\mathrm{en}}\to 0$, the non-zero terms being in degrees -1 and 0. Let

$$L=P\otimes_{P^{\mathrm{en}}}\Bigg(C_{P^{\mathrm{en}}}\Big(\mathrm{Tot}\big((M\otimes k[z_s])\otimes_P F_s\big)\Big)\Bigg).$$

By (2), we have

$$L\simeq P\otimes_{P^{\mathrm{en}}}\mathrm{Tot}^{13\to 1,2\to 2}\big(C_{P^{\mathrm{en}}}(M\otimes k[z_s])\otimes_P F_s\big)\simeq \mathrm{Tot}^{13\to 1,24\to 2}\big((F_s\otimes_{k[z_s]^{\mathrm{en}}}X)\otimes_{P^{\mathrm{en}}_{S\backslash s}}C_{P^{\mathrm{en}}_{S\backslash s}}(M)\big)$$

We have

$$F_s\otimes_{k[z_s]^{\mathrm{en}}}X\simeq \begin{array}{ccc}\theta_s & \xrightarrow{m} & P\\ {\scriptstyle z_s\otimes 1-1\otimes z_s}\downarrow & & \downarrow{\scriptstyle 0}\\ \theta_s\langle 1\rangle & \xrightarrow[m]{} & P\langle 1\rangle\end{array}\simeq \begin{array}{ccc}\theta_s & \xrightarrow{m} & P\\ {\scriptstyle \alpha_s\otimes 1-1\otimes \alpha_s}\downarrow & & \downarrow{\scriptstyle 0}\\ \theta_s\langle 1\rangle & \xrightarrow[m]{} & P\langle 1\rangle\end{array}$$

Indeed, there is $c\in k^*$ such that $s(z_s)-z_s=-2c\alpha_s$. Then $z_s-c\alpha_s\in(V^*)^s$, hence $z_s\otimes 1-1\otimes z_s$ and $c(\alpha_s\otimes 1-1\otimes\alpha_s)$ are equal in θ_s.

Lemma 5.2 below shows that, when $s\notin Z(W)$, then the exact sequence of P^{en}-modules

$$0\to P\xrightarrow{\alpha_s\otimes 1+1\otimes\alpha_s}\theta_s\xrightarrow{a\otimes b\mapsto as(b)} Ps\to 0$$

splits by restriction to $P^{\mathrm{en}}_{S\backslash s}$. Here, $Ps=P$ as a left P-module, and the right action of $a\in P$ is given by multiplication by $s(a)$. Note that when $s\in Z(W)$, then $P^s=P_{S\backslash s}\otimes k[\alpha_s^2]$, and the splitting of the sequence holds trivially.

We deduce that there is an isomorphism of complexes of graded $P^{\mathrm{en}}_{S\backslash s}$-modules

$$\big(0\to\theta_s\xrightarrow{\alpha_s\otimes 1-1\otimes\alpha_s}\theta_s\langle 1\rangle\to 0\big)\simeq P[1]\langle -1\rangle\oplus\big(0\to Ps\xrightarrow{a\mapsto a\alpha_s\otimes 1-a\otimes\alpha_s}\theta_s\to 0\big)\langle 1\rangle.$$

Let $Y_1 = \begin{tikzcd} P\langle -1\rangle \arrow[r, "2\alpha_s"] \arrow[d] & P \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$ and $Y_2 = \begin{tikzcd} Ps \arrow[r] \arrow[d, "a\mapsto a\alpha_s\otimes 1-a\otimes\alpha_s"'] & 0 \arrow[d] \\ \theta_s\langle 1\rangle \arrow[r] & P\langle 1\rangle \end{tikzcd}$

We have an exact sequence of bicomplexes of graded P^{en}-modules

$$0\to Y_1\to F_s\otimes_{k[\alpha_s]^{\mathrm{en}}} X\to Y_2\to 0.$$

It splits after restricting to $P^{\mathrm{en}}_{S\setminus s}$ and applying $?^{(i,*)}$. It follows that we have an exact sequence of complexes of graded P^{en}-modules

$$0\to H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(Y_1\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big)\to H^i_{d_2}(L)\to$$
$$\to H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(Y_2\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big)\to 0.$$

We have an exact sequence of P^{en}-modules

$$0\to Ps\xrightarrow{a\mapsto a\alpha_s\otimes 1-a\otimes\alpha_s}\theta_s\xrightarrow{m} P\to 0.$$

So, the morphism of bicomplexes $Y_2\to Y_2'$:

$$\begin{tikzcd} Ps \arrow[r] \arrow[d, "a\mapsto a\alpha_s\otimes 1-a\otimes\alpha_s"'] & 0 \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ \theta_s\langle 1\rangle \arrow[r, "m"'] \arrow[rr, bend right, "m"'] & P\langle 1\rangle \arrow[rr, bend right, "\mathrm{id}"'] & P\langle 1\rangle \arrow[r, "\mathrm{id}"] & P\langle 1\rangle \end{tikzcd}$$

induces an isomorphism of complexes

$$H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(Y_2\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big)\xrightarrow{\sim} H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(Y_2'\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big)$$

and these complexes vanish in $\mathrm{Ho}^b(P^{\mathrm{en}}\text{-modgr})$.

We deduce that

$$\begin{aligned} H^i_{d_2}(L) &\simeq H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(Y_1\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big) \\ &\simeq \rho^*_{S\setminus s}H^i_{d_2}\Big(\mathrm{Tot}^{13\to 1,24\to 2}\big(P_{S\setminus s}[(-1,1)]\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)\Big) \\ &\simeq \rho^*_{S\setminus s}H^{i+1}_{d_2}\big(P_{S\setminus s}\otimes_{P^{\mathrm{en}}_{S\setminus s}} C_{P^{\mathrm{en}}_{S\setminus s}}(M)\big)[-1] \end{aligned}$$

in $D^b(P^{\mathrm{en}}\text{-modgr})$. Note that the multiplication map $P^{\mathrm{en}}\to P$ is a split surjection of algebras. We deduce the first and last terms of the sequence of isomorphisms above are actually isomorphic in $D^b(P\text{-modgr})$. This shows the second assertion. The proof of the assertion involving F_s^{-1} is similar.

We have $k[z_s]\otimes_{k[z_s]^{\mathrm{en}}} X \simeq k[z_s]\langle -1\rangle[1]\oplus k[z_s]$ and the last assertion follows immediately. $\square$

Lemma 5.2. *Let $s\in S$. Assume $s\notin Z(W)$. Let $L=(V^{S\setminus s})^\perp\cap (V^*)^s$, a hyperplane of $(V^{S\setminus s})^\perp=(V_{S\setminus s})^*$. We have a commutative diagram of $P^{\mathrm{en}}_{S\setminus s}$-modules where the diagonal map is an isomorphism*

$$\begin{tikzcd} \theta_s \arrow[r, "a\otimes b\mapsto as(b)"] & Ps \\ P_{S\setminus s}\otimes_{S(L)} P_{S\setminus s} \arrow[u, "a\otimes b\mapsto a\otimes b"] \arrow[ur, "\sim", "a\otimes b\mapsto as(b)"'] & \end{tikzcd}$$

Here, Ps denotes the left $P_{S\setminus s}$-module P endowed with a right action of $a\in P_{S\setminus s}$ by multiplication by $s(a)$.

Proof. We will show that $\phi: P_{S\setminus s}\otimes_{S(L)} P_{S\setminus s}\to P,\ a\otimes b\mapsto as(b)$ is an isomorphism. It is a morphism of algebras, and a morphism of graded left $P_{S\setminus s}$-modules.

By assumption, there is $t\in S\setminus s$ such that $m_{st}\neq 2$, so that $s(\alpha_t)-\alpha_t$ is a non-zero multiple of α_s. It follows that $\phi(\alpha_t\otimes 1-1\otimes\alpha_t)\in k^\times\alpha_s$. Since $V^*=(V^{S\setminus s})^\perp\oplus k\alpha_s$, we have $P=P_{S\setminus s}\otimes k[\alpha_s]$ and we deduce that ϕ is surjective.

Since ϕ is a morphism of graded free left $P_{S\setminus s}$-modules with the same graded ranks $1+t+t^2+\cdots$, we deduce that ϕ is an isomorphism. $\square$

Remark 5.3. While we do not expect the category of Soergel bimodules to exist for complex reflection groups, we hope that its homotopy category does exist, as well as the 2-braid group. Its center would be a starting point for a structural approach to the construction of unipotent data in Broué–Malle–Michel's theory of spets [BroMaMi].

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