Algebra in the fourth dimension

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Algebra and low-dimensional topology

Classical representations $\mathfrak{g}$-mod $\to$ $\mathbb{C}$-mod commutes with $\otimes$ and swap $V \otimes W \cong W \otimes V$

(Lie algebras, groups..)

Quantum rep. 3d topology (Reshetikhin-Turaev) Forgetful functor $\otimes$, not swap: R-matrix

2-representations 4d topology (conj. Crane-Frenkel) Forgetful functor does

(alg. groups / $\mathbb{F}_1$?) Khovanov homology of knots not commute with $\otimes$

$\to$ Construction of new algebraic objects from given ones

$\sim$ moduli constructions in geometry

Enumerative invariants

$$\begin{tikzcd}
\text{variety } X \arrow[r, squiggly] \arrow[d] & \text{moduli space } M \arrow[r, "{\chi, H^*, K \dots}"] & \text{invariants} \\
\text{sheaves}/X \arrow[r, dashed, "?"] & \text{sheaves}/M \arrow[ru] &
\end{tikzcd}$$

More generally, category $\mathcal{X} \longrightarrow$ "vertex category" $\mathcal{M}$ $\overset{?}{\longrightarrow}$ invariants of 4d manifolds

$\mathcal{M} \longrightarrow$ invariants of $\mathcal{X}$

(unstable version of K-theory?)

Quantum groups formulas, R-matrices : arise from homological algebra

vertex operators : — homotopical — ?

Moduli spaces over

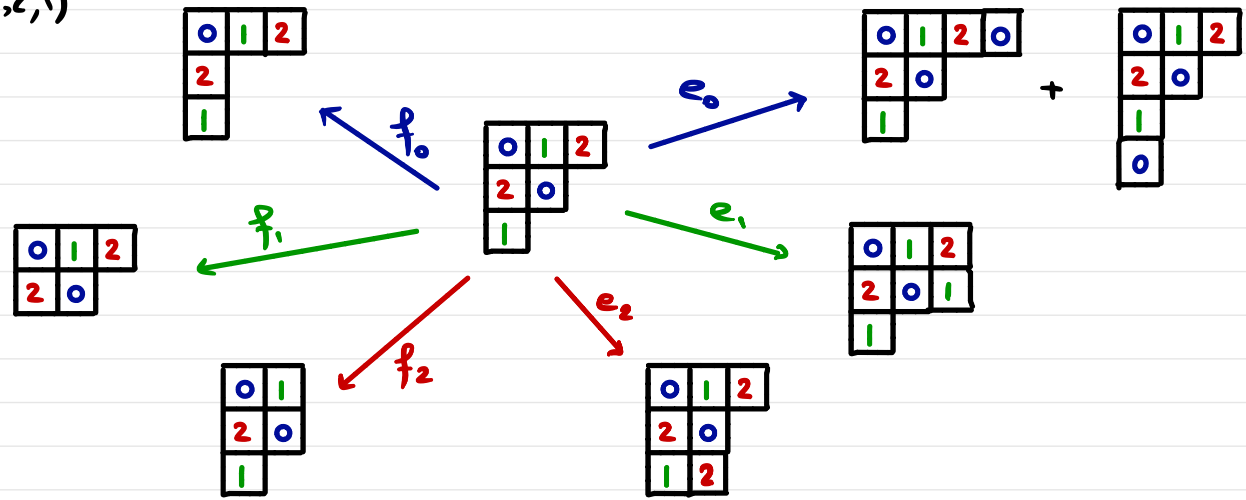
- $\mathbb{C}$ (flag var., quiver var.)
- ? $\mathbb{C}(t)$ (geo. Langlands)
- ? $\mathbb{F}_p(t)$ (Shtukas)
- ??? $\mathbb{Q}$ (beyond Shimura)

Fock space

$$\mathcal{F} = \bigoplus_{\lambda \text{ partition}} \mathbb{C}\cdot\lambda \circlearrowleft \widehat{\mathfrak{sl}}_d \qquad (d = 2, 3, \ldots \text{ or } \infty)$$

$e_i \,/\, f_i$ = add / remove a box of color i (mod d)

$d=3$, $\lambda=(3,2,1)$



$\mathcal{C}$ = Finite dim. rep. $\mathfrak{pgl}(V)$, V finite dim vect. sp. / k alg. closed field

d = char k or ∞

$E := V \otimes - : \mathcal{C} \to \mathcal{C}$ $\qquad T \curvearrowright E^2 = V \otimes V \otimes -$

$X_M : V \otimes M \to V \otimes M$ adjoint to action $\mathfrak{gl}(V) \otimes M = V^* \otimes V \otimes M \to M$

$E = \bigoplus_{i \in \mathbb{Z}/d} E_i$, $E_i(M)$ = gen. i-eigensp. of X_M. $F := V^* \otimes - = \bigoplus_i F_i$

$$\mathbb{C} \otimes K_0(\mathcal{C}) \xrightarrow{\sim} \mathcal{F}_1 \quad , \quad [E_i] = e_i, [F_i] = f_i$$
$$[S^\lambda V] \longleftarrow \lambda$$

$\boxed{\text{Data } \{E_i, F_i, X, T\} : \text{2-representation of } \hat{\mathfrak{sl}}_d \text{ on } \mathcal{C} \qquad \text{(Chuang-R)}}$

T induces $E_i F_j \xrightarrow{\sim} F_j E_i$ $\quad (i \neq j)$

and $E_i F_i |_{\mathcal{C}_\lambda} \xrightarrow{\sim} F_i E_i |_{\mathcal{C}_\lambda} \oplus \mathrm{id}_{\mathcal{C}_\lambda} \oplus \cdots \oplus \mathrm{id}_{\mathcal{C}_\lambda}$

$(\lambda \geq 0)$

$\varepsilon \quad \cdots$

$\varepsilon \circ X^{\lambda - 1} F_i$

$\mathcal{C} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{C}_\lambda$ weight decomp. for $\langle e_i, f_i \rangle = \mathfrak{sl}_2(\mathbb{C})$

Generalizes to Kac-Moody alg, $\langle X, T \rangle$: quiver Hecke alg.
(Khovanov-Lauda, R.)

- "Simple integrable 2-representations" $\mathcal{L}(\lambda)$, $\lambda \in X^+$
- Jordan-Hölder: integrable 2-rep. are extensions of $\mathcal{L}(\lambda) \otimes_{Z_\lambda} M_\lambda$'s.

$\mathcal{L}(\lambda) \simeq$ Zheng-microlocal perverse sheaves on quiver varieties
$\simeq$ Rep. of cyclotomic quiver Hecke alg. (Kang-Kashiwara)

Application : . rep. (super) gl, Cherednik algebras, S_n, Hecke
. H^*(Nakajima quiver varieties)

- Weyl group, chambers $\rightsquigarrow$ braid group, t-structures, stability conditions (Bezrukavnikov)

"Hopf categories"

commutative alg ⇝ quantization

$\wr$

symmetric $\otimes$ cat. ⇝ "quantized symm. $\otimes$ cat"

$\tau_{V,W} : V\otimes W \to W\otimes V$

$\tau_{V',W'}\circ(f\otimes g) = (g\otimes f)\,\tau_{V,W}$ mod $\mathrm{Hom}(V,W')\otimes\mathrm{Hom}(W,V')$

$\tau_{W,V}\circ\tau_{V,W} = \mathrm{id}$ mod $\mathrm{End}\,V\otimes\mathrm{End}\,W + \mathrm{Hom}(W,V)\otimes\mathrm{Hom}(V,W)\,\tau_{V,W}$

$$\begin{tikzcd}
 & V\otimes U\otimes W \arrow[r] & V\otimes W\otimes U \arrow[dr] & \\
U\otimes V\otimes W \arrow[ur] \arrow[dr] & & & W\otimes V\otimes U \\
 & U\otimes W\otimes V \arrow[r] & W\otimes\cdots \arrow[ur] &
\end{tikzcd}$$
mod ...

$\mathcal{C}$ $\otimes$-cat $\mathcal{C}\text{-Mod} = \{ \mathcal{V} \text{ cat}, \gamma : \mathcal{C} \to \mathrm{End}(\mathcal{V}) \text{ monoidal} \}$

$\mathrm{Hom}_{\mathcal{C}}(\mathcal{V}, \mathcal{V}') = \{ \phi : \mathcal{V} \to \mathcal{V}', \{ \alpha_c : \phi \circ \gamma(c) \longrightarrow \gamma'(c) \circ \phi \}_{c \in \mathcal{C}}$ such that

$\cdot\, \alpha_c$'s functorial in $\mathcal{C}$ and compatible with $\otimes$, α_c's invertible $\}$

$\mathcal{H}om(\mathcal{V}, \mathcal{V}') =$ as above, drop requirement α_c invertible

$$\begin{tikzcd}
(\mathcal{C}\text{-Mod})^{op} \times \mathcal{C}\text{-Mod} \arrow[rr, "\mathcal{H}om"] \arrow[dr, "\text{forget}"'] & & \mathrm{Cat} \\
& \mathrm{Cat}^{op} \times \mathrm{Cat} \arrow[ur, "\mathrm{Hom}"'] &
\end{tikzcd}$$

does **not** commute !

Data "quantized symmetric" structure $\rightsquigarrow$ action $\mathcal{C}$ on $\mathcal{H}om(\mathcal{V}, \mathcal{V}')$ | ⚠ Must use ∞-categories

+ Finiteness conditions : $\otimes$ for $\mathcal{C}$-Mod, R-matrix $\mathcal{V} \otimes \mathcal{V}' \xrightarrow{\sim} \mathcal{V}' \otimes \mathcal{V}$

Example $(\mathfrak{sl}_2)$ $\mathcal{L}(1) = \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}$, $\mathcal{L}(1)^{\otimes n} = \coprod_i \mathrm{Gr}(i, n)$.

$\rightsquigarrow$ Maulik-Okounkov stable bases and Yangians (general quiver var.)