Journal of Algebra 323 (2010) 59–82

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Dynkin diagram cohomology of finite Coxeter groups

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ARTICLE INFO

Article history:
Received 5 June 2008
Available online 22 October 2009
Communicated by Michel Broué

Keywords:
Quasi-Coxeter algebras
Dynkin diagram cohomology
Coxeter groups
Braid groups
KZ connection
Coxeter complex

ABSTRACT

Let D be a connected graph. The Dynkin complex $CD(A)$ of a D-algebra A was introduced by the second author to control the deformations of quasi-Coxeter algebra structures on A. In the present paper, we study the cohomology of this complex when A is the group algebra of a Coxeter group W and D is the Dynkin diagram of W. We compute this cohomology when W is finite and prove in particular the rigidity of quasi-Coxeter algebra structures on kW. For an arbitrary W, we compute the top cohomology group and obtain a number of additional partial results when W is affine. Our computations are carried out by filtering the Dynkin complex by the number of vertices of subgraphs of D. The corresponding graded complex turns out to be dual to the sum of the Coxeter complexes of all standard, irreducible parabolic subgroups of W.

1. Introduction

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra and D the corresponding Dynkin diagram. The notion of quasi-Coxeter algebra of type D was introduced in [TL2] to put the monodromy of the Casimir connection of $\mathfrak{g}$ [MTL,TL] and the quantum Weyl group representations arising from the quantum group $U_q\mathfrak{g}$ [Lu] on an equal footing, and allow for their comparison via a suitable deformation complex.

Roughly speaking, a quasi-Coxeter algebra A of type D is an algebra which carries representations of the generalised braid group B_D corresponding to D on its finite-dimensional modules. The

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1 Supported in part by NSF grants DMS-0707212 and DMS-0635607.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.09.023

deformation theory of such an algebra is controlled by a complex $CD(A)$ concentrated in degrees $0 \leqslant p \leqslant |D|$ called the *Dynkin complex* of A.

If W is the Weyl group of $\mathfrak{g}$, the complex group algebra $A = \mathbb{C}W$ possesses quasi-Coxeter algebra structures accounting respectively for the representations of B_D coming from the Hecke algebra $\mathcal{H}(W)$ of W and the monodromy of Cherednik's KZ connection [Ch] (see [TL2, Section 4] for details).

While in this case these representations may easily be shown to be equivalent, this raises nonetheless the question of computing the Dynkin diagram cohomology of $\mathbb{C}W$ and more generally of the group algebra kW of a Coxeter group over an arbitrary ground ring k, when the underlying D-algebra structure arises from the standard parabolic subgroups of W.

In the present paper, we answer this question when W is finite and k is a field of characteristic 0. Along the way, we also determine the top cohomology groups for arbitrary Coxeter groups and obtain partial results for affine ones.

We carry out our computations by filtering $CD(kW)$ by the number of vertices of subgraphs of D. Interestingly, and crucially for us, the associated graded complex $\mathsf{Gr}(CD(kW))$ turns out to be dual to the sum of the Coxeter complexes of all standard, irreducible, parabolic subgroups of W. This greatly simplifies the computation of $H^*(CD(kW))$ since the Coxeter complex is acyclic for affine Coxeter groups and has cohomology in one degree only for finite ones.

We turn now to a detailed description of the paper.

In Section 2, we review the definition of the Dynkin complex of a D-algebra and define its canonical filtration.

We then consider in Section 3 the case of the group algebra of a Coxeter group. In this case, the Dynkin complex has a direct sum decomposition parametrised by the conjugacy classes of W. Further, the graded complex $\mathsf{Gr}(CD(kW))$ determined by the filtration of $CD(kW)$ decomposes as a sum parametrised by connected subgraphs B of D. The summand associated to B is given by the morphisms from the Coxeter complex of W_B to kW_B endowed with the adjoint action of W_B.

In Section 4, we use the known description of the cohomology of the Coxeter complex for finite and affine Coxeter groups to compute the cohomology of $\mathsf{Gr}(CD(kW))$.

In Section 5, we consider finite Coxeter groups. We compute the induced differential on $H^*(\mathsf{Gr}(CD(kW)))$ and show that the resulting complex is quasi-isomorphic to CD.

We apply this result in Section 6 to prove the rigidity of quasi-Coxeter algebra structures on kW when W is finite.

The main result of Section 7 is the construction of a basis of the top Dynkin cohomology of the group algebra of an arbitrary Coxeter group: it is parametrised by cuspidal conjugacy classes such that the centraliser of an element of the class is in the kernel of the sign character.

Section 8 is devoted to the determination of the Dynkin cohomology of finite Coxeter groups. We proceed case by case, by explicit computation for W classical or of type G_2 and F_4, and by using the program GAP for the remaining exceptional groups. In types A and B, we provide a very simple formula for the corresponding generating series which, in type A turns out to be the product of a bosonic and fermionic partition function. We show moreover that, for W classical, the Dynkin cohomology spaces stabilise with the rank of W.

In the final Section 9, we describe the part of the Dynkin cohomology of an affine Weyl group corresponding to conjugacy classes of elements of infinite order.

2. *D*-algebras and the Dynkin complex

This section reviews the definition of D-algebras and of the Dynkin complex. With the exception of Section 2.4, the material is borrowed from [TL2].

2.1. *D-algebras [TL2, Section 3]*

Let D be a connected *diagram*, that is a nonempty undirected graph with no multiple edges or loops. We denote the set of vertices of D by $V(D)$ and set $|D| = |V(D)|$. By a *subdiagram* $B \subset D$ we shall mean a nonempty full subgraph of D, that is a graph consisting of a subset $V(B)$ of vertices of D, together with all edges of D joining any two elements of $V(B)$. We will often abusively identify such a B with its set of vertices and write $\alpha \in B$ to mean $\alpha \in V(B)$. Two subdiagrams $B_1, B_2 \subseteq D$ are *orthogonal* if they have no vertices in common and no two vertices $\alpha_1 \in B_1$, $\alpha_2 \in B_2$ are joined by an edge in D.

Let k be a fixed commutative ring with unit. By an algebra we shall mean a unital, associative k-algebra. All algebra homomorphisms are assumed to be unital. Recall the following

Definition 2.1. A D-algebra is an algebra A endowed with subalgebras A_B labelled by the connected subdiagrams B of D such that the following holds:

- $A_B \subseteq A_{B'}$ whenever $B \subseteq B'$.
- A_B and $A_{B'}$ commute whenever B and B' are orthogonal.

If $B_1, B_2 \subseteq D$ are subdiagrams with B_1 connected, we denote by $A_{B_1}^{B_2}$ the centraliser in A_{B_1} of the subalgebras $A_{B_2^i}$, where B_2^i runs over the connected components of B_2.

2.2. The Dynkin complex [TL2, Section 5]

For any $0 \leqslant p \leqslant n = |D|$, set

$$C^p(A) = \bigoplus_{\substack{\underline{\alpha} \subseteq B \subseteq D \\ |\underline{\alpha}|=p}} A_B^{B \setminus \underline{\alpha}}$$

where the sum ranges over all connected subdiagrams B of D and ordered subsets $\underline{\alpha} = \{\alpha_1, \dots, \alpha_p\} \subseteq V(B)$ of cardinality p and, by convention, $A_B^{\emptyset} = A_B$. We denote the component of $a \in C^p(A)$ along $A_B^{B\setminus\underline{\alpha}}$ by $a_{(B;\underline{\alpha})}$.

Definition 2.2. The group of Dynkin p-cochains on A is the subspace $CD^p(A) \subset C^p(A)$ of elements a such that

$$a_{(B;\sigma\underline{\alpha})} = (-1)^{\sigma} a_{(B;\underline{\alpha})}$$

where, for any $\sigma \in \mathfrak{S}_p$, $\sigma\{\alpha_1, \dots, \alpha_p\} = \{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}\}$.[2]

Note that

$$CD^0(A) = \bigoplus_{B \subseteq D} Z(A_B) \quad \text{and} \quad CD^n(A) \simeq A_D$$

For $1 \leqslant p \leqslant n-1$, define a map $d_D^p : C^p(A) \to C^{p+1}(A)$ by

$$(d_D a)_{(B;\underline{\alpha})} = \sum_{i=1}^{p+1} (-1)^{i-1} \bigl(a_{(B;\underline{\alpha}\setminus\alpha_i)} - a_{(\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i};\underline{\alpha}\setminus\alpha_i)}\bigr) \tag{1}$$

where $\underline{\alpha} = \{\alpha_1, \dots, \alpha_{p+1}\}$, $\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i}$ is the connected component of $B \setminus \alpha_i$ containing $\underline{\alpha} \setminus \alpha_i$ if one such exists and the empty set otherwise, and we set $a_{(\emptyset;-)} = 0$. For $p = 0$, define $d_D^0 : C^0(A) \to C^1(A)$ by

[2] In [TL2] Dynkin chains are defined with values in any D-bimodule M over A. We shall only need to consider $M = A$ in this paper and therefore restrict attention to this case.

$$d_D^0 a_{(B;\alpha_i)} = a_B - a_{B\setminus\alpha_i}$$

where $a_{B\setminus\alpha_i}$ is the sum of a_{B_2} with B_2 ranging over the connected components of $B \setminus \alpha_i$. Finally, set $d_D^n = 0$. The map d_D leaves $CD(A)$ invariant and satisfies $d_D^2 = 0$. The cohomology $HD(A)$ of $CD(A)$ with respect to d_D is called the *Dynkin diagram* cohomology of A.

2.3. Restriction [TL2, Section 5.2]

Let D' be a connected subgraph of D. We have a morphism of complexes

$$\mathrm{Res}_{D'}^D : CD(A) \to CD(A_{D'})$$
$$A_B^{B\setminus\underline{\alpha}} \ni a \mapsto \begin{cases} a & \text{if } B \subset D' \\ 0 & \text{otherwise} \end{cases}$$

2.4. The canonical filtration on $CD(A)$

Endow each chain group $C^p(A)$ with the $\mathbb{N}$-grading given by

$$C_q^p(A) = \bigoplus_{\substack{\underline{\alpha}\subset B\subset D \\ |\underline{\alpha}|=p,\ |B|=q}} A_B^{B\setminus\underline{\alpha}}$$

where $p \leqslant q \leqslant n$, and set $CD_q^p(A) = CD^p(A) \cap C_q^p(A)$. Since $|\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i}| < |B|$, the Dynkin differential d_D maps $CD_q^p(A)$ to

$$CD_{\geqslant q}^{p+1}(A) = \bigoplus_{r=q}^{n} CD_r^{p+1}(A)$$

This gives a decreasing $\mathbb{N}$-filtration on the Dynkin complex of A. The E_2-term of the corresponding spectral sequence is the cohomology of $CD(A)$ with respect to the differential

$$\left(d_D^0 a\right)_{(B;\underline{\alpha})} = \sum_{i=1}^{p+1} (-1)^{i-1} a_{(B;\underline{\alpha}\setminus\alpha_i)} \tag{2}$$

3. The Dynkin complex of a Coxeter group

3.1. Description

Let W be an irreducible Coxeter group with system of generators $S = \{s_i\}_{i\in I}$ and let D be the Coxeter graph of (W, S). For any subgraph $B \subseteq D$ with vertex set $I_B \subseteq I$, let $W_B \subseteq W$ be the standard parabolic subgroup generated by s_i, $i \in I_B$.

Regard the group algebra $A = kW$ as a D-algebra by setting $A_B = kW_B$. By choosing a total order on the vertices of D, we can identify the corresponding Dynkin complex with

$$CD^p = \bigoplus_{\substack{\underline{\alpha}\subset B\subset D \\ |\underline{\alpha}|=p}} kW_B{}^{W_{B\setminus\underline{\alpha}}}$$

where α now ranges over the unordered subsets of $V(B)$ and the $W_{B\setminus\underline{\alpha}}$-fixed points in kW_B are taken with respect to the diagonal (adjoint) action. The Dynkin differential on $kW_B{}^{W_{B\setminus\underline{\alpha}}}$ is the map

$$kW_B{}^{W_{B\setminus\underline{\alpha}}} \to \bigoplus_{\beta\in B\setminus\underline{\alpha}} kW_B{}^{W_{B\setminus(\underline{\alpha}\cup\{\beta\})}} \oplus \bigoplus_{\substack{B\subset B'\subset D\\ \beta\in B'\setminus\underline{\alpha}:\\ B=\mathsf{C}_{\underline{\alpha}}^{B'\setminus\{\beta\}}}} kW_{B'}{}^{W_{B'\setminus(\underline{\alpha}\cup\{\beta\})}}$$

given by

$$\left(\sum_{\beta}(-1)^{\mathrm{pos}(\beta,\underline{\alpha}\cup\{\beta\})-1}\mathrm{can}, \sum_{B',\beta}(-1)^{\mathrm{pos}(\beta,\underline{\alpha}\cup\{\beta\})}\mathrm{can}\right)$$

where can is the canonical inclusion map and for any $\beta\in\underline{\beta}\subset V(D)$, $\mathrm{pos}(\beta,\underline{\beta})\in\{1,\dots,|\underline{\beta}|\}$ is the position of β relative to the total order on $\underline{\beta}$.

3.2. Decomposition by conjugacy classes

Let $\mathcal{C}$ be the set of conjugacy classes of W. For any $c\in\mathcal{C}$, set

$$CD_c^p = \bigoplus_{\substack{\underline{\alpha}\subset B\subset D\\ |\underline{\alpha}|=p}} k(W_B\cap c)^{W_{B\setminus\underline{\alpha}}}$$

where, for any set X, $k(X)$ is the vector space with basis X. Then $CD_c=\bigoplus_p CD_c^p$ is a subcomplex of CD and

$$CD=\bigoplus_{c\in\mathcal{C}} CD_c$$

We denote the cohomology of the corresponding complex by $HD_c(kW)$.

3.3. Filtration

When filtered as in Section 2.4, the associated graded complex $\mathsf{Gr}(CD)$ (=the E_1-term of the spectral sequence) is the sum over all connected subdiagrams $B\subseteq D$ of the subcomplexes CD_B given by

$$\begin{aligned} CD_B^p &= \bigoplus_{\underline{\alpha}\subset B,\,|\underline{\alpha}|=p} kW_B{}^{W_{B\setminus\underline{\alpha}}} \\ &\simeq \bigoplus_{\underline{\alpha}\subset B,\,|\underline{\alpha}|=p} \mathrm{Hom}_{W_B}\big(k(W_B/W_{B\setminus\underline{\alpha}}), kW_B\big) \end{aligned} \qquad (3)$$

Recall that the Coxeter complex CC^B of W_B is the (homology) complex

$$CC_p^B = \bigoplus_{\underline{\alpha}\subset B,\,|\underline{\alpha}|=p} k(W_B/W_{B\setminus\underline{\alpha}})$$

with differential given by

$$\partial_C(wW_{B\setminus\underline{\alpha}}) = \sum_{i=1}^{p}(-1)^{i-1} wW_{(B\setminus\underline{\alpha})\cup\{\alpha_i\}}$$

where $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_p\}$ with $\alpha_1 < \cdots < \alpha_p$. The following immediate result identifies the B-component CD_B of $\mathsf{Gr}(CD)$ with the dual of the Coxeter complex of W_B with values in kW_B.

Proposition 3.1. *The isomorphism* (3) *induces an isomorphism of complexes*

$$CD_B \simeq \mathrm{Hom}_{W_B}\left(CC^B, kW_B\right)$$

4. The cohomology of CD_B

Assume henceforth that k is a field of characteristic 0.

4.1. Finite Coxeter groups

Assume in this subsection that W_B is finite. Let S_B be the unit sphere in the Euclidean reflection representation of W_B, and cellulate S_B by its intersections with the chambers of W_B. Then, the Coxeter complex CC^B is the cellular homology complex of S_B, reduced and shifted by one [Hu]. Thus, $H_p(CC^B)$ is zero if $p < |B|$ and the sign representation ε of W_B otherwise, so that, by Proposition 3.1, we have

$$H^p(CD_B) \simeq \begin{cases} kW_B{}^{\varepsilon} & \text{if } p = |B| \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $kW_B{}^{\varepsilon} \subset kW_B$ is the subspace transforming like the sign representation ε of W_B.

Consider $kW_B{}^{\varepsilon}[-|B|]$, a complex concentrated in degree $|B|$. We have a morphism of complexes $i_B : kW_B{}^{\varepsilon}[-|B|] \to CD_B$ given by the inclusion

$$kW_B{}^{\varepsilon} \hookrightarrow kW_B = kW_B{}^{W_{B\setminus B}}$$

in degree $|B|$. Let $\mathrm{Alt}_B : kW_B \to kW_B{}^{\varepsilon}$ be the projection given by

$$\mathrm{Alt}_B(f) = \frac{1}{|W_B|} \sum_{w \in W_B} \varepsilon(w) w f w^{-1}$$

Since Alt_B is zero on $\sum_{\alpha \in B} kW_B{}^{W_{B\setminus\alpha}}$, it defines a morphism of complexes $\rho_B : CD_B \to kW_B{}^{\varepsilon}[-|B|]$. Summarising, we have the following proposition.

Proposition 4.1. *If W_B is finite, the maps $i_B : kW_B{}^{\varepsilon}[-|B|] \to CD_B$ and $\rho_B : CD_B \to kW_B{}^{\varepsilon}[-|B|]$ are quasi-isomorphisms such that $\rho_B \circ i_B = \mathrm{id}$.*

4.2. Affine Coxeter groups

Assume now that W_B is an affine Coxeter group, and let E_B be the Euclidean space of dimension $|B| - 1$ cellulated by the alcoves of W_B. The Coxeter complex CC^B is the cellular homology complex of E_B, reduced and shifted by one [Hu], and is therefore acyclic. Its terms of positive degree are induced from the trivial representation of finite (parabolic) subgroups of W_B and are therefore projective. Thus, CC^B is a projective resolution of the trivial W_B-module. It follows from this, and Proposition 3.1, that

$$H^p(CD_B) \simeq \begin{cases} \mathrm{Ext}^{p-1}_{W_B}(k, kW_B) & \text{if } p \geqslant 2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where kW_B is endowed with the adjoint action of W_B.

5. The $\mathbf{E_2}$-term of the Dynkin complex of a Coxeter group

5.1. Finite Coxeter groups

We assume in this section that the Coxeter group W is finite.

Consider the complex

$$HC = \bigoplus_{B \subseteq D} k{W_B}^{\varepsilon} \tag{6}$$

where $k{W_B}^{\varepsilon}$ appears in degree $|B|$, endowed with the differential $d^{\#}$ given by

$$d^{\#} = \sum_{B'} (-1)^{(B';B)} \cdot \mathrm{Alt}_{B'} : k{W_B}^{\varepsilon} \to \bigoplus_{\substack{B \subset B' \\ |B'|=|B|+1}} k{W_{B'}}^{\varepsilon} \tag{7}$$

where, given $B' \subseteq D$ with ordered set of vertices $\alpha_1 < \cdots < \alpha_q$ and $B \subset B'$ such that $B' \setminus B = \{\alpha_i\}$, we set $(B';B) = i$.

Note that this complex is concentrated in degrees $\geqslant 2$ since, for $|B| = 1$, we have $k{W_B}^{\varepsilon} = k{\mathfrak{S}_2}^{\varepsilon} = 0$.

Consider the application $\phi : CD \to HC$ given by

$$k{W_B}^{W_{B \setminus \underline{\alpha}}} \ni a \mapsto \mathrm{Alt}_B(a) \tag{8}$$

Proposition 5.1. *The application ϕ is a quasi-isomorphism of complexes.*

Proof. Let $a \in k{W_B}^{W_{B\setminus\underline{\alpha}}}$. We have $\phi d_D(a) = 0 = \phi(a)$ if $\underline{\alpha} \neq B$. On the other hand, if $\underline{\alpha} = B$, we have

$$\phi d_D(a) = \sum_{\substack{B \subset B' \subset D \\ |B'|=|B|+1}} (-1)^{\mathrm{pos}(B' \setminus B, B')} \mathrm{Alt}(a) = d_D \phi(a)$$

so that ϕ is compatible with the differential.

Consider the filtration on HC given by $(HC_{\geqslant q})^p = HC^p$ for $p \geqslant q$ and $(HC_{\geqslant q})^p = 0$ for $p < q$. The morphism ϕ is a morphism of filtered complexes. Via the canonical isomorphisms of Section 3, the induced morphism $\bar{\phi}_p : CD_{\geqslant p}/CD_{\geqslant p+1} \to HC_{\geqslant p}/HC_{\geqslant p+1}$ becomes the sum over connected subdiagrams B of D of cardinality p of the morphisms ρ_B of Section 4.1. It follows that $\bar{\phi}_p$ is a quasi-isomorphism by Proposition 4.1. The proposition follows. □

Let D' be a connected subgraph of D and HC' the corresponding complex. Via the isomorphisms ϕ above, the restriction map of Section 2.3 becomes

$$\mathrm{Res}^D_{D'} : HC \to HC'$$

$$kW_B^{\varepsilon} \ni a \mapsto \begin{cases} a & \text{if } B \subset D' \\ 0 & \text{otherwise} \end{cases}$$

Remark 5.2. It seems an interesting problem to determine whether the complex (6)–(7) is the cellular cochain of a CW-complex or the Morse complex of a smooth manifold naturally associated to W.

5.2. Finite part

Assume now that W is an arbitrary Coxeter group. We proceed as in Section 5.1 for the subspace of CD corresponding to subdiagrams $B \subseteq D$ such that W_B is finite.

Let $CD_{\mathrm{inf}} \subset CD$ be the subcomplex

$$CD_{\mathrm{inf}} = \bigoplus_{\substack{B \subseteq D: \\ |W_B| = \infty}} CD_B$$

Let

$$HC_f = \bigoplus_{\substack{B \subset D: \\ |W_B| < \infty}} kW_B{}^{\varepsilon}$$

where B runs over connected subdiagrams of D which are Dynkin. Here, $kW_B{}^{\varepsilon}$ is in degree $|B|$, endowed with the differential $d^{\#}$ given by

$$\sum_{B'} (-1)^{(B';B)} \cdot \mathrm{Alt}_B : kW_B{}^{\varepsilon} \to \bigoplus_{\substack{B \subset B' \\ |B'| = |B|+1}} kW_{B'}{}^{\varepsilon}$$

Consider the application

$$\phi : CD \to HC_f$$
$$kW_B{}^{W_{B \setminus \underline{\alpha}}} \ni a \mapsto \begin{cases} \mathrm{Alt}_B(a) & \text{if } \underline{\alpha} = B \text{ and } |W_B| < \infty \\ 0 & \text{otherwise} \end{cases}$$

As in Proposition 5.1, one checks that ϕ is a morphism of complexes.

Proposition 5.3. *There is a distinguished triangle*

$$CD_{\mathrm{inf}} \xrightarrow{\mathrm{can}} CD \xrightarrow{\phi} HC_f \rightsquigarrow$$

Proof. One shows as in Proposition 5.1 that the map $CD/CD_{inf} \to HC_f$ induced by ϕ is a quasi-isomorphism. □

6. Rigidity of quasi-Coxeter algebra structures on *kW*

We apply below the results of Section 5 to show that quasi-Coxeter algebra structures on kW are rigid if W is finite. We begin by reviewing the definition of quasi-Coxeter algebras and their deformations.

6.1. Quasi-Coxeter algebras [TL2, Section 3]

Let R be a commutative ring with unit. Recall that a *quasi-Coxeter algebra* structure on RW is given by endowing it with the following data:

- **Local monodromies:** for each $i \in I$, an invertible element $S_i \in RW_{\alpha_i} \cong R\mathbb{Z}_2$;
- **Elementary associators:** for each connected subdiagram $B \subseteq D$ and vertices $\alpha_i \neq \alpha_j \in B$, an invertible element $\Phi_{(B;\alpha_i,\alpha_j)} \in RW$;

satisfying the following axioms:

- **Orientation:**

$$\Phi_{(B;\alpha_j,\alpha_i)} = \Phi^{-1}_{(B;\alpha_i,\alpha_j)}$$

- **Support:**

$$\Phi_{(B;\alpha_i,\alpha_j)} \in RW_B{}^{B\setminus\{\alpha_i,\alpha_j\}}$$

- **Braid relations:** for any connected subdiagram $B \subseteq D$ consisting of two vertices α_i, α_j such that the order m_{ij} of $s_i s_j \in W$ is finite, the following holds:

$$\mathrm{Ad}(\Phi_{(B;\alpha_i,\alpha_j)})(S_i) \cdot S_j \cdots = S_j \cdot \mathrm{Ad}(\Phi_{(B;\alpha_i,\alpha_j)})(S_i) \cdots \tag{9}$$

  where the number of factors on each side is equal to m_{ij};

as well as an additional axiom called the *generalised pentagon relations*, see [TL2, Section 3.17].

The above axioms are designed so that the elements S_i and $\Phi_{(B;\alpha_i,\alpha_j)}$ define a representation of the Tits braid group B_W on any W-module, with isomorphic quasi-Coxeter algebra structures yielding equivalent representations of B_W, see [TL2, Section 3.14].

6.2. *Deformations of quasi-Coxeter algebra structures [TL2, Section 5]*

Let now $R = k[[\hbar]]$ be the ring of formal power series in a variable $\hbar$. Let

$$\big(\{S_i\}, \{\Phi_{(B;\alpha_i,\alpha_j)}\}\big) \quad \text{and} \quad \big(\{S_i'\}, \{\Phi'_{(B;\alpha_i,\alpha_j)}\}\big)$$

be quasi-Coxeter algebra structures on $RW = kW[[\hbar]]$ such that, mod $\hbar$

$$\Phi_{(B;\alpha_i,\alpha_j)} = 1 = \Phi'_{(B;\alpha_i,\alpha_j)}$$

Assume further that $S_i = S_i'$ for any $\alpha_i \in D$, and that the two structures coincide mod $\hbar^n$ for some $n \geqslant 1$, that is that

$$\Phi'_{(B;\alpha_i,\alpha_j)} = \Phi_{(B;\alpha_i,\alpha_j)} + \hbar^n \varphi_{(B;\alpha_i,\alpha_j)} \mod \hbar^{n+1} \tag{10}$$

for any $\alpha_i \neq \alpha_j \in B \subseteq D$, where $\varphi_{(B;\alpha_i,\alpha_j)} \in kW_B{}^{B\setminus\{\alpha_i,\alpha_j\}}$.

Then, by [TL2, Theorem 5.22], $\varphi = \{\varphi_{(B;\alpha_i,\alpha_j)}\}$ is a 2-cocycle in the Dynkin complex $CD(kW)$ and the two structures are isomorphic mod $\hbar^{n+1}$ if, and only if, φ is a coboundary.

6.3. *Rigidity*

Let $\{S_i\}_{i\in I}$ be elements such that

$$S_i \in kW_{\alpha_i}[[\hbar]] \quad \text{and} \quad S_i = s_i \mod \hbar$$

Theorem 6.1. *Assume that W is finite. Then there exists, up to isomorphism, at most one quasi-Coxeter algebra structure on $kW[[\hbar]]$ with local monodromies given by the elements S_i and associators equal to* 1 mod $\hbar$.

Proof. Let $\varphi \in CD^2(kW)$ be the infinitesimal defined by (10). The linearisation of the braid relations (9) reads, for any connected subdiagram $B \subseteq D$ with vertex set $\{\alpha_i, \alpha_j\}$

$$\mathrm{Alt}_B(\varphi_{(B;\alpha_i,\alpha_j)}) = 0$$

where $\mathrm{Alt}_B : kW_B \to kW_B{}^{\varepsilon}$ is the antisymmetrisation operator.

The image of φ in $HC^2(kW) = \bigoplus_{B\subseteq D:\ |B|=2} kW_B{}^{\varepsilon}$ via the morphism (8) is therefore zero so that $[\varphi] = 0$ in $HD^2(kW)$ by Proposition 3.1. $\square$

6.4. When $k = \mathbb{C}$, one can endow $kW[[\hbar]]$ with two quasi-Coxeter algebra structures having local monodromies

$$S_i = s_i \cdot \exp(\pi\sqrt{-1}k_{\alpha_i}\hbar s_i)$$

where $q_{\alpha_i} \in \mathbb{C}$ are a set of complex weights invariant under W [TL2, Section 4]. The first structure comes from the standard one on the Iwahori–Hecke algebra $\mathcal{H}W$ obtained by quotienting the group algebra of the braid group B_W by the quadratic relations

$$(S_i - q_i)\big(S_i + q_i^{-1}\big) = 0$$

where $q_i = \exp(2\pi\sqrt{-1}k_{\alpha_i}\hbar)$, the second one underlies the monodromy of Cherednik's rational KZ connection [Ch].

By [TL2, Section 4.2.2], these two structures are isomorphic. Theorem 6.1 strengthens this result by showing that there are no other such structures with the above local monodromies.

7. Top dimensional Dynkin diagram cohomology of Coxeter groups

7.1. Sign-coinvariants of kW

Let W be a Coxeter group with system of generators $S = \{s_i\}_{i\in I}$. Let V be a W-module and set

$$\overline{V} = \sum_i V^{s_i} \quad \text{and} \quad V_\varepsilon = V/\overline{V} \tag{11}$$

Proposition 7.1.

(1) *$\overline{V}$ is invariant under W.*
(2) *For any $v \in V$ and $w \in W$,*

$$wv = \varepsilon(w)v \mod \overline{V}$$

where ε is the sign character of W. Thus, W acts on V_ε by ε.
(3) *If $V = \bigoplus_j V_j$ is a direct sum of W-submodules, then $\overline{V} = \bigoplus_j \overline{V}_j$. In particular,*

$$V_\varepsilon = \bigoplus_j (V_j)_\varepsilon$$

Proof. (i) Since $V^{s_i} = (1+s_i)V$, we have

$$s_j V^{s_i} = s_j(1+s_i)V \subset (1+s_j)(1+s_i)V + (1+s_i)V \subset V^{s_j} + V^{s_i}$$

(ii) Write $w = s_{i_1} \cdots s_{i_\ell}$. Then, for any $v \in V$

$$\begin{aligned} wv &= s_{i_1} \cdots s_{i_\ell} v \\ &= -s_{i_2} \cdots s_{i_\ell} v + (1+s_{i_1}) s_{i_2} \cdots s_{i_\ell} v \\ &= (-1)^\ell v + \sum_{j=1}^{\ell} (-1)^{j-1} (1+s_{i_j}) s_{i_{j+1}} \cdots s_{i_\ell} v \end{aligned}$$

(iii) is clear. □

Remark 7.2. Proposition 7.1 also follows from the fact that $V^{s_i} = (1+s_i)V$ so that $V_\varepsilon \simeq \varepsilon \otimes_{kW} V$.

Let now $V = kW$ endowed with the conjugation action of W and let $\mathcal{C}$ be the set of conjugacy classes of W. For any $c \in \mathcal{C}$, choose $w_c \in c$ and let A_{w_c} be the image of $w_c \in kW$ in kW_ε. Let $C_W(w_c)$ be the centraliser of w_c in W.

Proposition 7.3.

(1) *For any Coxeter group W, we have $A_{w_c} \neq 0$ when $\varepsilon(C_W(w_c)) = 1$ and*

$$kW_\varepsilon = \bigoplus_{\substack{c \in \mathcal{C}: \\ \varepsilon(C_W(w_c))=1}} kA_{w_c}$$

(2) *If W is finite, then given c such that $\varepsilon(C_W(w_c)) = 1$, the element $A^{w_c} = \sum_{w' \in W/C_W(w_c)} \varepsilon(w') w' w_c w'^{-1}$ is nonzero and*

$$kW^\varepsilon = \bigoplus_{\substack{c \in \mathcal{C}: \\ \varepsilon(C_W(w_c))=1}} kA^{w_c}$$

Proof. (i) Since $kW = \bigoplus_{c \in \mathcal{C}} \mathcal{F}_c$, where $\mathcal{F}_c = k(c)$ is the subspace spanned by elements of c, Proposition 7.1 yields $kW_W^\varepsilon = \bigoplus_{c \in \mathcal{C}} (\mathcal{F}_c)_\varepsilon$. Since W acts transitively on c, it follows from (ii) of Proposition 7.1 that $(\mathcal{F}_c)_\varepsilon$ is spanned by A_{w_c} and therefore at most one-dimensional. If w centralises w_c, then, by (ii) of Proposition 7.1,

$$A_{w_c} = A_{w w_c w^{-1}} = w A_{w_c} = \varepsilon(w) A_{w_c}$$

so that A_{w_c} is zero if the sign character is not trivial on the centraliser of w_c. Conversely, if $\varepsilon(C_W(w_c)) = 1$, the assignment $w w_c \to \varepsilon(w)$ consistently defines a nonzero linear form on $\mathcal{F}_c$ which descends to $(\mathcal{F}_c)_\varepsilon$ so that $A_{w_c} \neq 0$. (ii) readily follows from (i) and the fact that if W is finite, $V_\varepsilon \simeq V^\varepsilon$ for any W-module V. □

Consider $c \in \mathcal{C}$ such that $\varepsilon(C_W(w_c)) = 1$. Then, A_{w_c} depends only on c and we put $A_c = A_{w_c}$. Similarly, when W is finite we put $A^c = A^{w_c}$.

7.2. Top cohomology

Assume now that W is irreducible of rank n and let D be its Coxeter graph. Let $\mathcal{P}$ be the collection of proper, maximal connected subdiagrams B of D and $kW_{\mathcal{P}} \subset kW$ the span of kW_B as B varies in $\mathcal{P}$. For any $c \in \mathcal{C}$, let $\overline{A}_c$ be the class of A_c in $kW_{\varepsilon}/(kW_{\mathcal{P}} \cap \overline{kW})$.

Proposition 7.4. *We have a decomposition in one-dimensional subspaces*

$$HD^n(kW) = \bigoplus_{\substack{c\in\mathcal{C}:\\ \varepsilon(C_W(w_c))=1\\ c\cap W_B=\emptyset,\,\forall B\in\mathcal{P}}} k\overline{A}_{w_c}$$

Proof. The top degree Dynkin differential d_D^n is zero. Moreover,

$$d_D^{n-1} a_{(D;D)} = \sum_{i=1}^{n} (-1)^{i-1} \bigl(a_{(D;D\setminus\alpha_i)} - a_{(\complement_{D\setminus\alpha_i}^{D\setminus\alpha_i};D\setminus\alpha_i)}\bigr)$$

Since $a_{(D;D\setminus\alpha_i)} \in kW^{s_i}$ and $a_{(\complement_{D\setminus\alpha_i}^{D\setminus\alpha_i};D\setminus\alpha_i)} \in kW_{\complement_{D\setminus\alpha_i}^{D\setminus\alpha_i}}$,

$$\operatorname{Im} d_D^{n-1} = \sum_{i=1}^{n} kW^{s_i} + \sum_{B\in\mathcal{P}} kW_B$$

The result now follows from Proposition 7.3. $\square$

8. Finite Coxeter groups

8.1. Finite Coxeter groups of rank 2

Let $W = \mathsf{I}_2(m)$, $m \geqslant 3$, be the Coxeter group with generators s, t and relations $s^2 = 1 = t^2$ and $(st)^m = 1$. For $p = 0, \dots, m-1$, let c^p be the conjugacy class of $(st)^p$.

Proposition 8.1.

$$kW^{\varepsilon} = \bigoplus_{p=1}^{\lfloor\frac{m-1}{2}\rfloor} kA^{c^p}$$

Proof. The only conjugacy classes involved in the decomposition of Proposition 7.3 are those of words in s, t of even length and therefore those of the powers of st and ts. Since $ts = s(st)s$ we need only consider the classes c^p, $p = 1, \dots, m-1$. Moreover, since $(st)^{m-p} = (ts)^p = s(st)^p s$, $c^p = c^{m-p}$ and we may restrict our attention to $1 \leqslant p \leqslant m/2$. Finally, since for m even $(st)^{m/2}$ is central in W, the only possible relevant values of p are $1, \dots, \lfloor\frac{m-1}{2}\rfloor$. The proposition follows from the fact that for $p = 1, \dots, \lfloor\frac{m-1}{2}\rfloor$, the centraliser of $(st)^p$ in W is generated by st. $\square$

Since the differential $d^{\#}$ (7) is zero for W of rank 2, Proposition 8.1 implies the following

Theorem 8.2. *For any $m \geqslant 3$,*

$$\dim HD^p\big(k\mathsf{I}_2(m)\big) = \begin{cases} 0 & \text{if } p = 0 \\ 0 & \text{if } p = 1 \\ \lfloor \frac{m-1}{2} \rfloor & \text{if } p = 2 \\ 0 & \text{if } p \geqslant 3 \end{cases}$$

8.2. Type A_n

Let $W = \mathfrak{S}_{n+1}$ be the Weyl group of type A_n, $n \geqslant 1$. The conjugacy classes in W are parametrised by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n+1$, with c^λ the class of the product of cycles

$$\tau^\lambda = (1\,2 \cdots \lambda_1)(\lambda_1 + 1\,\lambda_1 + 2 \cdots \lambda_1 + \lambda_2) \cdots \left(\sum_{i=1}^{k-1} \lambda_i + 1 \cdots \sum_{i=1}^{k} \lambda_i\right)$$

8.2.1. For any $m \in \mathbb{N}^*$, let

$$\mathcal{O}_m = \{\lambda \vdash m \mid \lambda_i \in 2\mathbb{N} + 1,\ \forall i \text{ and } \lambda_i \neq \lambda_j,\ \forall i \neq j\} \tag{12}$$

be the set of partitions of m consisting of odd, distinct parts.

Proposition 8.3.

$$k\mathfrak{S}_{n+1}{}^{\varepsilon} = \bigoplus_{\lambda \in \mathcal{O}_{n+1}} kA^{c^\lambda}$$

Proof. Since $\varepsilon(m\,m+1 \cdots m+p-1) = (-1)^{p-1}$, the only conjugacy classes involved in the decomposition of Proposition 7.3 are those such that each λ_i is odd. Moreover, since the product

$$(m\,m+1 \cdots m+p-1)(m'\,m'+1 \cdots m'+p-1)$$

of two disjoint cycles of equal length is centralised by $\pi = (m\,m') \cdots (m+p-1\,m'+p-1)$ and $\varepsilon(\pi) = (-1)^p$, the λ_i's must all be distinct. When this last condition is fulfilled, any element centralising τ^λ is of the form

$$\zeta = (1\,2 \cdots \lambda_1)^{m_1}(\lambda_1 + 1\,\lambda_1 + 2 \cdots \lambda_1 + \lambda_2)^{m_2} \cdots \left(\sum_{i=1}^{k-1} \lambda_i + 1 \cdots \sum_{i=1}^{k} \lambda_i\right)^{m_k}$$

for some $0 \leqslant m_i \leqslant \lambda_i - 1$. Since $\varepsilon(\zeta) = \prod_i ((-1)^{\lambda_i - 1})^{m_i}$, the partitions arising in the decomposition of Proposition 7.3 are exactly those in $\mathcal{O}_{n+1}$. $\square$

8.2.2. Identify the connected subdiagrams of the Coxeter graph D of W with the subintervals of $[1, n]$ having integral endpoints so that $W_{[i,j]} \simeq \mathfrak{S}_{j-i+2}$. For any $2 \leqslant p \leqslant n$ and $1 \leqslant i \leqslant n-p+1$, let

$$A^{c^\lambda}_{[i,i+p-1]} = \mathrm{Alt}_{[i,i+p-1]}\big(\tau^\lambda_{[i,i+p-1]}\big) \in kW^{\varepsilon}_{[i,i+p-1]} \tag{13}$$

be the generator corresponding to $\lambda \in \mathcal{O}_{p+1}$. We shall need the following

Lemma 8.4.

$$\mathrm{Alt}_{[i,i+p]}\big(A^{c^\lambda}_{[i,i+p-1]}\big)=\begin{cases}A^{c^{\lambda\cup\{1\}}}_{[i,i+p]} & \textit{if } 1\notin\lambda\\ 0 & \textit{otherwise}\end{cases}$$

and

$$\mathrm{Alt}_{[i-1,i+p-1]}\big(A^{c^\lambda}_{[i,i+p-1]}\big)=\begin{cases}(-1)^{p+1}A^{c^{\lambda\cup\{1\}}}_{[i-1,i+p-1]} & \textit{if } 1\notin\lambda\\ 0 & \textit{otherwise}\end{cases}$$

provided $i\leqslant n-p$ *and* $i\geqslant 2$ *respectively.*

Proof. The first identity follows from the fact that under the inclusion $\mathfrak{S}_{[i,i+p-1]}\subset\mathfrak{S}_{[i,i+p]}$, $\tau^\lambda_{[i,i+p-1]}$ is mapped to $\tau^{\lambda\cup\{1\}}_{[i,i+p]}$. The second one follows from the first and the fact that, in $\mathfrak{S}_{[i-1,i+p-1]}$,

$$\tau^\lambda_{[i,i+p-1]}=\mathrm{Ad}\big((i-1\,i\cdots i+p)\big)\tau^\lambda_{[i-1,i+p-2]}$$

and $\varepsilon(i-1\,i\cdots i+p)=(-1)^{p+1}$. □

8.2.3. Label the nodes of D as in [Bo, Planche I] and order them as $\alpha_1<\cdots<\alpha_n$. For any $p=1,\dots,n$, let

$$d_p^\#:\bigoplus_{i=1}^{n-p+1}kW_{[i,i+p-1]}{}^\varepsilon\to\bigoplus_{j=1}^{n-p}kW_{[j,j+p]}{}^\varepsilon$$

be the differential of HC, where the right-hand side is understood to be 0 if $p=n$. Set

$$\mathcal{O}^*_{p+1}=\{\lambda\in\mathcal{O}_{p+1}\mid 1\notin\lambda\}\tag{14}$$

Proposition 8.5. *For any* $p=2,\dots,n$,

$$\mathrm{Im}\,d^\#_{p-1}=\bigoplus_{i=1}^{n-p+1}\bigoplus_{\substack{\lambda\in\mathcal{O}_{p+1}:\\1\in\lambda}}kA^{c^\lambda}_{[i,i+p-1]}\tag{15}$$

$$\mathrm{Ker}\,d^\#_p=\mathrm{Im}\,d^\#_{p-1}\oplus\bigoplus_{\lambda\in\mathcal{O}^*_{p+1}}kB^\lambda\tag{16}$$

where $B^\lambda=\sum_{i=1}^{n-p+1}A^{c^\lambda}_{[i,i+p-1]}$.

Proof. Since

$$\big([i-1,i+p-2];[i,i+p-2]\big)=1\quad\text{and}\quad\big([i,i+p-1];[i,i+p-2]\big)=p$$

Lemma 8.4 yields, for any $\bar{\lambda} \in \mathcal{O}_p$ and $i = 1, \ldots, n-p+2$,

$$d^{\#}_{p-1} A^{c^{\bar{\lambda}}}_{[i,i+p-2]} = \delta_{1 \notin \bar{\lambda}} \cdot (-1)^p \left(-\delta_{i \geqslant 2} \cdot A^{c^{\bar{\lambda} \cup \{1\}}}_{[i-1,i+p-2]} + \delta_{i \leqslant n-p+1} \cdot A^{c^{\bar{\lambda} \cup \{1\}}}_{[i,i+p-1]}\right) \tag{17}$$

It follows that, for any $\lambda \in \mathcal{O}^*_p$,

$$d^{\#}_{p-1}\left(\sum_{j=1}^{i} A^{c^{\bar{\lambda}}}_{[j,j+p-2]}\right) = (-1)^p A^{c^{\bar{\lambda} \cup \{1\}}}_{[i,i+p-1]} \tag{18}$$

which yields (15). It also follows from (17) that, for $\lambda \in \mathcal{O}^*_{p+1}$, B^{λ} is the unique linear combination of $A^{c^{\lambda}}_{[i,i+p-1]}$ such that $d^{\#}_p B^{\lambda} = 0$, which yields (16). □

8.2.4.

Theorem 8.6. *For any $0 \leqslant p \leqslant n$,*

$$\dim HD^p(k\mathfrak{S}_{n+1})_{c^{\lambda}} = \begin{cases} 1 & \text{if } \lambda = \lambda' \cup \{1^{n-p}\} \text{ and } \lambda' \in \mathcal{O}^*_{p+1} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Proof. (19) holds for $p \geqslant 2$ by Proposition 8.5 and therefore for $p \geqslant 0$ since $\dim HD^i(k\mathfrak{S}_{n+1}) = 0 = |\mathcal{O}^*_{i+1}|$ for $i = 0, 1$. □

8.2.5. Generating function

Theorem 8.7. *Set*

$$\chi^{\mathsf{A}}(q,t) = \sum_{n \geqslant 1, p \geqslant 0} q^n t^p \dim HD^p(k\mathfrak{S}_{n+1})$$

Then,

$$\chi^{\mathsf{A}}(q,t) = \frac{1}{1-q} \frac{\prod_{d \geqslant 1}(1+(qt)^{2d+1}) - 1}{qt}$$

Proof. Since $HD^p(k\mathfrak{S}_{n+1}) = 0$ if $p = 0$ or $p \geqslant n+1$, we have

$$\begin{aligned} \chi^{\mathsf{A}}(q,t) &= \sum_{n \geqslant p \geqslant 1} q^n t^p \dim HD^p(k\mathfrak{S}_{n+1}) \\ &= \sum_{p \geqslant 1} \frac{t^p q^p}{1-q} \left|\mathcal{O}^*_{p+1}\right| \\ &= \frac{1}{1-q} \frac{\prod_{d \geqslant 1}(1+(qt)^{2d+1}) - 1}{qt} \end{aligned}$$

where the last identity follows from the fact that

$$\sum_{m \geqslant 1} z^m \left|\mathcal{O}^*_m\right| = \sum_{m \geqslant 2} z^m \left|\mathcal{O}^*_m\right| = \left(1+z^3\right)\left(1+z^5\right)\left(1+z^7\right)\cdots - 1 \qquad \square \tag{20}$$

Remark 8.8. Up to a multiplication by qt, the generating function χ^{A} is the product of a Fermionic partition function by that of a one-dimensional harmonic oscillator. It would be interesting to know whether the direct sum

$$\bigoplus_{n\geqslant 1, p\geqslant 0} HD^p(k\mathfrak{S}_{n+1})$$

possesses a natural action of an infinite-dimensional Clifford algebra similar in spirit to that on the cohomology of the Hilbert schemes of points on a surface [Gr,Na].

8.3. Type B_n

Let now $W = \mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ be the Weyl group of type B_n, $n \geqslant 2$, and denote the generators of $\mathbb{Z}_2^n$ by ε_i, $i = 1, \dots, n$. The conjugacy classes in W are parametrised by ordered pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = n$, where $|\lambda| = \sum_i \lambda_i$ [GP, Proposition 3.4.7]. The class $c^{(\lambda,\mu)}$ is that of the product $\tau^\lambda \widetilde{\tau}^\mu$, where

$$\tau^\lambda = (1\,2\cdots\lambda_1)(\lambda_1+1\,\lambda_1+2\cdots\lambda_1+\lambda_2)\cdots\big(|\lambda|-\lambda_k+1\cdots|\lambda|\big) \tag{21}$$

and

$$\widetilde{\tau}^\mu = \big(|\lambda|+1\cdots|\lambda|+\mu_1\big)\varepsilon_{|\lambda|+\mu_1}\cdots(n-\mu_\ell+1\cdots n)\varepsilon_n \tag{22}$$

Proposition 8.9.

$$k\big(\mathfrak{S}_n \ltimes \mathbb{Z}_2^n\big)^\varepsilon = \begin{cases} \bigoplus_{\nu\vdash n/2} kA^{c^{(\emptyset,2\nu)}} & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. A necessary condition for a conjugacy class $c^{(\lambda,\mu)}$ to contribute to the decomposition of Proposition 7.3 is that the λ_i be odd and the μ_j even since $\varepsilon(\varepsilon_i) = -1$. Since $(m\,m+1\cdots m+p-1)$ is centralised by $\varepsilon_m\cdots\varepsilon_{m+p-1}$ and $\varepsilon(\varepsilon_m\cdots\varepsilon_{m+p-1}) = (-1)^p$, λ must in fact be the empty partition. In particular, n must be even.[3] There remains to show that if $\mu \vdash n$ only contains even parts, the sign character ε is trivial on the centraliser of $\widetilde{\tau}^\mu$. This is readily reduced to the case when μ only has one part which follows in turn from the following result. □

Lemma 8.10. *The centraliser of* $(1\cdots p)\varepsilon_p$ *in* $\mathfrak{S}_p \ltimes \mathbb{Z}_2^p$ *is the group* $\mathbb{Z}_{2p}$ *generated by* $(1\cdots p)\varepsilon_p$.

Proof. If $w \in \mathfrak{S}_p \ltimes \mathbb{Z}_2^p$ centralises $(1\cdots p)\varepsilon_p$, its projection in $\mathfrak{S}_p$ centralises $(1\cdots p)$ and is therefore equal to $(1\cdots p)^s$ for some $0 \leqslant s \leqslant p-1$. Thus $((1\cdots p)\varepsilon_p)^{-s}w$ centralises $(1\cdots p)\varepsilon_p$ and lies in $\mathbb{Z}_2^p$ from which it readily follows that it is either equal to 1 or to $\varepsilon_1\cdots\varepsilon_p = ((1\cdots p)\varepsilon_p)^p$. □

[3] This also follows from the fact that the central element $\zeta = \varepsilon_1\cdots\varepsilon_n$ is of sign $(-1)^n$ so that

$$kW^\varepsilon \subseteq \big\{f \in kW \bigm| \zeta f \zeta = (-1)^n f\big\}$$

is zero if n is odd.

Proposition 8.11. *For any $0 \leqslant p \leqslant n$, we have*

$$\dim HD^p(kW_{B_n})_{c^{\lambda,\mu}} = \begin{cases} 1 & \text{if } (\mu = 0 \text{ and } \lambda = \lambda' \cup \{1^{n-p-1}\} \text{ with } \lambda' \in \mathcal{O}^*_{p+1}) \\ & \text{or } (\lambda = (1^{n-p}) \text{ and } \mu = 2\nu \text{ with } \nu \vdash p/2) \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$\dim HD^p(W_{B_n}) = \dim HD^p(k\mathfrak{S}_n) + \delta_{p \in 2\mathbb{N}^*} \cdot P(p/2)$$

where P is the partition function.

Proof. Identify the connected subdiagrams of the Coxeter graph D of W with the subintervals of $[1,n]$ having integral endpoints so that

$$W_{[i,j]} \simeq \begin{cases} \mathfrak{S}_{j-i+2} & \text{if } j \leqslant n-1 \\ \mathfrak{S}_{n-i+1} \ltimes \mathbb{Z}_2^{n-i+1} & \text{if } j = n \end{cases}$$

For $1 \leqslant p \leqslant n-1$ and $1 \leqslant i \leqslant n-p$, let

$$A^{c^\lambda}_{[i,i+p-1]} \in kW^{\varepsilon}_{[i,i+p-1]}$$

be the generator corresponding to $\lambda \in \mathcal{O}_{p+1}$, as in (13). By Proposition 8.9, $\mathrm{Alt}_{[i,n]}(A^{c^\lambda}_{[i,n-1]}) = 0$. Since, in addition $kW_{[n-p+1,n]}{}^{\varepsilon}$ is zero whenever p is odd and of dimension $|\{\nu \vdash p/2\}| = P(p/2)$ when p is even, the complex $(\bigoplus_{B \subseteq D} kW_B{}^{\varepsilon}; d^{\#})$ of Section 5 decomposes as the direct sum of the corresponding complex for $\mathfrak{S}_n$ and a complex concentrated in positive, even degrees with chain groups of dimension $\delta_{p \in 2\mathbb{N}^*} \cdot P(p/2)$. □

Theorem 8.12. *Set $\chi^{\mathsf{B}}(q,t) = \sum_{n \geqslant 2,\, p \geqslant 0} q^n t^p \dim HD^p(kW_{B_n})$. Then,*

$$\chi^{\mathsf{B}}(q,t) = \frac{\prod_{d \geqslant 1}(1 + (qt)^{2d+1}) - 1}{(1-q)t} + \frac{\prod_{d \geqslant 1}(1 - (qt)^{2d})^{-1} - 1}{1-q}$$

Proof. This follows from Proposition 8.11, Theorem 8.7 and the fact that

$$\sum_{m \geqslant 0} z^m P(m) = \prod_{d \geqslant 1} \left(1 - z^d\right)^{-1} \qquad \square$$

8.4. Type D_n

Let $\mathbb{Z}^n_{2,+} \subset \mathbb{Z}^n_2$ be the kernel of the sign character and $W = \mathfrak{S}_n \ltimes \mathbb{Z}^n_{2,+}$ the Weyl group of type D_n, $n \geqslant 3$. The conjugacy classes in W fall into two types [GP, Proposition 3.4.12]:

- type I. These are labelled by ordered pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = n$ and the number of parts $[\mu]$ of μ is even. The corresponding class $c^{(\lambda,\mu)}$ is that of the product $\tau^\lambda \widetilde{\tau}^\mu$, where τ^λ, $\widetilde{\tau}^\mu$ are given by (21)–(22).
- type II. These are labelled by partitions λ of n all of whose parts are even, with $c^{\lambda,\mathrm{II}}$ the class of

$$\tau^{\lambda,\mathrm{II}} = (1\,2 \cdots \lambda_1) \cdots (n - \lambda_{k-1} - \lambda_k + 1 \cdots n - \lambda_k)(n - \lambda_k + 1 \cdots n)\varepsilon_{n-1}\varepsilon_n$$

8.4.1.

Proposition 8.13.

$$\left(k\mathfrak{S}_n \ltimes \mathbb{Z}_{2,+}^n\right)^{\varepsilon} = \bigoplus_{\substack{0\leqslant m\leqslant n\\ \lambda\in\mathcal{O}_m\\ \mu\in\mathcal{O}_{n-m}:\\ [\mu]\in 2\mathbb{N}}} kA^{c^{(\lambda,\mu)}} \oplus \begin{cases} \bigoplus_{\nu\vdash n/2:\ [\nu]\in 2\mathbb{N}} kA^{c^{(\emptyset,2\nu)}} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} \tag{23}$$

Proof. Conjugacy classes of type II do not contribute to the decomposition of Proposition 7.3 since $\tau^{\lambda,\mathrm{II}}$ is centralised by

$$w = (n-\lambda_k+1\cdots n)\varepsilon_{n-1}\varepsilon_n$$

and $\varepsilon(w) = (-1)^{\lambda_k-1} = -1$. Consider now a class $c^{(\lambda,\mu)}$ of type I which contributes to the decomposition of Proposition 7.3. As in Proposition 8.3, the λ_i must be odd and distinct if λ is nonempty. Moreover, since for any $1\leqslant i<j\leqslant \ell$

$$w_{i,j} = \left(|\lambda| + \sum_{a=1}^{i-1}\mu_a + 1\cdots|\lambda| + \sum_{a=1}^{i}\mu_a\right)\varepsilon_{|\lambda|+\sum_{a=1}^{i}\mu_a} \cdot \left(|\lambda| + \sum_{a=1}^{j-1}\mu_a + 1\cdots|\lambda| + \sum_{a=1}^{j}\mu_a\right)\varepsilon_{|\lambda|+\sum_{a=1}^{j}\mu_a} \in W$$

centralises $\tau^{\lambda}\widetilde{\tau}^{\mu}$ and $\varepsilon(w_{i,j}) = (-1)^{\mu_i+\mu_j}$, all μ_i must be of the same parity. Finally, since for λ_1 odd,

$$w = (1\,2\cdots\lambda_1)\varepsilon_1\cdots\varepsilon_{\lambda_1}\cdot\left(|\lambda|+1\cdots|\lambda|+\mu_1\right)\varepsilon_{|\lambda|+\mu_1}$$

lies in W, centralises $\tau^{\lambda}\widetilde{\tau}^{\mu}$ and $\varepsilon(w) = (-1)^{\mu_1-1}$, all μ_i must be odd, and therefore distinct, if λ is nonempty.

There remains to show that if c is a conjugacy class appearing on the right-hand side of (23), then ε is trivial on the centraliser of any element of c. If $c = c^{(\emptyset,\mu)}$ with all μ_i even, the centraliser of $\widetilde{\tau}^{\mu}$ in $\mathbb{Z}_2^n \rtimes \mathfrak{S}_n$ lies in the kernel of ε by Proposition 8.9. *A fortiori*, this is true in $\mathbb{Z}_{2,+}^n \rtimes \mathfrak{S}_n$. If, on the other hand, $c = c^{(\lambda,\mu)}$, where λ and μ are either empty or consist of odd, distinct parts, it follows from Lemma 8.14 below that the component $\sigma\in\mathfrak{S}_n$ of any $w\in W$ centralising $\tau^{\lambda}\widetilde{\tau}^{\mu}$ is of the form

$$(1\,2\cdots\lambda_1)^{m_1}\cdots\left(|\lambda|-\lambda_k+1\cdots|\lambda|\right)^{m_k} \cdot\left(|\lambda|+1\cdots|\lambda|+\mu_1\right)^{m_1'}\cdots(n-\mu_\ell+1\cdots n)^{m_\ell'}$$

so that $\varepsilon(w) = \varepsilon(\sigma) = 1$. □

Lemma 8.14. *The centraliser of* $(1\cdots p)(p+1\cdots 2p)\varepsilon_{2p}$ *in* $\mathfrak{S}_{2p}\ltimes\mathbb{Z}_2^{2p}$ *is the product of the centralisers of* $(1\cdots p)$ *and* $(p+1\cdots 2p)\varepsilon_{2p}$ *in* $\mathfrak{S}_{\{1,\dots,p\}}\ltimes\mathbb{Z}_2^p$ *and* $\mathfrak{S}_{\{p+1,\dots,2p\}}\ltimes\mathbb{Z}_2^p$ *respectively.*

8.4.2. Label now the nodes of the Dynkin diagram D of W as in [Bo, Planche IV], so that α_{n-2} is the trivalent node of D if $n \geqslant 4$, and order them as $\alpha_1 < \cdots < \alpha_{n-2} < \alpha_{n-1} < \alpha_n$. For any $1 \leqslant i \leqslant j \leqslant n-2$, let $[i,j] \subset D$ be the connected subdiagram with vertices $\alpha_i, \ldots, \alpha_j$. For any $i = 1, \ldots, n-2$, let $B_i^{\pm} \subset D$ be the connected subdiagrams with vertices $\alpha_i, \ldots, \alpha_{n-2}$ and α_{n-1} (resp. α_n) and $B_i^Y \subseteq D$ the subdiagram with vertices $\alpha_i, \ldots, \alpha_n$. Thus,

$$W_{[i,j]} \simeq \mathfrak{S}_{j-i+2}, \quad W_{B_i^{\pm}} \simeq \mathfrak{S}_{n-i+1} \quad \text{and} \quad W_{B_i^Y} \simeq \mathfrak{S}_{n-i+1} \ltimes \mathbb{Z}_{2,+}^{n-i+1}$$

Let $\sigma \in \mathrm{Aut}(W)$ be the involution induced by fixing the nodes $\alpha_1, \ldots, \alpha_{n-2}$ and permuting α_{n-1} and α_n so that $\sigma(W_{B_i^+}) = W_{B_i^-}$. For any $\lambda \in \mathcal{O}_{n-i+1}$, set $A_{B_i^-}^{c^\lambda} = \sigma A_{B_i^+}^{c^\lambda}$.

Proposition 8.15. *The following holds for* $3 \leqslant p \leqslant n$,

$$\begin{aligned} \mathrm{Im}\, d_{p-1}^{\#} = & \bigoplus_{\substack{\lambda \in \mathcal{O}_{p+1}:\\ 1 \in \lambda}} \bigoplus_{i=1}^{n-p-1} kA_{[i,i+p-1]}^{c^\lambda} \\ & \oplus \bigoplus_{\substack{\lambda \in \mathcal{O}_{p+1}:\\ 1 \in \lambda}} V^\lambda \\ & \oplus \bigoplus_{\substack{1 \leqslant m \leqslant p \\ \overline{\lambda} \in \mathcal{O}_m \\ \overline{\mu} \in \mathcal{O}_{p-m}: \\ 1 \in \overline{\lambda}, [\overline{\mu}] \in 2\mathbb{N}}} kA_{B_{n-p+1}^Y}^{c^{(\overline{\lambda}, \overline{\mu})}} \end{aligned} \tag{24}$$

where the first summand only arises if $p \leqslant n-2$ *and*

$$V^\lambda = \begin{cases} \langle A_{B_{n-p}^+}^{c^\lambda} + A_{B_{n-p}^-}^{c^\lambda}, A_{B_{n-p}^+}^{c^\lambda} - A_{B_{n-p+1}^Y}^{c^{(\lambda \setminus \{1\}, \emptyset)}}, A_{B_{n-p}^-}^{c^\lambda} + A_{B_{n-p+1}^Y}^{c^{(\lambda \setminus \{1\}, \emptyset)}} \rangle & \text{if } p \leqslant n-1 \\ kA_{B_1^Y}^{c^{(\lambda \setminus \{1\}, \emptyset)}} & \text{if } p = n \end{cases} \tag{25}$$

Proof. By (18), the image of the restriction of $d_{p-1}^{\#}$ to

$$\bigoplus_{j=1}^{n-p-1} kW_{[j,j+p-2]}{}^{\varepsilon}$$

is the span of the generators $A_{[i,i+p-1]}^{c^\lambda}$, $i = 1, \ldots, n-p-1$, as λ runs through the elements of $\mathcal{O}_{p+1}$ containing 1. This accounts for the first summand in (24). Further, since

$$\big([n-p-1, n-2]; [n-p, n-2]\big) = 1 \quad \text{and} \quad \big(B_{n-p}^{\pm}; [n-p, n-2]\big) = p$$

Lemma 8.4 yields, for any $\overline{\lambda} \in \mathcal{O}_p$,

$$d_{p-1}^{\#}\big(A_{[n-p,n-2]}^{c^{\overline{\lambda}}}\big) = (-1)^p \cdot \delta_{1 \notin \overline{\lambda}} \cdot \big(-A_{[n-p-1,n-2]}^{c^{\overline{\lambda} \cup \{1\}}} + A_{B_{n-p}^+}^{c^{\overline{\lambda} \cup \{1\}}} + A_{B_{n-p}^-}^{c^{\overline{\lambda} \cup \{1\}}}\big)$$

Thus, if $p \leqslant n-1$ and $\lambda \in \mathcal{O}_{p+1}$ contains 1,

$$A^{c^{\lambda}}_{B^+_{n-p}} + A^{c^{\lambda}}_{B^-_{n-p}} \in \operatorname{Im} d^{\#}_{p-1} \tag{26}$$

To proceed, we need the following

Lemma 8.16. *For any* $\overline{\lambda} \in \mathcal{O}_p$,

$$\operatorname{Alt}_{B^Y_{n-p+1}}\big(A^{c^{\overline{\lambda}}}_{B^{\pm}_{n-p+1}}\big) = A^{c^{(\overline{\lambda},\emptyset)}}_{B^Y_{n-p+1}}$$

Proof. The "+" identity follows from the fact that, under the inclusion $W_{B^+_{n-p+1}} \subset W_{B^Y_{n-p+1}}$, $\tau^{\overline{\lambda}}_{B^+_{n-p+1}}$ is mapped to $\tau^{\overline{\lambda}}_{B^Y_{n-p+1}}$. The "−" one follows by applying the automorphism σ and noticing that $A^{c^{(\overline{\lambda},\emptyset)}}_{B^Y_{n-p+1}}$ is fixed by σ. Indeed if $1 \in \overline{\lambda}$, $\tau^{\overline{\lambda}}_{B^Y_{n-p+1}}$ lies in $W_{[n-p+1,n-2]}$ and is therefore fixed by σ. If on the other hand $1 \notin \overline{\lambda}$, the cycle $(n-\overline{\lambda}_k+1\cdots n)$ is the product $s_{n-\overline{\lambda}_k+1}\cdots s_{n-1}$, so that

$$\begin{aligned}\sigma(n-\overline{\lambda}_k+1\cdots n) &= s_{n-\overline{\lambda}_k+1}\cdots s_{n-2}s_n\\ &= (n-\overline{\lambda}_k+1\cdots n)\varepsilon_{n-1}\varepsilon_n\\ &= \operatorname{Ad}(\varepsilon_{n-\overline{\lambda}_k+1}\cdots\varepsilon_{n-1})(n-\overline{\lambda}_k+1\cdots n)\end{aligned}$$

whence

$$\sigma A^{c^{(\overline{\lambda},\emptyset)}}_{B^Y_{n-p+1}} = \operatorname{Alt}_{B^Y_{n-p+1}}\big(\operatorname{Ad}(\varepsilon_{n-\overline{\lambda}_k+1}\cdots\varepsilon_{n-1})\tau^{\overline{\lambda}}_{B^Y_{n-p+1}}\big) = A^{c^{(\overline{\lambda},\emptyset)}}_{B^Y_{n-p+1}}$$

since $\varepsilon(\varepsilon_{n-\overline{\lambda}_k+1}\cdots\varepsilon_{n-1}) = (-1)^{\overline{\lambda}_k-1}$ and $\overline{\lambda}_k$ is odd. □

Since

$$\big(B^{\pm}_{n-p}; B^{\pm}_{n-p+1}\big) = 1, \qquad \big(B^Y_{n-p+1}; B^+_{n-p+1}\big) = p \quad \text{and} \quad \big(B^Y_{n-p+1}; B^-_{n-p+1}\big) = p-1$$

Lemmas 8.4 and 8.16 imply that, for any $\overline{\lambda} \in \mathcal{O}_p$,

$$d^{\#}_{p-1}\big(A^{c^{\overline{\lambda}}}_{B^{\pm}_{n-p+1}}\big) = (-1)^p\big(-\delta_{1\notin\overline{\lambda}}\cdot A^{c^{\overline{\lambda}\cup\{1\}}}_{B^{\pm}_{n-p}} \pm A^{c^{(\overline{\lambda},\emptyset)}}_{B^Y_{n-p+1}}\big) \tag{27}$$

Choosing $\overline{\lambda} \in \mathcal{O}_p$ such that $1 \notin \overline{\lambda}$ in (27) and using (26) accounts for the second summand in (24). To conclude, we need the following straightforward consequence of Lemma 8.4.

Lemma 8.17. *For any* $4 \leqslant p \leqslant n$,

$$\operatorname{Alt}_{B^Y_{n-p+1}}\big(A^{c^{(\lambda,\mu)}}_{B^Y_{n-p+2}}\big) = \begin{cases} 0 & \text{if } 1 \in \lambda \text{ or } \mu_i \in 2\mathbb{N}^*\ \forall i\\ (-1)^{|\lambda|}\cdot A^{c^{(\lambda\cup\{1\},\mu)}}_{B^Y_{n-p+1}} & \text{otherwise}\end{cases}$$

Since $d^{\#}_{p-1}(f) = -\operatorname{Alt}_{B^Y_{n-p+1}}(f)$ for any $f \in kW_{B^Y_{n-p+2}}{}^{\varepsilon}$, Lemma 8.17 accounts for the third summand in (24). □

8.4.3.

Proposition 8.18. *The following holds for any $2 \leqslant p \leqslant n$,*

$$\operatorname{Ker} d_p^\# = \operatorname{Im} d_{p-1}^\# \oplus \bigoplus_{\substack{\nu \vdash p/2: \\ [\nu] \in 2\mathbb{N}}} kA_{B_{n-p+1}^Y}^{c^{(\emptyset,2\nu)}} \oplus \begin{cases} \bigoplus_{\lambda \in \mathcal{O}_{p+1}^*} k\widetilde{B}_\lambda & \text{if } p \leqslant n-1 \\ \bigoplus_{0 \leqslant m < n,\, \lambda \in \mathcal{O}_m^*,\, \mu \in \mathcal{O}_{n-m}:\, [\mu] \in 2\mathbb{N}} kA_{B_1^Y}^{c^{(\lambda,\mu)}} & \text{if } p = n \end{cases} \tag{28}$$

where the second summand only arises if p is even and greater or equal to 3, $\mathcal{O}_q^* = \{\lambda \in \mathcal{O}_q \mid 1 \notin \lambda\}$ for $q \geqslant 1$, $\mathcal{O}_0^* = \{\emptyset\}$ and

$$\widetilde{B}_\lambda = \sum_{i=1}^{n-p-1} A_{[i,i+p-1]}^{c^\lambda} + A_{B_{n-p}^+}^{c^\lambda} + A_{B_{n-p}^-}^{c^\lambda}$$

Proof. For $p \geqslant 3$ even, the subspace spanned by $A_{B_{n-p+1}^Y}^{c^{(\emptyset,2\nu)}}$, $\nu \vdash p/2$, lies in $\operatorname{Ker} d_p^\#$ by Lemma 8.17 and is in direct sum with $\operatorname{Im} d_{p-1}^\#$ by Proposition 8.15. This accounts for the second summand in (28). Let now $f \in \operatorname{Ker} d_p^\#$ be such that all components of $f_{B_{n-p+1}^Y}$ of type $c^{(\emptyset,2\nu)}$, $\nu \vdash p/2$, are zero. If $p = n$, f lies in the span of the elements $A_{B_1^Y}^{c^{(\lambda,\mu)}}$, $(\lambda,\mu) \in \mathcal{O}_m \times \mathcal{O}_{n-m}$ and $0 \leqslant m \leqslant n$. Since $d_n^\# = 0$ and

$$\operatorname{Im} d_{n-1}^\# = \bigoplus_{\lambda \in \mathcal{O}_n^*} kA_{B_1^Y}^{c^{(\lambda,\emptyset)}} \oplus \bigoplus_{\substack{1 \leqslant m \leqslant n \\ \overline{\lambda} \in \mathcal{O}_m \\ \overline{\mu} \in \mathcal{O}_{n-m}: \\ 1 \in \overline{\lambda}, [\overline{\mu}] \in 2\mathbb{N}}} kA_{B_1^Y}^{c^{(\overline{\lambda},\overline{\mu})}}$$

by Proposition 8.15, (28) holds for $p = n$. Assume now that $p \leqslant n-1$. By Lemmas 8.16–8.17, the restriction of $d_p^\#$ is injective on the span of the elements $A_{B_{n-p+1}^Y}^{c^{(\lambda,\mu)}}$ where $(\lambda,\mu) \in \mathcal{O}_m \times \mathcal{O}_{p-m}$, $0 \leqslant m \leqslant p$, are such that λ does not contain 1 and μ is nonempty. The corresponding components of f are therefore zero. Since $A_{B_{n-p+1}^Y}^{c^{(\lambda,\mu)}}$ lies in $\operatorname{Im} d_{p-1}^\#$ if $1 \in \lambda$ by Lemma 8.17, we may therefore assume that $f_{B_{n-p+1}^Y}$ only has components along $A_{B_{n-p+1}^Y}^{c^{(\lambda,\emptyset)}}$, $\lambda \in \mathcal{O}_p^*$. Working modulo the first summand of $\operatorname{Im} d_{p-1}^\#$ given by Proposition 8.15, we may further assume that f lies in the span of

$$\bigoplus_{\lambda \in \mathcal{O}_{p+1}^*} \bigoplus_{i=1}^{n-p-1} kA_{[i,i+p-1]}^{c^\lambda} \oplus \bigoplus_{\lambda \in \mathcal{O}_{p+1}} \left(kA_{B_{n-p}^+}^{c^\lambda} \oplus kA_{B_{n-p}^+}^{c^\lambda}\right) \oplus \bigoplus_{\overline{\lambda} \in \mathcal{O}_p^*} kA_{B_{n-p+1}^Y}^{c^{(\overline{\lambda},\emptyset)}} \tag{29}$$

Let $\lambda \in \mathcal{O}_{p+1}^*$. It readily follows from (16) and (27) that $\widetilde{B}_\lambda \in \operatorname{Ker} d_p^\#$. Next, applying (16) to the c^λ-components of f along $kW_{[i,i+p-1]}{}^\varepsilon$, $i = 1, \ldots, n-p-1$ and $kW_{B_{n-p}^\pm}{}^\varepsilon$, we see that these components are equal to $a_\lambda A_{[i,i+p-1]}^{c^\lambda}$ and $a_\lambda A_{B_{n-p}^\pm}^{c^\lambda}$ respectively, for some constant a_λ. Thus, subtracting $a_\lambda \widetilde{B}_\lambda$ to f, we may assume that all these components are equal to zero and therefore that f lies in

$$\bigoplus_{\substack{\lambda \in \mathcal{O}_{p+1}: \\ 1 \in \lambda}} \left(kA_{B_{n-p}^+}^{c^\lambda} \oplus kA_{B_{n-p}^+}^{c^\lambda} \oplus kA_{B_{n-p+1}^Y}^{c^{(\lambda \setminus \{1\},\emptyset)}}\right) \tag{30}$$

By (27) and Lemma 8.17 any solution of $d_p^\# f = 0$ with values in (30) lies in the subspace $\bigoplus_{\lambda\in\mathcal{O}_{p+1}:\ 1\in\lambda} V^\lambda$ defined by (25) and therefore in $\operatorname{Im} d_{p-1}^\#$. □

8.4.4.

Theorem 8.19. *For any $0 \leqslant p < n$, we have*

$$\dim HD^p(kW_{D_n})_c = \begin{cases} 1 & \text{if } (c = c^{(1^{n-p},2\nu)} \text{ for some } \nu \vdash p/2 \text{ with } [\nu] \in 2\mathbb{N}) \\ & \text{or } (c = c^{(\lambda\cup\{1^{n-p-1}\},\emptyset)} \text{with } \lambda \in \mathcal{O}_{p+1}^*) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim HD^n(kW_{D_n})_c = \begin{cases} 1 & \text{if } c = c^{\lambda,\mu} \text{ with } \lambda \in \mathcal{O}_m^* \text{ and } \mu \in \mathcal{O}_{n-m} \text{ and } [\mu] \in 2\mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

8.5. Stabilisation

Assume that W is of classical type $\mathsf{X} = \mathsf{A}$, B or D. Then, using the description of the Dynkin cohomology of kW given in this section, one readily checks that for $n \geqslant m$, the restriction map on Dynkin diagram cohomology defined in Section 2.3

$$\operatorname{Res}_{\mathsf{X}_m}^{\mathsf{X}_n} : HD^p(kW_{\mathsf{X}_n}) \to HD^p(kW_{\mathsf{X}_m})$$

is an isomorphism for $p \leqslant m$.

8.6. Exceptional groups

8.6.1. G_2

The Weyl group of type $\mathsf{G}_2 = \mathsf{I}(6)$ was treated in Section 8.1.

8.6.2. F_4

Let W be the Weyl group of type F_4 and label the connected subdiagrams of D by the subintervals of $[1,4]$ with integral endpoints. Thus, $W_{[1,3]}$ and $W_{[2,4]}$ are of type B_3, $W_{[1,2]}$ and $W_{[3,4]}$ are of type A_2, and $W_{[2,3]}$ is of type B_2. It follows from Propositions 8.3 and 8.9 that

$$kW_{[1,3]}{}^\varepsilon = 0, \qquad kW_{[2,4]}{}^\varepsilon = 0$$
$$kW_{[1,2]}{}^\varepsilon \simeq k, \qquad kW_{[2,3]}{}^\varepsilon \simeq k, \qquad kW_{[3,4]}{}^\varepsilon \simeq k$$

The differential $d^\#$ of HC is therefore equal to zero, so that

$$\dim HD^p(kW) = \begin{cases} 0 & \text{if } p = 0 \\ 0 & \text{if } p = 1 \\ 3 & \text{if } p = 2 \\ 0 & \text{if } p = 3 \\ \dim kW^\varepsilon & \text{if } p = 4 \\ 0 & \text{if } p \geqslant 5 \end{cases}$$

8.6.3. GAP calculations

For the groups H_3, H_4, E_6, E_7 and E_8, the dimension of the cohomology spaces of the complex HC can be readily computed with the computer algebra package [GAP]. For each B, we enumerate conjugacy classes of W_B and select those whose elements have a centraliser in the kernel of ε. Then, we compute the matrices corresponding to the differentials and determine their rank. We indicate the results in the following table, where the columns provide $\dim HD^i(kW)$, $i = 2, \dots, n$.

$i=$	2	3	4	5	6	7	8
E_6	1	0	2	0	4		
E_7	1	0	2	0	7	0	
E_8	1	0	2	0	6	1	17
F_4	3	0	5				
G_2	2						
H_3	3	0					
H_4	3	0	16				

9. Affine Coxeter groups

The methods of Sections 4.2 and 5.2 provide a partial computation of the Dynkin cohomology of affine Weyl groups. Let D_0 be a finite crystallographic Dynkin diagram with n vertices and let D be its completion. Let V be the reflection representation of $W_0 = W_{D_0}$ and let $Q \subset V$ be the coroot lattice, so that $W \simeq Q \rtimes W_0$.

We have canonical Serre duality isomorphisms

$$\mathrm{Ext}^i_{kQ}(k, M) \xrightarrow{\sim} \mathrm{Ext}^{n-i}_{kQ}(M, k)^* \otimes \Lambda^n V^*$$

for any finitely generated kQ-module M and any integer i. Thus, given a finitely generated kW-module M and an integer i, we have isomorphisms

$$\mathrm{Ext}^i_{kW}(k, M) \simeq \mathrm{Ext}^i_{kQ}(k, M)^{W_0} \simeq \left(\mathrm{Ext}^{n-i}_{kQ}(M, \varepsilon)^*\right)^{W_0} \simeq \mathrm{Ext}^{n-i}_{kW}(M, \varepsilon)^* \tag{31}$$

Let us describe the conjugacy classes of W. Given $v \in W_0$, let $Q_v \subseteq Q$ be the sublattice given by

$$Q_v = \{x - v(x) \mid x \in Q\}$$

Proposition 9.1. *Representatives of conjugacy classes of W are given by elements tv where v runs over representatives of conjugacy classes of W_0 and t runs over representatives of $(Q/Q_v)/C_{W_0}(v)$.*

Proof. Given $t, t' \in Q$ and $v, v' \in W_0$, if tv and $t'v'$ are conjugate in W, then v and v' are conjugate in W_0. Let then $g = \tau x$ with $\tau \in Q$ and $x \in W_0$ be such that $gtvg^{-1} = t'v$. Then, $x \in C_{W_0}(v)$. We have $gtvg^{-1} = xtx^{-1} \cdot \tau \cdot v\tau^{-1}v^{-1} \cdot v$ and the proposition follows. □

Let c be a conjugacy class of W, let $w \in c$ and let $\bar{w}$ be the image of w in W_0. The quotient $\overline{C_W(w)} = C_W(w)/Q^{\bar{w}}$ is a subgroup of $C_{W_0}(\bar{w})$. Denoting as customary the vector space spanned by the elements of c by $k(c)$, we have

$$\begin{aligned}\mathrm{Ext}^i_{kW}\big(k, k(c)\big) &\simeq \mathrm{Ext}^{n-i}_{kW}\big(k(c), \varepsilon\big)^* \\ &\simeq \mathrm{Ext}^{n-i}_{kC_W(w)}(k, \varepsilon)^* \\ &\simeq \left(\mathrm{Ext}^{n-i}_{kQ^{\bar{w}}}(k, k)^* \otimes \varepsilon\right)^{\overline{C_W(w)}} \\ &\simeq \left(\Lambda^{n-i}\big(V^{\bar{w}}\big) \otimes \varepsilon\right)^{\overline{C_W(w)}}\end{aligned}$$

where the first isomorphism uses (31), the second one Frobenius reciprocity and the last one the fact that, given a finitely-generated free abelian group L, we have the Koszul isomorphism $H^*(L,k) \simeq \Lambda^*(L^* \otimes_{\mathbb{Z}} k)$.

Theorem 9.2. *If the elements in c have infinite order, then*

$$HD_c^i \simeq \left(\Lambda^{n+1-i}\left(V^{\bar{w}}\right) \otimes \varepsilon\right)^{\overline{C_W(w)}}$$

Proof. The assumption on c shows that $c \cap W_B = \emptyset$ for any proper subset B of D. Thus, the subcomplex CD_c defined in Section 3.2 is concentrated in degree n. The theorem now follows from Section 4.2 and the isomorphisms above. □

Remark 9.3. If the elements of c have finite order, then c has a nonempty intersection with W_B for some proper connected subdiagram B of D [Hu]. In that case, there is a distinguished triangle

$$\bigoplus_i \left(\Lambda^{n-i+1}\left(V^{\bar{w}}\right) \otimes \varepsilon\right)^{\overline{C_W(w)}}[-i] \to CD_c \to HC_{f_c} \rightsquigarrow$$

where HC_{f_c} is the subcomplex of HC_f given by $HC_{f_c}^i = \bigoplus_B k(c \cap W_B)^\varepsilon$ and B runs over the proper subdiagrams of D of size i.

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