NOTES ON THE TATE, SHAFAREVICH, AND MORDELL CONJECTURES

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Contents

1. Introduction

This is an outline of the connections between the Tate, Shafarevich and Mordell conjectures in arithmetic geometry. The focus is mainly on the work of Faltings, who proved versions of all three of these conjectures. The Tate and Shafarevich conjectures are about abelian varieties so in the first section I cover some of the relevant basics of abelian varieties.

2. Background on abelian varieties

In this section I will introduce some of the theory of abelian varieties. The standard reference is Mumford's book, *Abelian Varieties*; Milne's notes are also good.

One way to think about abelian varieties is as higher-dimensional generalizations of elliptic curves. Indeed an elliptic curve is just an abelian variety of dimension 1. Alternatively, one can think of abelian varieties as projective algebraic groups.

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Definition 1. A *group scheme* is a group object in the category of schemes (over a field K). An *algebraic group* is a group scheme of finite type over K. An *abelian variety* is a smooth, proper, connected algebraic group.

Remark 2. Abelian varieties are automatically projective and geometrically integral.

Here are some related definitions. Recall that a torus is an algebraic group that is a twist of $\mathbb{G}_m^n$ as a group scheme.

Definition 3. A *semiabelian variety* is an extension G of an abelian variety A by a torus T:

$$0 \to T \to G \to A \to 0 \tag{2.1}$$

Definition 4. A *semiabelian scheme* of relative dimension g over a base scheme S is a smooth group scheme over S with whose geometric fibers are connected semiabelian varieties of dimension g. An *abelian scheme* is a semiabelian scheme whose fibers are abelian varieties.

It is important to distinguish between a "morphism" and "homomorphism" of abelian varieties. A *morphism* is simply a morphism in the category of schemes, while a *homomorphism* is a morphism that preserves the group structure and is thus a morphism in the category of abelian varieties.

Example 5. For any integer n, there is a "multiplication by n" homomorphism $n : A \to A$. The kernel of this homomorphism is denoted $A[n]$.

Example 6. Let A be an abelian variety over a field K, and let $a \in A(K)$. Then there is a morphism of *translation by* a, $t_a : A \to A$; on rational points, t_a sends $b \in A(K)$ to $b + a \in A(K)$. Clearly a nontrivial translation is not a homomorphism.

Proposition 7. *Every morphism $f : A \to B$ of abelian varieties over K is the composite of a homomorphism and a translation t_b, where $b = -f(0) \in B(K)$.*

Proof. This is a consequence of the rigidity lemma. See Corollary 2.2 in Milne's notes on Abelian Varieties in Cornell-Silverman. □

Corollary 8. *If a morphism $A \to B$ of abelian varieties sends the identity to the identity, it is automatically a homomorphism.*

Corollary 9. *The group law on an abelian variety A is commutative.*

Proof. There is an inverse morphism $f : A \to A$ sending $a \mapsto -a \in A(K)$. Since this preserves the identity, it is a homomorphism. Since the map taking an element to its inverse is a homomorphism, the group law is commutative. □

2.1. **Tate module.**

Definition 10. Let l be a prime different from the characteristic of a field K. Let A be an abelian variety defined over K. Then the *Tate module* $T_l(A)$ is the inverse limit over the abelian groups $A[l^n](\overline{K})$, connected by group homomorphisms $A[l^{n+1}](\overline{K}) \to A[l^n](\overline{K})$ which are multiplication by l. The inverse limit has the structure of a $\mathbb{Z}_l$-module, and is equipped with the action of the Galois group $\mathrm{Gal}(\overline{K}, K)$. We also define the $\mathbb{Q}_l$-Tate module by $V_l A := \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l(A)$.

Remark 11. As a $\mathbb{Z}_l$-module, $T_l(A) \cong \mathbb{Z}_l^2$ (this can be easily seen over $\mathbb{C}$); what is interesting is the Galois action. Also the Tate module is actually isomorphic to dual of the first étale cohomology group of the abelian variety.

2.2. Good and semistable reduction.

Example 12. For motivation, consider an elliptic curve E defined over $\mathbb{Q}$. This can be defined using a minimal Weierstrass equation

$$y^2 = x^3 + Ax + B, \quad \Delta = 4a^3 + 27b^2 \neq 0 \tag{2.2}$$

Reducing this equation mod p defines an algebraic group E_0 over $\mathbb{F}_p$. There are three possibilities:

1. Good reduction: E_0 is an elliptic curve; this happens when $p \nmid \Delta$
2. Multiplicative reduction: E_0 has a node, and with its node removed is isomorphic to $\mathbb{G}_m$ (possibly after base change)
3. Additive reduction: E_0 has a cusp, and with its cusp removed is isomorphic to $\mathbb{G}_a$

Cases (2) and (3) together are called *bad reduction.* Cases (1) and (2) together are called *semistable reduction.* Essentially, the elliptic curve has good reduction at all primes except for those that divide the discriminant, where it has one of two kinds of bad reduction: multiplicative or additive. If the only kind of bad reduction it has is multiplicative reduction (i.e. it has semistable reduction everywhere), then it is called a *semistable elliptic curve.*

We would like to generalize this to abelian varieties. Here is the definition of good reduction for abelian varieties.

Definition 13. Let R be a discrete valuation ring, K its field of fractions. Let A be an abelian variety defined over K. Then A has *good reduction* if there exists an abelian scheme $\mathcal{A}$ over R such that there is an isomorphism $\mathcal{A}_K \cong A$ of K-schemes; in other words, a smooth proper *model* for A over R.

Remark 14. Note that the special fiber of $\mathcal{A}$ over the residue field $k = R/\mathfrak{m}$ is thus also an abelian variety.

To define the notion of semistable reduction, one uses the theory of Néron models, which are smooth models that are not necessarily proper. Even though the Néron model of an abelian variety may not be proper, we still have a bijection between the R-points of the Néron model and the K-points of the abelian variety. See the book of Bosch et. al. for the theory of Néron models.

Definition 15. Let R be a discrete valuation ring, K its field of fractions. Let A be an abelian variety defined over K. Let $\mathcal{A}$ be the Néron model of A and $\mathcal{A}^0$ the connected component of the identity of $\mathcal{A}$. Then A has *semistable reduction* if the special fiber of $\mathcal{A}^0$ is a semiabelian variety.

For a number field K, for each archimedean place v of K, we can ask what sort of reduction the abelian variety has at that place. In the context of Faltings' theorem, two phrases that come up a lot are "good reduction outside a finite set of places S" and "semistable reduction everywhere" (i.e. at all places).

Theorem 16. *Every abelian variety has semistable reduction after base change to a finite extension of K.*

Remark 17. This is difficult fact (due to Grothendieck?) that is usually quoted without proof. We will use it later. I remember it being discussed during Roy's talk.

2.3. **Isogeny.**

Definition 18. A homomorphism of abelian varieties $f : A \to B$ is an *isogeny* if it is surjective and its kernel is a finite group scheme. The degree of f is the order of the kernel of f as a finite group scheme.

Example 19. The multiplication by n map $A \to A$ is an isogeny of degree n^{2g}, where g is the dimension of A. (this is easy to see over $\mathbb{C}$)

Remark 20. If there exists an isogeny $A \to B$, we say A and B are isogenous. Isogenous abelian varieties must be of the same dimension. Isogeny is actually an equivalence relation on abelian varieties. In fact if there is an isogeny $f : A \to B$ of degree n, there is another isogeny $g : B \to A$ such that the composite $g \circ f$ is multiplication by n.

Remark 21. Let $\mathbf{AbelVar}_K$ be the category whose objects are abelian varieties over a field K and whose morphisms are homomorphisms of abelian varieties. $\mathbf{AbelVar}_K$ is a preadditive category, meaning that Hom-sets are $\mathbb{Z}$-modules. Let $\mathbb{Q} \otimes \mathbf{AbelVar}_K$ be the category with the same objects as $\mathbf{AbelVar}_K$, but with $\mathrm{Hom}_{\mathbb{Q}\otimes\mathrm{AbelVar}}(A, B) := \mathbb{Q} \otimes \mathrm{Hom}_{\mathrm{AbelVar}}(A, B)$. Then $\mathbb{Q} \otimes \mathbf{AbelVar}_K$ is a $\mathbb{Q}$-linear tensor category, in which isogenies are isomorphisms. One can even define characteristic polynomials of endomorphisms in this category and prove the Weil conjectures for abelian varieties. In fact, $\mathbb{Q} \otimes \mathbf{AbelVar}_K$ is a semisimple abelian category.

2.4. **Polarization.**

Associated to an abelian variety A is its dual abelian variety $A^\vee$. Roughly speaking, $A^\vee$ parametrizes line bundles of degree zero on A. Here is a precise definition.

Definition 22. Given an abelian variety A over K, $A^\vee$ is the *dual abelian variety* of A, and $\mathcal{P}$ is the *Poincaré line bundle* on $A \times A^\vee$ if:

(1) $\mathcal{P}|_{\{0\}\times A^\vee}$ is trivial and $\mathcal{P}|_{A\times\{a\}}$ for $a \in A^\vee$ is a line bundle of degree 0 on $A \times \{a\}$, and
(2) For every K-scheme T and line bundle $\mathcal{L}$ on $A \times T$ such that $\mathcal{L}|_{\{0\}\times T}$ is trivial and $\mathcal{L}|_{A\times\{t\}}$ for $t \in T$ is a line bundle of degree 0, and there is a unique morphism $f : T \to A^\vee$ such that $(1 \times f) * \mathcal{P} \cong \mathcal{L}$.

Remark 23. We can think of this as follows: (1) says that $\mathcal{P}$ is a family of degree zero line bundles on A parametrized by $A^\vee$, and (2) states that $\mathcal{P}$ over $A \times A^\vee$ is the universal such family of degree zero line bundles on A.

Remark 24. Given a line bundle $\mathcal{L}$ on A, there is a homomorphism of abelian varieties associated to $\mathcal{L}$ called $\phi_\mathcal{L} : A \to A^\vee$. On the level of K-points, it sends $a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$.

Now we come to a very important definition.

Definition 25. A *polarization* on A is an isogeny $\lambda : A \to A^\vee$ such that $\lambda_{\overline{K}} = \phi_\mathcal{L}$ for some ample line bundle on $A_{\overline{K}}$. The *degree* of the polarization is its degree as an isogeny. If λ is of degree 1 it is called a *principal polarization*. An abelian variety along with the data of polarization is called a *polarized abelian variety* (if the polarization is of degree 1, *principally polarized abelian variety*).

Remark 26. There is an obvious notion of morphism of polarized abelian variety, so polarized abelian varieties over K form a category with a "forgetful functor" to the category of abelian varieties over K.

The following proposition says that given a polarizable abelian variety, there are only finitely many essentially distinct ways to equip it with a polarization.

Proposition 27. *Let A be an abelian variety over K and d an integer. Then there are only finitely many isomorphism classes of polarized abelian varieties (A, λ), where λ is a polarization of degree d.*

Proof. See Theorem 18.1, Milne's notes on Abelian Varieties in Cornell-Silverman. □

Zarhin's trick, as described in Zarhin's paper "A remark on endomorphisms of abelian varieties over function fields of finite characteristic", establishes that for any abelian variety A, $(A \times A^\vee)^4$ is principally polarizable. Interestingly, Zarhin's trick relies on Lagrange's four-square theorem and the quaternions.

Definition 28. A *direct factor* of an abelian variety A is an abelian subvariety $B \subset A$ such that there exists another abelian subvariety $C \subset A$ and an isomorphism $A \cong B \times C$.

Proposition 29 (Finiteness of direct factors)**.** *An abelian variety A has only finitely many direct factors up to isomorphism.*

Remark 30. By the proposition, this implies that the map from g-dimensional abelian varieties (up to isomorphism) to $8g$-dimensional principally polarized abelian varieties (up to isomorphism) sending $A \mapsto (A \times A^\vee)^4$ is finite-to-one.

This is very important, because it allows us to turn finiteness statements about principally polarized abelian varieties into finiteness statements about abelian varieties. We will use Zarhin's trick in this way multiple times.

2.5. **Siegel moduli space.** One reason to be interested in polarizations is that polarized abelian varieties can be parametrized by a moduli space.

Proposition 31. *Let $F_{g,d}$ be the functor which sends a (locally noetherian) scheme S to the set of isomorphism classes of polarized abelian schemes over S of dimension g and degree d. Then this functor has a coarse moduli space $\mathcal{A}_{g,d}$, defined over $\mathbb{Z}$, which is a quasi-projective variety over $\mathbb{Q}$.*

Remark 32. Let A and B be polarized abelian varieties of dimension g and degree d over K. So A and B correspond to elements of the set $F_{g,d}(K)$ which are distinct if A is not isomorphic to B. Then consider the following diagram:

$$\begin{array}{ccc} F_{g,d}(K) & \longrightarrow & \mathcal{A}_{g,d}(K) \\ \downarrow & & \downarrow \\ F_{g,d}(\overline{K}) & \xrightarrow{\cong} & \mathcal{A}_{g,d}(\overline{K}) \end{array} \tag{2.3}$$

The downward arrow $F_{g,d}(K) \to F_{g,d}(\overline{K})$ sends (the K-isomorphism class of) an abelian variety A to (the $\overline{K}$-isomorphism class of) $A_{\overline{K}}$. Thus A and B correspond to the same point of $\mathcal{A}_{g,d}$ if and only if $A_{\overline{K}} \cong B_{\overline{K}}$ i.e. B is a twist of A. Also recall that the twists of A are classified by the Galois cohomology group $\mathrm{H}^1(K, \mathrm{Aut}(A/K))$.

3. Tate conjecture

Let K be a a finitely-generated field (i.e. a field that is finitely-generated over its prime subfield, either $\mathbb{Q}$ or $\mathbb{F}_p$). Let $\overline{K}$ be the separable closure of K. The *Tate conjecture* is the following statement.

Conjecture 33. *Let A and B be abelian varieties over a field K, l a prime different from the characteristic of K, and T_l is the l-adic Tate module, which is naturally equipped with a linear action of $G = \mathrm{Gal}(\overline{K}/K)$. Let Hom_G is the module of G-equivariant homomorphisms. Then the natural map*

$$\mathbb{Z}_l \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}_G(T_l(A), T_l(B)) \tag{3.1}$$

is bijective.

Remark 34. Here is another way to think about this. Conjecture 33 states that the Tate module functor defines a *fully faithful embedding* of the category $\mathbb{Z}_l \otimes \mathbf{AbelVar}_K$ into the category of l-adic Galois representations (i.e. representations of the Galois group on $\mathbb{Z}_l$-modules). Thus in some sense it is saying the study of abelian varieties (after inverting prime-to-l isogenies) can be reduced to the study of l-adic representations.

Let us discuss some equivalent statements.

Lemma 35. *For abelian varieties A and B, the map*

$$\mathbb{Z}_l \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}_G(T_l(A), T_l(B)) \tag{3.2}$$

is injective with torsion-free cokernel.

Proof. See Theorem 12.5 in Milne's notes on Abelian Varieties in Cornell-Silverman. □

Proposition 36. *Conjecture 33 is equivalent to the following: For abelian varieties A and B, the natural map*

$$\mathbb{Q}_l \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}_G(V_l(A), V_l(B)) \tag{3.3}$$

is bijective.

Proof. This map is obtained by tensoring 3.2 with $\mathbb{Q}_l$, which is flat over $\mathbb{Z}_l$. Thus 3.3 is bijective if 3.2 is. The converse is a consequence of 35, since if M is a torsion-free $\mathbb{Z}_l$-module and $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} M$ is zero, M must be zero. □

Proposition 37. *Conjecture 33 is equivalent to the following: For an abelian variety A, the natural map*

$$\mathbb{Q}_l \otimes \mathrm{End}(A) \to \mathrm{End}_G(V_l(A)) \tag{3.4}$$

is bijective.

Proof. This is a consequence of the previous proposition and the following facts: $\mathrm{End}(A \times B) \cong \mathrm{End}(A) \times \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, A) \times \mathrm{End}(B)$, and like wise for $\mathrm{End}_G(V_l(A) \times V_l(B))$. Also $V_l(A \times B) \cong V_l(A) \times V_l(B)$. □

Now identify $\mathbb{Q}_l \otimes \mathrm{End}(A)$ with its image $\mathrm{End}(V_l(A))$, the algebra consisting of all (not necessarily G-equivariant) $\mathbb{Z}_l$-endomorphisms. Let $\mathbb{Q}_l[G]$ be the subalgebra of $\mathrm{End}(V_l(A))$ generated by automorphisms of $V_l(A)$ given by elements of the Galois group G.

Proposition 38. *If $\mathbb{Q}_l[G]$ is a semisimple algebra, Proposition 3.2 is equivalent to the statement that $\mathbb{Q}_l[G]$ is the centralizer of $\mathbb{Q}_l \otimes \mathrm{End}(A)$.*

Proof. This follows from the semisimplicity of $\mathbb{Q}_l[G]$ and the double centralizer theorem. (because bijectivity of 3.2 just means that $\mathbb{Q}_l \otimes \mathrm{End}(A)$ is the centralizer of $\mathbb{Q}_l[G]$). □

In the 1960s, Tate studied endomorphisms of abelian varieties over finite fields. He proved the Tate conjecture for finite fields. Zarhin extended Tate's work to function fields over finite fields. In his breakthrough paper, Faltings proved the Tate conjecture for number fields. He used this to prove the Shafarevich conjecture for number fields and the Mordell conjecture, which will be discussed later in this paper.

3.1. **The general Tate conjecture.** There is a much more general conjecture called the *Tate conjecture on algebraic cycles*. Let X be a smooth projective variety of dimension d over K. Let l be a prime different from the characteristic of K. Then for all integers m, i, with $0 \leq i \leq 2d$, there is the *étale cohomology group* $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_l(m))$: this is a $\mathbb{Q}_l$-vector space equipped with the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$.

To any subvariety $Z \subset X$ of codimension i we can associate an étale cohomology class in $H^{2i}_{et}(X_{\overline{K}}, \mathbb{Q}_l(i))$. We say that the cohomology class is *represented* by the subvariety Z.

The Tate conjecture is a characterization of which étale cohomology classes are represented by algebraic subvarieties. It is the following statement:

Conjecture 39 (Tate)**.** *The $\mathbb{Q}_l$-subspace of Galois-invariant cohomology classes in $H^{2i}_{et}(X_{\overline{K}}, \mathbb{Q}_l(i))$ is spanned by classes represented by subvarieties of X.*

This conjecture, for divisors on abelian varieties (i.e. when $i = 1$ and X is an abelian variety) is equivalent to our Tate conjecture, Conjecture 33. The relationship comes from the fact that the Tate module is naturally isomorphic to the dual of the first étale cohomology group with coefficients in $\mathbb{Z}_l$, and the relationship between divisors and homomorphisms of abelian varieties given by

$$NS(A \times B) \cong NS(A) \oplus NS(B) \oplus Hom(A, B^\vee) \tag{3.5}$$

where $NS(X)$ is the *Neron-Severi group*, the group of divisors on X modulo algebraic equivalence.

The Tate conjecture is one of the "big" conjectures on algebraic cycles in the arithmetic context. It is closely related to a number of other famous conjecures: the Birch and Swinnerton-Dyer conjecture, Hodge conjecture, standard conjectures, finiteness of the Tate-Shafarevich group, etc. For a survey, see Totaro's expository article "Recent progress on the Tate conjecture".

4. Shafarevich conjecture

The Shafarevich conjecture is a finiteness statement in algebraic geometry; it states there are only finitely many families (up to isomorphism) of a certain kind. There are analogues in the function field setting, but we will be interested in the formulation over number fields. There are actually many finiteness statements involved in Faltings' work on the Mordell conjecture and related work. To keep track of them all, here are some acronyms:

Proposition 40. *There are finitely many isomorphism classes of g-dimensional abelian varieties*

(1) $(*P_d)$ *equipped with a degree d polarization (e.g.* $(*P_1)$ *equipped with a principal polarization)*
(2) (P_d) *that can be given a degree d polarization*
(3) (SS) *with semistable reduction everywhere*
(4) (F) *of bounded Faltings height*
(5) (I) *that are isogenous to A*
(6) (I_l) *that are isogenous to A, with the isogeny of l-power degree*
(7) (G) *with good reduction outside a finite set of places S*

Finally, let () be the finiteness statement "there are finitely many isomorphism classes of g-dimensional abelian varieties".

Statement (G) is called the *Shafarevich conjecture* (for abelian varieties): there are finitely many isomorphism classes of g-dimensional abelian varieties with good reduction outside a finite set of places S.

Remark 41. Note that the statement $(*P_d)$ is different from the rest; because it is not just a statement about a restricted class of abelian varieties, but abelian varieties with further structure. It is a finiteness statement for polarized abelian varieties, not abelian varieties.

I will represent various finiteness statements with codes, for example, $(*P_1SSF)$ is the statement that "there are finitely many isomorphism classes of principally polarized abelian varieties that have semistable reduction everywhere and of bounded Faltings height".

Remark 42. Obviously, "$(A) \implies (AB)$", if you know what I mean: a finiteness theorem with more conditions is implied by a finiteness theorem with less conditions.

Remark 43. The statements $(*P_d)$ and (P_d) are actually equivalent because of Proposition 27; given a polarizable abelian variety, there are only finitely many ways to make it a polarized abelian variety.

Remark 44. If we have a finiteness statement starting with $(*P_1)$ (equivalently (P_1)), we can often remove the $*P_1$ using Zarhin's trick.

Tate's work is an important precursor to Faltings work. Tate showed $(P_{d^2}I_l)$ implies the Tate conjecture, and then proved $(P_{d^2}I_l)$ over finite fields. A better way is to simply notice that the Siegel modular variety over a finite field, being a quasi-projective variety over a finite field, has finitely many rational points, establishing the finiteness statement $(*P_d)$. Using Zarhin's trick for $(*P_1)$ gives us (). Then using Tate's arguments we can show that () implies (P_dI_l) and (I_l). So, the story over finite fields looks like the following:

$$(*P_d) \implies () \implies (I_l) \implies Tate \tag{4.1}$$

(I) is called "Finiteness I". Statement (G) is called "Finiteness II", or the *Shafarevich conjecture for abelian varieties.* Faltings' proof proceeds by showing, essentially:

$$(*P_dSSF) \implies (*P_1SSF) \implies (SSF) \implies (I) \implies Tate \implies (G) \implies (*P_dG) \tag{4.2}$$

The first implication is trivial. The second implication, removing the principal polarizations on the abelian varieties, again uses Zarhin's trick. Proving $(SSF) \implies (I)$ is a technical step that involves earlier theories, specifically Raynaud's work on finite group schemes. (G) is really all we need for the Mordell conjecture, we don't need (P_dG). But I should remark that the implication $(*P_dG) \implies Tate$ was known before Faltings. Faltings idea is to replace $(*P_dG)$ with the slightly weaker statement $(*P_dSSF)$, use that to prove the Tate conjecture, and then use the Tate conjecture to finally establish $(*P_dG)$ in the end.

Remark 45. If I'm not mistaken, from (SSF) you can actually deduce (F). At least, this is suggested in Faltings-Chai. On the other hand, infinitely many quadratic twist seem to be a counterexample; wtf is going on?

Tate's work uses deduces the Tate conjecture over finite fields from a version of the Shafarevich conjecture. Faltings turns this around by first proving a weak version of the Shafarevich conjecture, then deducing the Tate conjecture, then proving the full Shafarevich conjecture.

A closely related statement is the *Shafarevich conjecture for curves*: Up to isomorphism, there are only finitely many curves of genus g with good reduction outside a finite set of places S.

Proposition 46. *The Shafarevich conjecture for abelian varieties implies the Shafarevich conjecture for curves.*

Proof. The connection is made by taking the Jacobian of the curve. The Torelli theorem says that given a Jacobian variety, the original curve can be recovered up to isomorphism. □

5. Mordell conjecture

5.1. **Digression on arithmetic of curves.** The Mordell conjecture is a basic statement about the arithmetic of curves. The arithmetic of an algebraic curve defined over a number field is controlled by an invariant called the *genus*, which is a nonnegative integer.

The arithmetic of genus 0 curves is completely understood: genus 0 curves are precisely the Brauer-Severi curves i.e. twists of $\mathbb{P}^1_k$. If C is a nontrivial twist, it has no rational points; otherwise $C \cong \mathbb{P}^1_k$. By the fundamental exact sequence of class field theory, it is possible to compute whether C is a nontrivial twist of $\mathbb{P}^1_k$.

Genus 1 curves are torsors under their Albanese varieties, which are elliptic curves. If C is a nontrivial torsor, it has no rational points; otherwise C is an elliptic curve. Using the theory of the Selmer group, it is possible to compute whether C is a nontrivial torsor. Thus the arithmetic of genus 1 curves reduces to the arithmetic of elliptic curves. This is a big subject. By the Mordell-Weil theorem, the set of rational points on an elliptic curve is a finitely generated abelian group. The *rank* of an elliptic curve is the rank of its group of rational points, as a $\mathbb{Z}$-module. So an elliptic curve has finitely many rational points if and only if its rank is zero.

Mazur's torsion theorem is an interesting result for elliptic curves defined over $\mathbb{Q}$. Mazur's torsion theorem lists the possible torsion of the group of rational points; in particular, the torsion group must have at most 16 elements. (This implies that an elliptic curve with finitely many rational points can have at most 16 rational points!). On the other hand, understanding the rank of elliptic curves is a major area of current research connected to the subject of L-functions. Conjecturally, the rank is given by the Birch and Swinnerton-Dyer conjecture, which states an equality between the rank of an elliptic curve and the order of vanishing of the L-function of the elliptic curve $L(s)$ at $s = 1$ (the "analytic rank").

So much for curves of genus 0 and 1. What about curves of higher genus? The Mordell conjecture states that curves of genus at least 2 defined over number fields have finitely many rational points.

There are four "routes" toward the Mordell conjecture, three of which are complete proofs. There is Faltings' proof of the Mordell conjecture, which is what this paper is about. It earned Faltings a Fields medal, and used a variety of advanced mathematics, including moduli stacks, p-adic Hodge theory, p-divisible groups, etc. There is Vojta's proof, using Diophantine approximation. Then there is the Chabauty-Coleman-Kim method, which is a p-adic method; it still does not yield a complete proof of the Mordell conjecture. Finally, there is the recent work of Lawrence and Venkatesh which proves the Mordell conjecture using p-adic period maps.

5.2. **Parshin's trick.** The Mordell conjecture (now Faltings' theorem) is the following statement:

Theorem 47 (Faltings)**.** *Let C be a smooth, projective and geometrically integral curve of genus $g \geq 2$ defined over a number field K. Then $C(K)$ is a finite set.*

The Mordell conjecture can be deduced from Shafarevich's conjecture for curves using a geometric construction due to Parshin known as Parshin's trick. Given a curve C over K and a point $P \in C(K)$, Parshin's trick produces a finite cover $\phi_P : C_P \to C$ that ramifies only at P. ϕ_P is defined over a finite extension L/K, and C_P is of bounded genus and has good reduction outside a finite set of primes of L.

Consider the maps

$$C(K) \to \{\text{possible } C_P \to C, \text{ up to isomorphism}\} \to \{\text{possible } C_P\} \tag{5.1}$$

The first map is injective, and the second map is finite-to-one when $g \geq 2$ by a classical theorem of de Franchis stating that, for fixed C' and C, there are only finitely many maps $C' \to C$ when C has genus greater than 2. (This is the only place where $g \geq 2$ is used!) Thus the composite is finite-to-one, and the latter set is finite by Shafarevich's conjecture. Thus the first set is finite, yielding Mordell's conjecture.

6. Tate's work

Here, using the simplification provided by Zarhin's trick, review the Tate and Shafarevich conjectures over finite fields.

Theorem 48. *There are finitely many isomorphism classes of abelian varieties over a finite field.*

Proof. First note that the existence of the Siegel moduli space implies that there are only finitely many isomorphism classes of *principally polarized* abelian varieties over a finite field. Then using Zarhin's trick and finiteness of direct factors, the result follows. □

Theorem 49. *The Tate conjecture (33) holds over finite fields. In addition if A is an abelian variety over a finite field K, the $\mathbb{Q}_l$-Tate module $V_l(A)$ is a semisimple l-adic representation of G_K.*

Proof. See Propositions 1 and 2 in Tate's paper, "Endomorphisms of abelian varieties over finite fields". The main idea of the proof is construct a sequence of abelian varieties B_n with isogenies $f_n : B_n \to A$, whose kernel is (at most) of size l^n. By the previous proposition, there are infinitely many of the B_n's in a single isomorphism class; these can be used to define elements of $\mathbb{Z}_l \otimes \mathrm{End}(A)$, which accumulate to a a desired element of $\mathbb{Z}_l \otimes \mathrm{End}(A)$ which is the preimage of the endomorphism of Tate modules under $\mathbb{Z}_l \otimes \mathrm{End}(A) \to \mathrm{End}(T_l(A))$. □

7. Faltings' work

To attack the problem in the setting of number fields, we would like to define a notion of the "height" $h(A)$ of an abelian variety A, a nonnegative real number, such that there are finitely many isomorphism classes of abelian varieties (satisfying some hypotheses) whose height is less than a fixed constant C.

There is a classical notion of the height of a point of projective space $\mathbb{P}^n_K$ for K a number field, such that there are finitely many rational points of bounded height. So one way to define a notion of the height of a principally polarized abelian variety is by considering an embedding of the Siegel moduli space into projective space. This is well-defined up to a constant, depending on the choice of embedding. This is called the *naive height*.

Faltings key innovation is a new notion of the height of an abelian variety. Unlike the naive height it is defined intrinsically using the data of the abelian variety. The definition is motivated by the philosophy of Arakelov theory.

Definition 50. Let $p : A \to S$ be a semiabelian scheme. Let $s : S \to A$ be the zero section. Then define the *Hodge line bundle* $\omega_{A/S}$ to be the sheaf $s^*(\bigwedge^k \Omega^1_{A/S})$.

In the following let K be a number field, $\mathcal{O}_K$ the ring of integers of K, M_K the set of places of K, and M_K^∞ the set of infinite places of K.

Definition 51. A *metrized line bundle* on $\operatorname{Spec} \mathcal{O}_K$ is an invertible $\mathcal{O}_K$-module equiped with a norm $|\cdot|_v$ on $K_v \otimes M$ for all $v \in M_K^\infty$.

Definition 52. If $(M, |\cdot|_v)$ is a metrized line bundle, its *Arakelov degree* is defined as

$$\log(|M/(\mathcal{O}_K \cdot m)|) - \sum_{v \in M_K^\infty} \log ||m||_v \tag{7.1}$$

where $0 \neq m \in M$ can be any nonzero element in M.

The Hodge bundle described earlier can be given the structure of a metrized line bundle as follows: If $v \in M_K^\infty$, then we can define a norm $|\cdot|_v$ on $K_v \otimes M \subseteq \mathbb{C} \otimes M$ as follows:

$$|\alpha|_v = \left(\left(\frac{i}{2g} \right)^g \int_{A(\mathbb{C})} |\alpha \wedge \overline{\alpha}| \right)^{\frac{1}{2}} \tag{7.2}$$

Definition 53. Let A be an abelian variety over a number field K. Then let ω_A be the Hodge line bundle of the Néron model of A, with the structure of a metrized line bundle. Then the *Faltings height* $h_F(A)$ is defined to be the Arakelov degree of ω_A.

The first main finiteness theorem is the following:

Proposition 54 (Height I, or $(*P_dSSF)$)**.** *There are only finitely isomorphism classes of polarized abelian varieties (A, λ) over K of dimension g, polarization of degree d, semistable reduction everywhere, and bounded Faltings height.*

Proof. This is the hardest part of Faltings paper. It involves a close study of certain moduli stacks and is proved by a comparison theorem comparing the naive height and Faltings height. □

Proposition 55 ((SSF))**.** *There are only finitely isomorphism classes of abelian varieties (A, λ) over K of dimension g with semistable reduction everywhere and bounded Faltings height.*

Proof. This follows from $(*P_1SSF)$ and Zarhin's trick, using a theorem of Raynaud that says that $h_F(A) = h_F(A^\vee)$. □

Proposition 56 (Height II)**.** *Let A be an abelian variety over K with semistable reduction everywhere. Then the set*

$$\{h(B) \mid B \text{ is isogenous to } A\} \tag{7.3}$$

is finite.

Proof. This is also difficult. It involves a long computation with p-divisible groups, and uses some p-adic Hodge theory and theorems of Raynaud and Tate on p-divisible groups. □

Theorem 57. *Let A be an abelian variety. Then there are only finitely many isomorphism classes of abelian varieties isogenous to A.*

Proof. The previous two propositions imply the theorem when A has semistable reduction everywhere. To remove this assumption, note that there is a finite extension L/K such that A acquires semistable reduction over L. The finiteness follows from the fact that there are only finitely isomorphism classes of abelian varieties defined over K that become isomorphic to each other over L, because the Galois cohomology group $\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{Aut}(A_L/L))$ is a finite group. □

Theorem 58. *The Tate conjecture holds over number fields. For an abelian variety A over a number field K, the representation $V_l(A)$ is semisimple.*

Proof. This uses essentially the same argument as Tate's proof, cf 49. □

Theorem 59 (Shafarevich, or (G))**.** *There are only finitely many isomorphism classes of abelian varieties with good reduction outside a finite set S of places of K.*

Proof. The idea of the proof is to show that, since the Tate modules are semisimple, that they give rise to only a finite number of trace functions. Here we can use the Weil conjectures, which imply the traces are integers with bounded absolute value. This proves finiteness up to isogeny. To prove it up to isomorphism we have to again prove a "boundedness within isogeny classes" result like Height II, using work of Raynaud and other technical tools. □

The Shafarevich conjecture for abelian varieties implies the Shafarevich conjecture for curves. By Parshin's trick, this implies the Mordell conjecture.

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