THE RIEMANN HYPOTHESIS FOR CURVES

ROHAN JOSHI

Abstract. In the 1940s, Weil proved an analogue of the Riemann hypothesis for curves over finite fields. This result became the basis for the celebrated Weil conjectures, which give a bound on the number of points of a smooth projective variety over a finite field. In this paper I will give an exposition of the Weil conjectures for curves and sketch a proof of the Riemann hypothesis for curves along the lines of Weil's original proof using intersection theory.

Contents

1. Introduction

The Riemann zeta function is defined by the power series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{1}$$

This series only converges for $\mathrm{Re}(s) > 1$, but the function can be extended to the whole complex plane via analytic continuation.

An important property that the Riemann zeta function satisfies is the functional equation. The Riemann zeta function captures various properties of the distribution of prime numbers in the location of its zeros.

One of the most important open questions about the Riemann zeta function is the Riemann hypothesis. It is one of the oldest and most central open problems in number theory. Proposed by Riemann in 1859, it states that

Date: December 2019.

Key words and phrases. Algebraic geometry, number theory.

Conjecture 1 (Riemann hypothesis)**.** *If* $\zeta(s) = 0$, $s = -2, -4, -6\ldots$ *or* $\Re(s) = \frac{1}{2}$.

The Riemann hypothesis is equivalent to the following fairly concrete asymptotic statement about the distribution of prime numbers:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x) \tag{2}$$

where $\pi(x)$ is the number of prime numbers less than x. For more on the Riemann hypothesis, its history and consequences, see [MS16].

In this paper we will explore an analogue of this mathematics in the context of function fields, where the analogue of the Riemann hypothesis has proved more tractable. Indeed, there is a "function field" Riemann zeta function, and it's corresponding functional equation was proved by the German school in the 1930's. The analogue of the Riemann hypothesis was proved in the 1940's by Andre Weil. Weil's proof used algebraic geometry over finite fields, and it was this work that spurred him to rewrite the foundations of algebraic geometry in his work *Foundations of Algebraic Geometry* [Wei62]. This work also inspired his highly influential proposals, the Weil conjectures, which motivated much future work in algebraic geometry and the French school's further rewriting of the foundations with the theory of schemes, as well as the theory of étale cohomology.

We will use the language of schemes for convenience but not in any serious way, since almost the ideas here are classical and are due to Weil and his predecessors.

2. Riemann's zeta function

To define the analogue of the Riemann zeta function in the function field context, we first note that the Riemann zeta function can be equivalently defined in terms of an Euler product. Indeed,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_p \left(\frac{1}{1 - p^{-s}}\right) \tag{3}$$

Definition 2. Let K be a global function field (i.e. a finite extension of $\mathbb{F}_p(t)$), and let k be the algebraic closure of $\mathbb{F}_p$ in K. Then define the Riemann zeta function of K

$$\zeta(K, s) = \prod_{\mathfrak{p}} \frac{1}{1 - |\mathcal{O}_K/\mathfrak{p}|^{-s}} \tag{4}$$

where $\mathfrak{p}$ runs over the nonzero prime ideals of the integral closure of $\mathcal{O}_K$ of $k[x]$ in K; we also include factors for primes $\mathfrak{p}$ of the integral closure $k[x^{-1}]$ that are not primes of $\mathcal{O}_K$. Again this extends to the whole complex plane via analytic continuation.

The reason to include the "extra" primes of the integral closure of $\mathcal{O}_K$ is motivated by algebraic geometry. Indeed, since a global function field can also be defined as the function field of a smooth projective algebraic curve over a finite field, it is natural to take the product over the all closed points of the curve, including those "at infinity". These in fact correspond to archimedean valuations of $\mathcal{O}_K$, while the primes of $\mathcal{O}_K$ correspond to non-archimedean valuations.

Thus it is natural to interpret this zeta function and the analogue of the Riemann hypothesis in terms of algebraic curves. This also suggests generalizations to higher-dimensional algebraic varieties; however that will not concern us right now.

In all that follows, C_0 will be a smooth projective curve over a finite field $\mathbb{F}_q$. Given a global function field K, there is up to isomorphism a unique curve C_0 whose function field is K, where $k = \mathbb{F}_q$. On the other hand, given C_0, we obtain K as its function field.

Definition 3.

$$\zeta(C_0, s) := \prod_{p \in C_0(cl)} \frac{1}{1 - |k(p)|^{-s}} \tag{5}$$

where $C_0(cl)$ is the set of closed points of C_0 and $k(p)$ is the residue field of p.

Thus $\zeta(K, s) = \zeta(C_0, s)$ if K is the function field of C_0.

The main theorem we will prove in this paper is the following (note the absence of "trivial zeroes"):

Theorem 4 (Analogue of Riemann hypothesis)**.** *If* $\zeta(C_0, s) = 0$, $\Re(s) = \frac{1}{2}$.

3. Rational Points of Curves over Finite Fields

Remarkably, the previous theorem can actually be interpreted as a statement about rational points. It gives a bound on the number of $\mathbb{F}_{q^r}$ points of the curve C_0. For convenience (and respect to convention) we will introduce another "zeta function" Z which is simply a change of variable:

Definition 5.

$$Z(C, T) = \prod_{p \in C_{cl}} \frac{1}{1 - T^{\deg p}} \tag{6}$$

where $\deg p := [k(p) : \mathbb{F}_q]$ is the degree of the residue field over the base field $\mathbb{F}_q$. Note that $Z(C, t) = \zeta(C, q^{-s})$.

Furthermore, $\log Z(C, T) = -\sum_p \log(1 - T^{\deg p})$. Recall the power series for $\log(1 - x)$,

$$\log(1 - x) = -\sum_{k=1}^{\infty} \frac{a^k}{k}. \tag{7}$$

Thus

$$\log Z(C, T) = \sum_p \sum_{k=0}^{\infty} \frac{(T^{\deg p})^k}{k} = \sum_p \sum_{k=0}^{\infty} \frac{T^{(\deg p)k}}{(\deg p)k} (\deg p) \tag{8}$$

Now, we can reorganize this sum as follows:

$$\sum_{r=1}^{\infty} \sum_{k \in \mathbb{N}, p (\deg p)k = r} \frac{T^r}{r} (\deg p) = \sum_{n=1}^{\infty} \frac{T^r}{r} \sum_{(\deg p)k = r} \deg p. \tag{9}$$

However, note that $\sum_{(\deg p)k=r} \deg p = \sum_{\deg p | r} \deg p$. This sum counts each closed point p with multiplicity $\deg p$, if $\deg p$ divides r. But $\deg p = [k(p) : \mathbb{F}_q]$; since all extensions of finite fields are separable, this degree equals the separability degree of the extension $k(x)/\mathbb{F}_q$, which is the cardinality of the set of homomorphisms $k(x) \to \overline{\mathbb{F}_q}$ over $\mathbb{F}_q$. All such homomorphisms will have image which lies in $\mathbb{F}_{q^r}$, so we are counting homomorphisms of the residue field into $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$. Crucially, this is equal to the number of morphisms $\operatorname{Spec}\mathbb{F}_q \to C$ which send the point of $\operatorname{Spec}\mathbb{F}_q$ to the closed point p, by [Har77, II Exercise 2.7]. Thus

$$\sum_{(\deg p)k=n} \deg p = \sum_{\deg p \mid n} \deg p = |C_0(\mathbb{F}_{q^r})|. \tag{10}$$

Let $N_r := |C_0(\mathbb{F}_{q^r})|$. We therefore have

$$\log Z(C,T) = \sum_{r=1}^{\infty} N_r \frac{T^r}{r}. \tag{11}$$

So,

$$Z(C,T) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{T^r}{r}\right). \tag{12}$$

Thus $Z(C,T)$ is a sort of generating function for the numbers N_r, which count the number of points C_0 has over all finite extensions of the base field $\mathbb{F}_q$. Note that this is very concrete: C_0 is could be defined by a single homogeneous polynomial in three variables, and so this counts how many solutions this polynomial has when the variables take values in finite fields.

We will later see how the Riemann hypothesis implies a bound on N_r called the Hasse-Weil inequality.

4. Rationality and the Functional Equation

Before we get to the Hasse-Weil inequality and the analogue of the Riemann hypothesis, we will first prove that $Z(C_0,T)$ is in fact a rational function of T, and that it satisfies a functional equation, albeit one quite different looking from the one the classical Riemann zeta function satisfies.

To do this, we will use a little of the theory of divisors. Let C_0 be the curve over $\mathbb{F}_q$. Recall that a Weil divisor on a curve (cf. [Har77, II.6] for general Weil divisors) is a finite integer combination of closed points of C_0. The difference in the case of curves over non-algebraically closed fields is in the degree map: the degree of a divisor $D = \sum n_i p_i$ is $\sum n_i(\deg p_i)$ as opposed to $\sum n_i$ (for a closed point p, $\deg p = [k(p) : \mathbb{F}_q]$). Let d_r be the cardinality of the set of effective divisors of degree n.

We will let $\operatorname{Div}(C_0)$ be the group of divisors on C_0, and $\operatorname{Div}^+(C_0)$ the set of effective divisors (those with nonnegative coefficients). $\operatorname{Pic}(C_0)$ is the group of divisors up to linear equivalence. Let $\operatorname{Div}^n(C_0)$ be the set of divisors of degree n, and $\operatorname{Pic}^n(C_0)$ the set of linear equivalence classes of divisors of degree n.

This is useful to us for the following reason. First, we will reformulate the zeta function Z to look more like the power series for the original zeta function. Recall that

$$Z(C_0,T) = \prod_{p \in C_0(cl)} \frac{1}{1 - T^{\deg p}} \tag{13}$$

Thus,

$$Z(C_0, T) = \prod_{p \in C_0(cl)} \sum_{n=1}^{\infty} T^{(\deg p)n} \tag{14}$$

The coefficients of the T^n term of this convolution will be the number of sequences $\{n_{p_i}\}$ of nonnegative integers such that $\prod_i T^{(\deg p_i)(n_{p_i})} = T^n$, or in other words $\sum_i n_{p_i}(\deg p_i) = n$. This is exactly the number of effective divisors on C_0 of degree n! Thus

$$Z(C_0, T) = \sum_{n=0}^{\infty} d_n T^n. \tag{15}$$

If D is a divisor, $|D|$ is the set of all effective divisors linearly equivalent to D; as in the case of curves over an algebraically closed field [Har77, IV.1] these divisors are in bijection with elements of the quotient set $H^0(C_0, D) - \{0\}/\mathbb{F}_q^\times$. Thus the size of $|D|$ is $(q^{l(D)} - 1)/(q-1)$. Say the degree of D is n. So,

$$d_n = \sum_{D \in \mathrm{Pic}^n(X_0)} \frac{q^{l(D)} - 1}{q - 1} \tag{16}$$

The main tool we need is the Riemann-Roch theorem.

Theorem 6 (Riemann-Roch)**.** *Let D be a divisor of degree n and let K be the canonical divisor on a curve of genus g. Then*

$$l(D) - l(K - D) = n + 1 - g. \tag{17}$$

Proof. See [Har77, IV.1.3] □

Corollary 7. *If $n > 2g - 2$, $l(D) = n + 1 - g$.*

Proof. If $n > 2g - 2$, $K - D$ is a divisor of negative degree, so $l(K - D) = 0$. □

Corollary 8.

$$d_n = |Pic^n(C_0)| \frac{q^{n+1-g} - 1}{q - 1} \tag{18}$$

for $r > 2g - 2$.

Proof. Combine 6 and 16. □

Theorem 9 (Rationality)**.** *$Z(C_0, T)$ is a rational function of T. In particular, there exists a polynomial $P(T)$ such that*

$$Z(C_0, T) = \frac{P(T)}{(1 - T)(1 - qT)}. \tag{19}$$

Proof.

$$Z(C_0,T)=\sum_{n=1}^{\infty} d_nT^n=\sum_{n=1}^{2g-2} d_nT^n+\sum_{n>2g-2}|\mathrm{Pic}^n(C_0)|\frac{q^{n+1-g}-1}{q-1}T^n \tag{20}$$
$$=\sum_{n=1}^{2g-2} d_nT^n+\frac{|\mathrm{Pic}^n(C_0)|}{q-1}\sum_{n>2g-2}(q^{n+1-g}-1)T^n$$
$$=\sum_{n=1}^{2g-2} d_nT^n+\frac{|\mathrm{Pic}^n(C_0)|}{q-1}T^{2g-1}\left(\frac{q^g}{1-qT}-\frac{1}{1-T}\right).$$

This can be written as fraction whose numerator is a polynomial and whose denominator is $(1-T)(1-qT)$. □

Lemma 10. $d_n - q^{n+1-g}d_{2g-2-n} = |Pic^0(C_0)|\frac{q^{n+1-g}-1}{q-1}$

Proof. Recall that $\mathrm{Pic}^n(C_0)$ is the fiber of the surjective map $\deg : \mathrm{Pic}(C_0) \to \mathbb{Z}$ over n, and therefore $|\mathrm{Pic}^0(C_0)| = |\mathrm{Pic}^n(C_0)|$. Also, there is an explicit bijection $\mathrm{Pic}^n(C_0) \xrightarrow{\cong} \mathrm{Pic}^{2g-2-n}(C_0)$ provided by $D \mapsto K - D$. Thus

$$d_n - q^{n+1-g}d_{2g-2-n}=\sum_{D\in \mathrm{Pic}^n(C_0)}\frac{q^{l(D)}-1}{q-1}-q^{n+1-g}\sum_{D'\in \mathrm{Pic}^{2g-2-n}(C_0)}\frac{q^{l(D')}-1}{q-1} \tag{21}$$
$$=\sum_{D\in \mathrm{Pic}^n(C_0)}\left(\frac{q^{l(D)}-1}{q-1}-\frac{q^{n+1-g+l(K-D)}-q^{n+1-g}}{q-1}\right)$$
$$=|\mathrm{Pic}^n(C_0)|\frac{q^{n+1-g}-1}{q-1}=|\mathrm{Pic}^0(C_0)|\frac{q^{n+1-g}-1}{q-1}.$$

□

Theorem 11 (Functional Equation)**.** $Z(C_0,\frac{1}{qT}) = q^{1-g}T^{2-2g}Z(C_0,T)$

Proof. Note that $d_n = 0$ for $n < 0$. First,

$$Z(C_0,\frac{1}{qT})=\sum_{n\in\mathbb{Z}} d_nq^{-n}T^{-n}=\sum_{n\in\mathbb{Z}} d_{-n}q^nT^n \tag{22}$$

(swapping $n \mapsto -n$). Furthermore we have

$$q^{g-1}T^{2g-2}Z(C_0,\frac{1}{qT})=\sum_{n\in\mathbb{Z}} q^{n+1-g}d_{2g-2-n}T^n \tag{23}$$

via $n \mapsto n + 2 - 2g$. Thus

$$Z(C_0,T)-q^{g-1}T^{2g-2}Z(C_0,\frac{1}{qT)}=\frac{|\mathrm{Pic}^0(C_0)|}{q-1}\sum_{n\in\mathbb{Z}}(q^{n+1-g}-1)T^n \tag{24}$$

Finally we must inspect the series $\sum_{n\in\mathbb{Z}}(q^{n+1-g}-1)T^n = q^{1-g}\sum_{n\in\mathbb{Z}}(qT)^n - \sum_{n\in\mathbb{Z}}T^n$. The left series is annihilated by $1-T$ and the right series is annihilated by $1-T$. Thus $Z(C_0,T)-q^{g-1}T^{2g-2}Z(C_0,\frac{1}{qT)} = 0$ at all but two points; since Z is continuous and defined on the whole complex plane, it is identically zero: so $Z(C_0,\frac{1}{qT}) = q^{1-g}T^{2-2g}Z(C_0,T)$. □

Corollary 12. *There exist constants $\alpha_i, \dots \alpha_{2g}$ such that $P(T) = (1-\alpha_1 T)\cdots(1-\alpha_{2g}T)$, where $\alpha_i\alpha_{2g-i} = q$.*

Proof. The functional equation implies that $P(\frac{1}{qT}) = q^{-g}T^{-2g}P(T)$. Note that this implies that P is of degree (at most) $2g$. Let $P(T) = (1-\alpha_i T)\cdots(1-\alpha_{2g}T)$. Then the functional equation implies, up to rearrangement, that the factors $qT^2 - \alpha_i T$ are the same as the factors $1-\alpha_i T$. Thus, rearranging if necessary, we must have $qT^2 - \alpha_i T = 0 \iff 1 - \alpha_{2g-i}T = 0$. So $\frac{1}{\alpha_{2g-i}} = \frac{\alpha_i}{g}$, so $\alpha_i\alpha_{2g-i} = q$. □

5. Statement of Riemann hypothesis and the Hasse-Weil Inequality

Recall that we have established that

$$Z(C_0, T) = \frac{(1-\alpha_1 T)\cdots(1-\alpha_{2g}T)}{(1-T)(1-qT)}. \tag{25}$$

Thus the zeros $Z(C_0,T)$ are $\frac{1}{\alpha_1}, \dots \frac{1}{\alpha_{2g}}$. Thus to show that the roots of Z have absolute value $q^{-\frac{1}{2}}$, it suffices to show that $|\alpha_i| = \sqrt{q}$.

Furthermore, we have that $\alpha_i\alpha_{2g-i} = q$. So notice that the Riemann hypothesis will follow from simply the inequality $|\alpha_i| \le \sqrt{q}$.

We will prove the Riemann hypothesis via the **Hasse-Weil inequality**, which is an inequality that puts an explicit bound on N_r. The Hasse-Weil inequality states that

$$|N_r - (1+q^r)| \le 2g\sqrt{q^r} \tag{26}$$

which is actually a pretty good bound. Why does the Hasse-Weil inequality imply the Riemann hypothesis? Well, if we take the logarithm of $Z(C,T)$ and use the power series for $\log(1-x)$, regrouping terms gives us

$$N_r = 1 + q^r - \sum_{i=1}^{2g}\alpha_i^r \implies |\alpha_1^r + \cdots \alpha_{2g}^r| \le 2g\sqrt{q^r} \tag{27}$$

In other words,

$$\left|\left(\frac{\alpha_1}{\sqrt{q}}\right)^r + \cdots + \left(\frac{\alpha_1}{\sqrt{q}}\right)^r\right| \tag{28}$$

is bounded.

Letting $r \to \infty$, we have $\max\left|\frac{\alpha_i}{\sqrt{q}}\right| \le 1$, so $\alpha_i \le \sqrt{q}$ for all i as desired. This works, with some care, even if the α_i are not distinct.

6. Example

Example 13 (Projective line)**.** Let $C_0 = \mathbb{P}^1_{\mathbb{F}_q}$. Then clearly $N_r = q^r + 1$. Thus the zeta function is

$$Z(C_0,T)=\exp\left(\sum_{r=1}^{\infty}\frac{(q^r+1)T^r}{r}\right)=\exp\left(\sum_{r=1}^{\infty}\frac{T^r}{r}\right)\exp\left(\sum_{r=1}^{\infty}\frac{(qT)^r}{r}\right) \tag{29}$$
$$=\exp(-\log(1-T))\exp(-\log(1-qT))=\frac{1}{(1-T)(1-qT)}.$$

This is indeed what rationality predicts in the genus 0 case. To verify the functional equation, note that

$$Z(C_0,\frac{1}{qT})=\frac{1}{(1-\frac{1}{qT})(1-\frac{1}{T})}=\frac{qT^2}{(qT-1)(T-1)}=qT^2Z(C_0,T), \tag{30}$$

as desired. Finally, the Riemann hypothesis holds trivially since there are no values of α. Note that in the genus 0 case, the Hasse-Weil bound reduces to an equality $N_r=q^r+1$; the projective line satisfies this.

7. Proof of the Hasse-Weil Inequality

Now, we will prove the Hasse-Weil inequality using intersection theory. Let C be the base extension of C_0 to the algebraic closure of $\mathbb{F}_q$ i.e. $C=C_0\times_{\operatorname{Spec}\mathbb{F}_q}\operatorname{Spec}\overline{\mathbb{F}_q}$. So C is a curve over an algebraically closed field, and we can think of it essentially as a classical algebraic variety.

Then there is the Frobenius map $\mathrm{Frob}_r : C\to C$. If we embed C into projective space, then Frob_r sends $[x_0:\cdots:x_n]\mapsto[x_0^{q^r}:\cdots:x_n^{q^r}]$. We can interpret N_r as the size of the set of fixed points of Frob_r. Our plan then to use inequalities from intersection theory to bound the intersection of Γ_{Frob_r} and Δ (the diagonal) in $C\times C$.

First, let us set up the intersection theory we need. This material is from Chapter V.1 of Hartshorne, on surfaces.

Theorem 14 (Intersection pairing on a surface)**.** *Let X be a surface. There exists a symmetric bilinear pairing $PicX\times PicX\to\mathbb{Z}$ (where the product of divisors C and D is denoted $C.D$) such that if C, D are smooth curves intersecting transversely, then $C.D=|C\cap D|$.*

Theorem 15 (Hodge index)**.** *Let H be an ample divisor on X and D a nonzero divisor, with $D.H=0$. Then $D^2\le 0$. (D^2 denotes $D.D$)*

Now let us begin with some general set up. Let C_1 and C_2 be two curves, and let $X=C_1\times C_2$. Identify C_1 with $C_1\times *$ and C_2 with $*\times C_2$. Notice that $C_1.C_1=C_2.C_2=0$ and $C_1.C_2=1$. Thus $(C_1+C_2)^2=2\ge 0$.

Let D be a divisor on X. Let $d_1=D.C_1$ and $d_2=D.C_2$;

Proposition 16 (Castelnuovo-Severi inequality)**.** *def$(D):=2d_1d_2-D^2\ge 0$*

Proof. $(D-d_2C_1-d_1C_2).(C_1+C_2)=0$ (expand it out). The Hodge index theorem implies then that $(D-d_2C_1-d_1C_2)^2\le 0$. Expanding this out yields $D^2\le 2d_1d_2$. □

Thus we may define $\mathrm{def}(D):=2d_1d_2-D^2\ge 0$.

Proposition 17. *Let D and D' be divisors. Then $|D.D'-d_1d_2'-d_2d_1'|\le\sqrt{def(D)def(D')}$.*

Proof. Expand out $\mathrm{def}(mD+nD') \geq 0$, for $m,n \in \mathbb{Z}$. We can let $\frac{m}{n}$ become arbitrarily close to $\sqrt{\frac{\mathrm{def}(D')}{\mathrm{def(D)}}}$, yielding the inequality. □

Lemma 18. *Consider a map $f : C_1 \to C_2$. If Γ_f is the graph of f on $C_1 \times C_2$, then def(Γ_f) $= 2g_2$ deg(f) (where g_2 is the genus of C_2).*

Proof. The adjunction formula ([Har77, V.1.5] states that $K_{\Gamma_f} = (K_V + \Gamma_f).\Gamma_f$. Since $K_V = K_{C_1} \times C_2 + C_1 \times K_{C_2}$, we have that

$$2g_1 - 2 = (\Gamma_f)^2 + (2g_1-2)(1) + (2g_2-2)\deg f. \tag{31}$$

Thus, $\mathrm{def}(\Gamma_f) = 2g_2 \deg f$. □

Now we have what we need: we will do intersection theory on $C \times C$. The Frobenius map $f = \mathrm{Frob}_r : C \to C$ is a map of degree q^r, so $\mathrm{def}(\Gamma_f) = 2gq^r$. We might as well think of Δ as the graph of the identity map, so $\mathrm{def}(\Delta) = 2g$. Finally, $d_2' = d_2 = d_1' = 1$ and $d_1 = q^r$. Plugging it into the inequality, we get

$$|\Gamma_f.\Delta - q^r - 1| \leq \sqrt{(2gq^r)(2g)} \tag{32}$$

yielding the Hasse-Weil inequality

$$|N_r - (1+q^r)| \leq 2g\sqrt{q^r}. \tag{33}$$

This proves the Riemann hypothesis for curves over finite fields.

8. The Weil conjectures

After Weil proved the Hasse-Weil inequality and thus the Riemann hypothesis, he proposed what are now called the Weil conjectures. The Weil conjectures basically generalized the story for curves to higher-dimensional algebraic varieties. Furthermore, they establish an even stronger link with topology.

Let V_0 be a smooth projective variety of dimension n over a finite field $\mathbb{F}_q$. Let $N_r = |V_0(\mathbb{F}_{q^r})|$. Define the local zeta function

$$Z(V_0, T) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{T^r}{r}\right) \tag{34}$$

Proposition 19 (Weil conjectures)**.** *There are four parts:*

(1) ***Rationality****: $Z(V_0,T)$ is a rational function of T. More precisely, there exist polynomials $P_0 \dots P_{2n}$ such that*

$$Z(V_0,T) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)} \tag{35}$$

Also, $P_0(T) = 1-T$, $P_{2n}(T) = 1-q^nT$, and for $1 \leq i \leq 2n-1$, $P_i(T) = \prod_j(1-\alpha_{ij}T)$ for some numbers α_{ij}.

(2) ***Functional Equation***

$$Z\left(V_0, \frac{1}{q^nT}\right) = \pm q^{nE/2}T^E Z(V_0,T) \tag{36}$$

where E is the top Chern class of the tangent bundle of V.

(3) ***Riemann hypothesis*** *$|\alpha_{ij}| = q^{i/2}$ for all $1 \leq i \leq 2i-1$ and all j.*
(4) ***Betti numbers*** *If V_0 was obtained from an arithmetic variety over a number ring via reduction to a prime, then one can consider the original variety before reduction, and by embedding the ring into $\mathbb{C}$, consider it over the complex numbers. The degree of P_i is ith Betti number of the associated complex variety, considered as a complex-analytic space.*

9. Further Reading

For information about the classical Riemann hypothesis, see [MS16]. The proofs of rationality and the functional equation are drawn from a set of course notes [ET11]. The proof of the Riemann hypothesis and much else is drawn from a highly recommended expository paper of Milne [Mil16]. We encourage the reader to read this paper to learn about the Weil conjectures and all sorts of future developments inspired by the mathematics described in this paper. For the theory of algebraic surfaces, as well as intersection theory, see [Har77]. A fast run down of some of the contents of this paper may be found in the blog post [Hil17].

Acknowledgments

Thanks to Professor Sug Woo Shin for motivation. Thanks to Kai Shaikh for reviewing this paper.

References

[ET11] B. Edixhoven and L. Taelman, Algebraic geometry, (2011). Available at `http://www.math.leidenuniv.nl/~edix/teaching/2010-2011/AG-mastermath/ag.pdf`.
[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157.
[Hil17] A. Hilado, The riemann hypothesis for curves over finite fields, (2017). Available at `https://ahilado.wordpress.com/2017/02/24/the-riemann-hypothesis-for-curves-over-finite-fields/`.
[MS16] B. Mazur and W. Stein, *Prime numbers and the Riemann hypothesis*, Cambridge University Press, Cambridge, 2016. MR 3616260. `https://doi.org/10.1017/CBO9781316182277`.
[Mil16] J. S. Milne, The Riemann hypothesis over finite fields: from Weil to the present day [Reprint of 3525903], *ICCM Not.* **4** no. 2 (2016), 14–52. MR 3635684. `https://doi.org/10.4310/ICCM.2016.v4.n2.a4`.
[Wei62] A. Weil, *Foundations of algebraic geometry*, American Mathematical Society, Providence, R.I., 1962. MR 0144898.

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840
Email address: `rohanjoshi@berkeley.edu`