Spherical Trigonometry

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1 Introduction

The sides of a spherical triangle are arcs of great circles. A great circle is the intersection of a sphere with a central plane, a plane through the center of that sphere. The angles of a spherical triangle are measured in the plane tangent to the sphere at the intersection of the sides forming the angle.

To avoid conflict with the antipodal triangle, the triangle formed by the same great circles on the opposite side of the sphere, the sides of a spherical triangle will be restricted between 0 and π radians. The angles will also be restricted between 0 and π radians, so that they remain interior.

To derive the basic formulas pertaining to a spherical triangle, we use plane trigonometry on planes related to the spherical triangle. For example, planes tangent to the sphere at one of the vertices of the triangle, and central planes containing one side of the triangle.

Unless specified otherwise, when projecting onto a plane tangent to the sphere, the projection will be from the center of the sphere. Since each side of a spherical triangle is contained in a central plane, the projection of each side onto a tangent plane is a line. We will also assume the radius of the sphere is 1. Thus, the length of an arc of a great circle, is its angle.

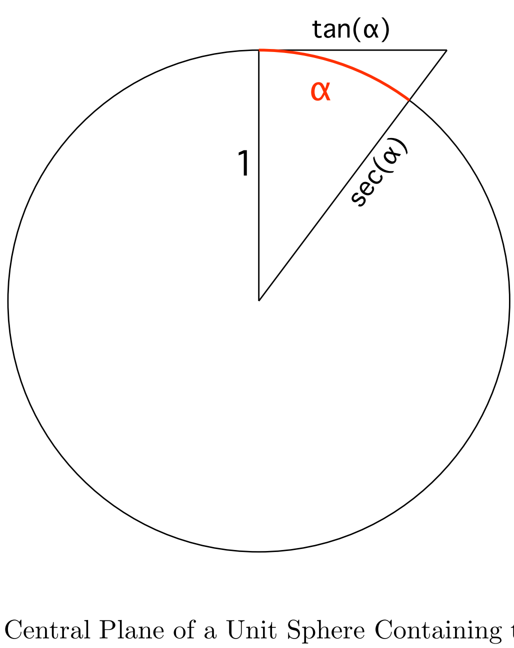

Figure 1: Central Plane of a Unit Sphere Containing the Side α

One of the simplest theorems of Spherical Trigonometry to prove using plane trigonometry is The Spherical Law of Cosines.

Theorem 1.1 (The Spherical Law of Cosines): *Consider a spherical triangle with sides α, β, and γ, and angle Γ opposite γ. To compute γ, we have the formula*

$$cos(\gamma) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)cos(\Gamma) \tag{1.1}$$

Proof: Project the triangle onto the plane tangent to the sphere at Γ and compute the length of the projection of γ in two different ways. First, using the plane Law of Cosines in the plane tangent to the sphere at Γ, we see that the length of the projection of γ is

$$tan^2(\alpha) + tan^2(\beta) - 2tan(\alpha)tan(\beta)cos(\Gamma) \tag{1.2}$$

Whereas if we use the plane Law of Cosines in the plane containing the great circle of γ, we get that the length of the projection of γ is

$$sec^2(\alpha) + sec^2(\beta) - 2sec(\alpha)sec(\beta)cos(\gamma) \tag{1.3}$$

By applying Figure 1 to α and β, Figure 2 illustrates these two methods of computing the length of the projection of γ onto the plane tangent at Γ, that is, the red segment:

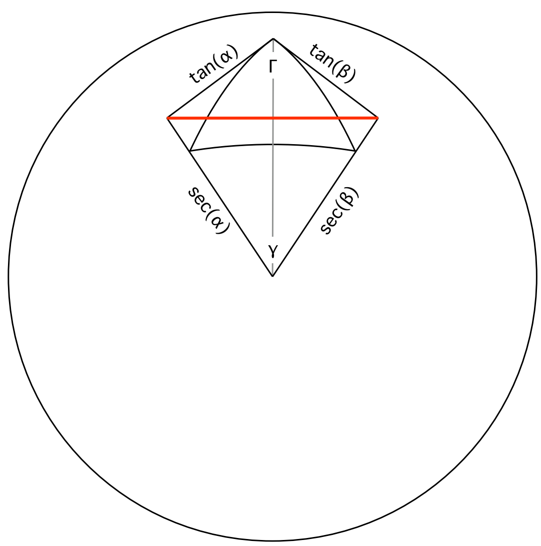

Figure 2: Two ways to measure the red segment

Subtracting equation (1.2) from equation (1.3), we get that

$$0 = 2 + 2tan(\alpha)tan(\beta)cos(\Gamma) - 2sec(\alpha)sec(\beta)cos(\gamma) \tag{1.4}$$

Solving for $cos(\gamma)$, gives The Spherical Law of Cosines:

$$cos(\gamma) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)cos(\Gamma) \tag{1.5}$$

□

Corollary 1.2: *Given the spherical triangle* ΑΒΓ *with opposing sides* α, β, *and* γ, *we have the following:*

$$sin(\alpha)cos(\mathrm{B}) = cos(\beta)sin(\gamma) - sin(\beta)cos(\gamma)cos(\mathrm{A}) \tag{1.6}$$

Proof: Extend the side γ to $\frac{\pi}{2}$ radians as in Figure 3:

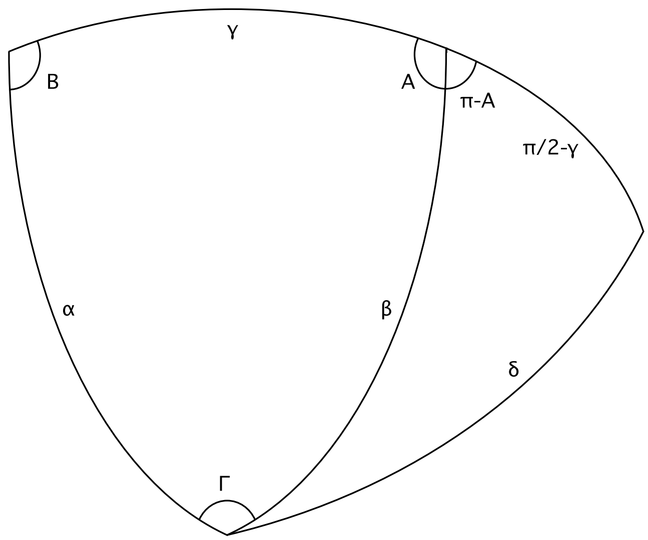

Figure 3

Using The Spherical Law of Cosines, there are two ways of computing $cos(\delta)$:

$$cos(\delta) = cos(\alpha)cos(\pi/2) + sin(\alpha)sin(\pi/2)cos(\mathrm{B}) \tag{1.7a}$$

$$= sin(\alpha)cos(\mathrm{B}) \tag{1.7b}$$

$$cos(\delta) = cos(\beta)cos(\pi/2 - c) + sin(\beta)sin(\pi/2 - c)cos(\pi - \mathrm{A}) \tag{1.8a}$$

$$= cos(\beta)sin(\gamma) - sin(\beta)cos(\gamma)cos(\mathrm{A}) \tag{1.8b}$$

Equating (1.7b) and (1.8b), we get the corollary:

$$sin(\alpha)cos(\mathrm{B}) = cos(\beta)sin(\gamma) - sin(\beta)cos(\gamma)cos(\mathrm{A}) \tag{1.9}$$

□

2 Duality: Equators and Poles

For every great circle, there are two antipodal points which are $\frac{\pi}{2}$ radians from every point on that great circle. Call these the poles of the great circle. Similarly, for each pair of antipodal points on a sphere, there is a great circle, every point of which is $\frac{\pi}{2}$ radians from the pair. Call this great circle the equator of these antipodal points. The line containing the poles is perpendicular to the plane containing the equator. Thus, a central plane contains both poles if and only if it is perpendicular to the equatorial plane. Therefore, any great circle containing a pole is perpendicular to the equator, and any great circle perpendicular to the equator contains both poles.

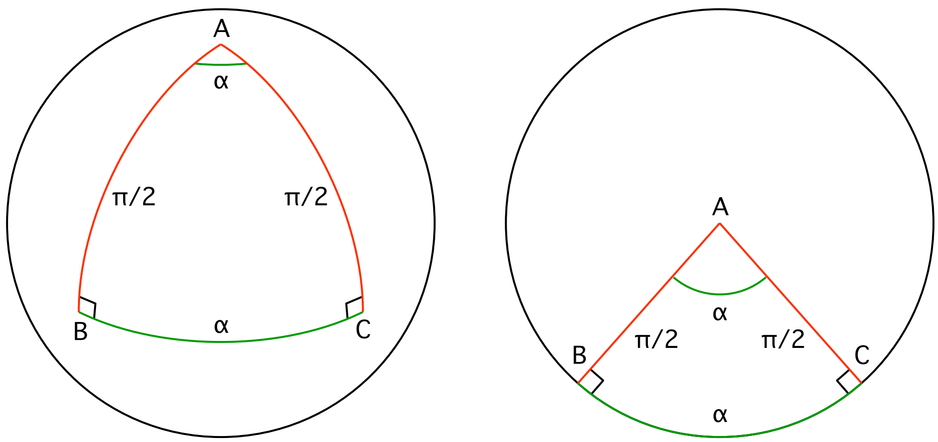

Figure 4: Semilunar Triangle
$\overline{BC}$ is an Arc of the Equator for the Pole A

In Figure 4, $\angle BAC$ is the angle between the plane containing $\overline{AB}$ and the plane containing $\overline{AC}$. As is evident in the view from above A, the length of $\overline{BC}$ is the same as the size of $\angle BAC$.

Definition 2.1 (Semilune): *A triangle in which one of the vertices is a pole of the opposing side is called a semilunar triangle, or a semilune.*

As described above, the angle at the pole has the same measure as the opposing side. All of the other sides and angles measure $\frac{\pi}{2}$ radians.

Lemma 2.2 (Semilunar Lemma): *If any two parts, a part being a side or an angle, of a spherical triangle measure $\frac{\pi}{2}$ radians, the triangle is a semilune.*

Proof: There are four cases:

1. two right sides
2. two right angles
3. opposing right side and right angle
4. adjacent right side and right angle

We will handle these cases in order.

Case 1 (two right sides):
Suppose both $\overline{AB}$ and $\overline{AC}$ have a length of $\frac{\pi}{2}$ radians. The Spherical Law of Cosines says

$$cos(\overline{BC}) = cos(\overline{AB})cos(\overline{AC}) + sin(\overline{AB})sin(\overline{AC})cos(\angle BAC) \tag{2.1a}$$
$$= cos(\frac{\pi}{2})cos(\frac{\pi}{2}) + sin(\frac{\pi}{2})sin(\frac{\pi}{2})cos(\angle BAC) \tag{2.1b}$$
$$= cos(\angle BAC) \tag{2.1c}$$

Thus $\angle BAC$ and opposing side $\overline{BC}$ are equal. Furthermore,

$$cos(\overline{AC}) = cos(\overline{AB})cos(\overline{BC}) + sin(\overline{AB})sin(\overline{BC})cos(\angle ABC) \tag{2.2a}$$
$$cos(\frac{\pi}{2}) = cos(\frac{\pi}{2})cos(\overline{BC}) + sin(\frac{\pi}{2})sin(\overline{BC})cos(\angle ABC) \tag{2.2b}$$
$$0 = sin(\overline{BC})cos(\angle ABC) \tag{2.2c}$$

Since $\overline{BC}$ is between 0 and pi radians, $sin(\overline{BC}) \neq 0$; thus, $cos(\angle ABC) = 0$, and $\angle ABC$ must be $\frac{\pi}{2}$ radians. By a similar argument, $\angle ACB$ must also be $\frac{\pi}{2}$ radians.

Case 2 (two right angles):
Suppose both $\angle ABC$ and $\angle ACB$ are right angles. The Spherical Law of Cosines says that

$$cos(\overline{AC}) = cos(\overline{AB})cos(\overline{BC}) + sin(\overline{AB})sin(\overline{BC})cos(\angle ABC) \tag{2.3a}$$
$$= cos(\overline{AB})cos(\overline{BC}) + sin(\overline{AB})sin(\overline{BC})cos(\frac{\pi}{2}) \tag{2.3b}$$
$$= cos(\overline{AB})cos(\overline{BC}) \tag{2.3c}$$

Similarly, $cos(\overline{AB}) = cos(\overline{AC})cos(\overline{BC})$. Plugging this formula for $cos(\overline{AB})$ into equation (2.3), we get

$$cos(\overline{AC}) = cos(\overline{AC})cos^2(\overline{BC}) \tag{2.4}$$

Subtracting the right side of equation (2.4) from both sides yields

$$cos(\overline{AC})sin^2(\overline{BC}) = 0 \tag{2.5}$$

Since $\overline{BC}$ is between 0 and π radians, $sin(\overline{BC}) \neq 0$. Therefore, $cos(\overline{AC}) = 0$, and $\overline{AC}$ is $\frac{\pi}{2}$ radians. By the same argument, $\overline{AB}$ is also $\frac{\pi}{2}$ radians. Now apply Case 1.

Case 3 (opposing right side and right angle):
Suppose both $\angle ABC$ and $\overline{AC}$ measure $\frac{\pi}{2}$ radians. equation (2.3) says that

$$cos(\overline{AC}) = cos(\overline{AB})cos(\overline{BC}) \tag{2.6a}$$
$$0 = cos(\overline{AB})cos(\overline{BC}) \tag{2.6b}$$

Therefore, one of $\overline{AB}$ or $\overline{BC}$ must be $\frac{\pi}{2}$ radians, and we are back to Case 1.

Case 4 (adjacent right side and right angle):
Suppose both $\angle ABC$ and $\overline{AB}$ measure $\frac{\pi}{2}$ radians. The Spherical Law of Cosines says that

$$cos(\overline{AC}) = cos(\overline{AB})cos(\overline{BC}) + sin(\overline{AB})sin(\overline{BC})cos(\angle ABC) \tag{2.7a}$$
$$= cos(\frac{\pi}{2})cos(\overline{BC}) + sin(\frac{\pi}{2})sin(\overline{BC})cos(\frac{\pi}{2}) \tag{2.7b}$$
$$= 0 \tag{2.7c}$$

Thus, $\overline{AC}$ is $\frac{\pi}{2}$ radians, and we are back to Case 1. □

3 Dual Triangles

Definition 3.1 (Dual Triangle): *Given a spherical triangle ABC, let $\triangle A'B'C'$ be the triangle whose vertices are the poles of the sides opposite the corresponding vertices of $\triangle ABC$ in the same hemisphere as $\triangle ABC$ (i.e. A' is on the same side of $\overline{BC}$ as A, etc.). $\triangle A'B'C'$ is the dual of $\triangle ABC$.*

As in Figure 5, let a, b, and c be the sides opposite A, B, and C respectively, and a', b', and c' the sides opposite A', B', and C'. Since A', B', and C' are the poles of a, b, and c, all the red arcs measure $\frac{\pi}{2}$ radians. By construction, $\triangle ABC'$, $\triangle AB'C$, and $\triangle A'BC$ are semilunes. However, by Lemma 2.2, so are $\triangle A'B'C$, $\triangle A'BC'$, and $\triangle AB'C'$. Thus, the vertices of $\triangle ABC$ are poles of the sides of $\triangle A'B'C'$, in the proper hemispheres. Therefore, $\triangle ABC$ is the dual of $\triangle A'B'C'$.

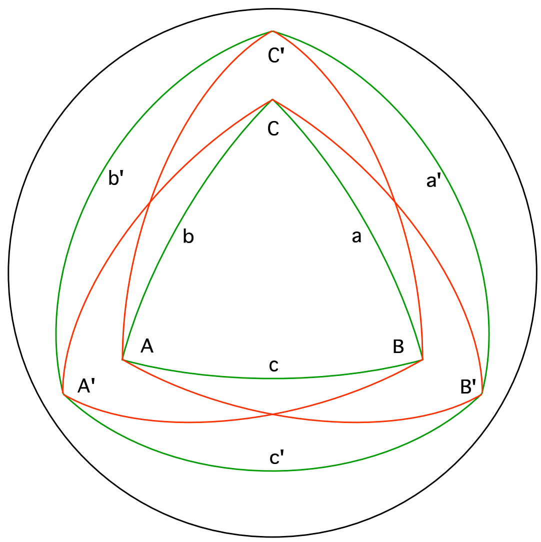

Figure 5: Dual Triangles

Theorem 3.2 (Angle and Side Duality): *The measure of an angle in a spherical triangle and the length of the corresponding side in its dual are supplementary.*

Proof: Given $\triangle ABC$, let $\triangle A'B'C'$ be its dual as constructed above. By the duality of the construction, we need only consider one side and the angle at its corresponding pole, which is a vertex of the dual triangle. Consider $\angle ACB$ and c' in Figure 5. As noted above, $\triangle A'CB'$, $\triangle AB'C$, and $\triangle A'BC$ are semilunes. Thus, $\angle A'CB'$ and c' are the same size. Furthermore, $\angle A'CB$ and $\angle ACB'$ are right angles. Therefore,

$$c' + \angle ACB = \angle A'CB' + \angle ACB \tag{3.1}$$
$$= \angle A'CB + \angle ACB' \tag{3.2}$$
$$= \frac{\pi}{2} + \frac{\pi}{2} \tag{3.3}$$
$$= \pi \tag{3.4}$$

Thus, we have that side c' and $\angle ACB$ are supplementary. □

Applying The Spherical Law of Cosines to the dual of a spherical triangle, we get

Theorem 3.3 (The Law of Cosines for Angles): *Given a spherical triangle with two angles* A *and* B *and the side* γ *between them, we can compute the cosine of opposite angle,* Γ*, with*

$$cos(\Gamma) = -cos(\mathrm{A})cos(\mathrm{B}) + sin(\mathrm{A})sin(\mathrm{B})cos(\gamma) \tag{3.5}$$

Proof: Consider $\triangle \mathrm{A'B'}\Gamma'$, the dual of $\triangle \mathrm{AB}\Gamma$, with sides α', β', and γ'. Apply The Spherical Law of Cosines to compute γ':

$$cos(\gamma') = cos(\alpha')cos(\beta') + sin(\alpha')sin(\beta')cos(\Gamma') \tag{3.6}$$

Use Theorem 3.2 to replace each angle and side with the supplement of the corresponding side and angle in the dual

$$cos(\pi - \Gamma) = cos(\pi - \mathrm{A})cos(\pi - \mathrm{B}) + sin(\pi - \mathrm{A})sin(\pi - \mathrm{B})cos(\pi - \gamma) \tag{3.7}$$

Since $cos(\pi - x) = -cos(x)$ and $sin(\pi - x) = sin(x)$, this becomes

$$cos(\Gamma) = -cos(\mathrm{A})cos(\mathrm{B}) + sin(\mathrm{A})sin(\mathrm{B})cos(\gamma) \tag{3.8}$$

□

Theorem 3.4 (Incircle and Circumcircle Duality): *The incenter of a spherical triangle is the circumcenter of its dual. The inradius of a spherical triangle is the complement of the the circumradius of its dual.*

Proof: Given $\triangle ABC$, as in Figure 6, let G be the center of its incircle, and D, E, and F be the points of tangency of the incircle with sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. Let $\triangle A'B'C'$ be the dual of $\triangle ABC$.

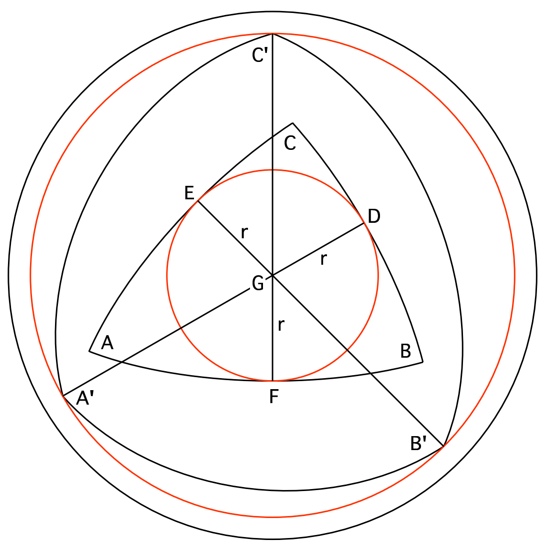

Figure 6: Incircle and Dual Circumcircle

Since any radius of a circle is perpendicular to the circle, $\overline{GD}$ is perpendicular to $\overline{BC}$. Therefore, if we extend $\overline{DG}$, it passes through A', the pole of $\overline{BC}$, and $\overline{DA'}$ has length $\frac{\pi}{2}$. The same is true for the other points of tangency. Thus, $\overline{GA'}$, $\overline{GB'}$, and $\overline{GC'}$ are complementary to r, the inradius of $\triangle ABC$, and hence, equal. We can then conclude that the incenter of $\triangle ABC$ is the circumcenter of $\triangle A'B'C'$ and the inradius of $\triangle ABC$ is complementary to the circumradius of $\triangle A'B'C'$. □

4 Right Spherical Triangles

As in plane trigonometry, many facts about spherical triangles can be derived using right spherical triangles.

Theorem 4.1 (Projecting Right Angles): *Projecting a right spherical triangle onto a plane tangent to any of its vertices preserves the right angle.*

Proof: The case of projecting at the right angle is trivial. Therefore, consider the right triangle ABC in Figure 7.

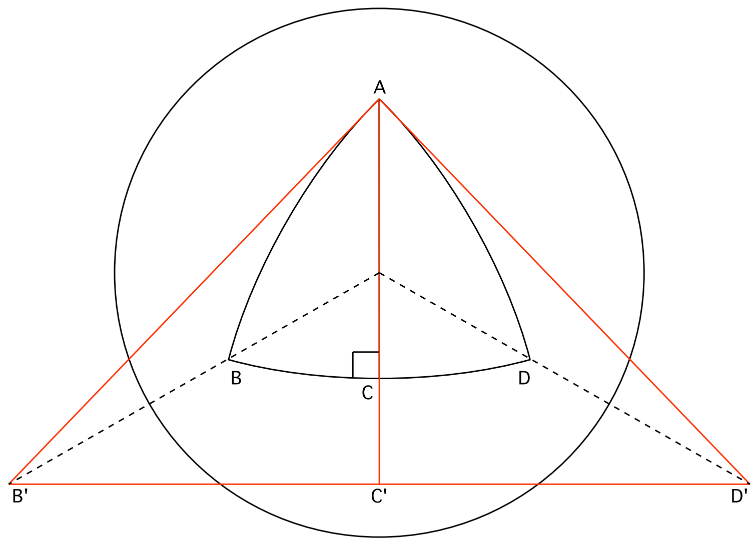

Figure 7

Construct the right triangle ADC, congruent to ABC, only reflected across side AC. Made of two right angles, $\angle BCD$ is a straight angle; thus, C is on $\overline{BD}$. Let B', C', and D' be the projections of B, C, and D onto the plane tangent at A; thus, C' is on $\overline{B'D'}$. Since $\triangle ABC$ is congruent to $\triangle ADC$, $\triangle AD'C'$ is congruent to $\triangle AB'C'$; that is, $\overline{AC'} = \overline{AC'}$, $\overline{AB'} = tan(\overline{AB}) = tan(\overline{AD}) = \overline{AD'}$, and $\angle B'AC' = \angle BAC = \angle DAC = \angle D'AC'$. Therefore, $\angle AC'B' = \angle AC'D'$, yet since C' is on $\overline{B'D'}$, $\angle AC'B'$ and $\angle AC'D'$ are supplementary. Thus, each is a right angle. ☐

Theorem 4.1 says that right angles are preserved when projecting a spherical triangle onto a plane tangent at any vertex of the given triangle. Corollary 4.2 tells what happens to other sizes of angles in a spherical triangle.

Corollary 4.2 (Projecting Angles): *Given a spherical triangle projected onto a plane tangent at one angle, the tangent of the projection of any other angle in the triangle is the tangent of the corresponding spherical angle times the cosine of the edge connecting the angles.*

Proof: Consider $\triangle ABD$ in Figure 7. From A, drop the perpendicular, $\overline{AC}$, to $\overline{BD}$. Consider the projection of $\triangle ABC$ onto the plane tangent at A, $\triangle AB'C'$. Since $\triangle ABC$ is a right triangle, Theorem 4.1 assures that $\triangle AB'C'$ is also a right triangle. The Law of Cosines for Angles says that

$$cos(\angle ACB) = -cos(\angle CAB)\, cos(\angle B) + sin(\angle CAB)\, sin(\angle B)\, cos(\overline{AB}) \tag{4.1}$$

Since $\angle ACB$ is a right angle, $cos(\angle ACB) = 0$, leaving us with

$$tan(\angle CAB)\,tan(\angle B)\,cos(\overline{AB}) = 1 \tag{4.2}$$

Now in the plane, $\angle CAB = \angle C'AB'$ is complementary to $\angle B'$, therefore

$$tan(\angle CAB)\,tan(\angle B') = 1 \tag{4.3}$$

Combining equation (4.2) and equation (4.3), we get that

$$tan(\angle B') = tan(\angle B)\,cos(\overline{AB}) \tag{4.4}$$

□

In plane trigonometry, the Law of Cosines is usually derived from the Pythagorean Theorem. Here, we prove things the other way around.

Theorem 4.3 (Spherical Pythagorean Theorem): *Given a right triangle with legs α and β and hypotenuse γ, we have the relation*

$$cos(\alpha)cos(\beta) = cos(\gamma) \tag{4.5}$$

Proof: In Figure 8, $\angle\Gamma$ is a right angle.

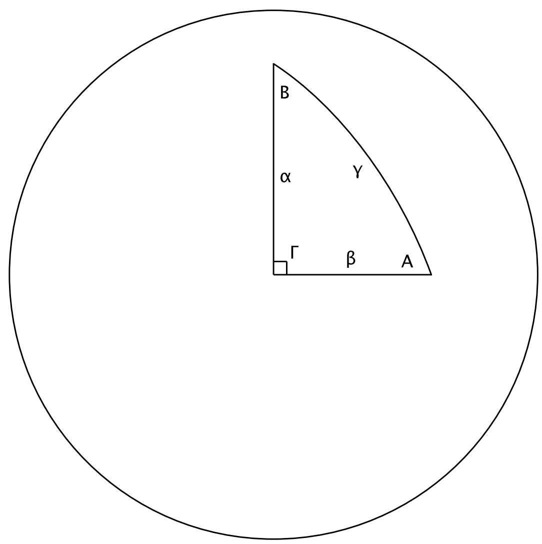

Figure 8

Applying The Spherical Law of Cosines, we get

$$cos(\gamma) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)cos(\Gamma) \tag{4.6a}$$

$$= cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)cos(\frac{\pi}{2}) \tag{4.6b}$$

$$= cos(\alpha)cos(\beta) \tag{4.6c}$$

□

Theorem 4.4 (Right Spherical Trigonometric Identities): *Given a right triangle* ABΓ *with legs* α *and* β *opposite angles* A *and* B*, and hypotenuse* γ *opposite the right angle* Γ*, we have*

$$sin(\mathrm{A}) = \frac{sin(\alpha)}{sin(\gamma)} \qquad sin(\mathrm{B}) = \frac{sin(\beta)}{sin(\gamma)} \tag{4.7a}$$

$$cos(\mathrm{A}) = \frac{tan(\beta)}{tan(\gamma)} \qquad cos(\mathrm{B}) = \frac{tan(\alpha)}{tan(\gamma)} \tag{4.7b}$$

$$tan(\mathrm{A}) = \frac{tan(\alpha)}{sin(\beta)} \qquad tan(\mathrm{B}) = \frac{tan(\beta)}{sin(\alpha)} \tag{4.7c}$$

Proof: Refer to Figure 8. If we project the triangle onto the plane tangent at A, we get $\triangle \mathrm{AB}'\Gamma'$. By Theorem 4.1, Γ' is a right angle. $\mathrm{AB}' = tan(\gamma)$ and $\mathrm{A}\Gamma' = tan(\beta)$. Using the plane trigonometric formulae, we get

$$cos(\mathrm{A}) = \frac{tan(\beta)}{tan(\gamma)} \tag{4.8}$$

By similarity, we also have

$$cos(\mathrm{B}) = \frac{tan(\alpha)}{tan(\gamma)} \tag{4.9}$$

Projecting onto the plane tangent at Γ, we get the right triangle $\mathrm{A}'\mathrm{B}'\Gamma$. Using this projection, plane trigonometric formulae say that $tan(\mathrm{A}') = \frac{tan(\alpha)}{tan(\beta)}$. Corollary 4.2 says that $tan(\mathrm{A}') = tan(\mathrm{A})\, cos(\beta)$. Therefore, we get

$$tan(\mathrm{A}) = \frac{tan(\alpha)}{sin(\beta)} \tag{4.10}$$

By similarity, we also have

$$tan(\mathrm{B}) = \frac{tan(\beta)}{sin(\alpha)} \tag{4.11}$$

Multiplying equation (4.8) and equation (4.10), we get

$$sin(\mathrm{A}) = \frac{tan(\alpha)}{sin(\beta)} \frac{tan(\beta)}{tan(\gamma)} \tag{4.12a}$$

$$= \frac{sin(\alpha)}{sin(\gamma)} \frac{cos(\gamma)}{cos(\alpha)cos(\beta)} \tag{4.12b}$$

$$= \frac{sin(\alpha)}{sin(\gamma)} \tag{4.12c}$$

where the last step in equation (4.12) follows by The Spherical Pythagorean Theorem.

By similarity, we also have

$$sin(\mathrm{B}) = \frac{sin(\beta)}{sin(\gamma)} \tag{4.13}$$

□

5 Spherical Trigonometric Formulas

Theorem 5.1 (The Spherical Law of Sines): *Given* $\triangle ABC$ *with sides* a *and* b *opposite angles* A *and* B*, we have the following:*

$$\frac{sin(a)}{sin(A)} = \frac{sin(b)}{sin(B)} \tag{5.1}$$

Proof: As in Figure 9, drop the perpendicular $\overline{CD}$ onto $\overline{AB}$. Using equations (4.7a), compute $sin(h)$ in two ways: $sin(B)sin(a) = sin(h) = sin(A)sin(b)$. Dividing by $sin(A)sin(B)$, we get equation (5.1). □

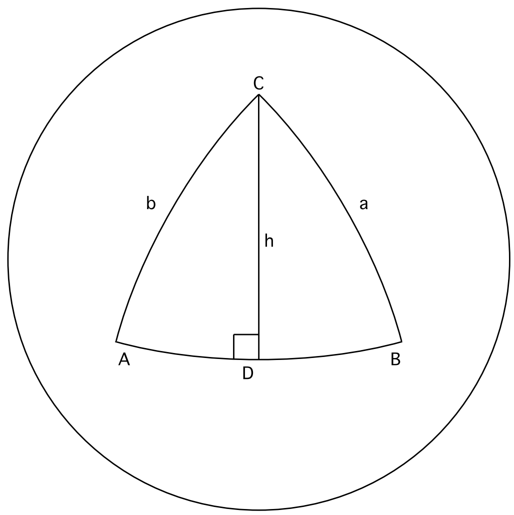

Figure 9

Given two sides and their included angle, The Spherical Law of Cosines yields the cosine of the remaining side. As a complement, the following theorem yields the tangents of the other angles.

Theorem 5.2: *Given the spherical triangle ABC with opposing sides a, b, and c, we have the following:*

$$tan(A) = \frac{tan(a)sec(b)sin(C)}{tan(b) - tan(a)cos(C)} \tag{5.2}$$

Proof: The Spherical Law of Sines followed by Corollary 1.2 says

$$\begin{aligned} tan(A) &= \frac{sin(A)}{cos(A)} && (5.3a)\\ &= \frac{sin(a)sin(C)}{sin(c)cos(A)} && (5.3b)\\ &= \frac{sin(a)sin(C)}{cos(a)sin(b) - sin(a)cos(b)cos(C)} && (5.3c)\\ &= \frac{tan(a)sec(b)sin(C)}{tan(b) - tan(a)cos(C)} && (5.3d) \end{aligned}$$

□

6 Trigonometric Identities

Let us recall some trigonometric identities that we will need.

Theorem 6.1:

$$sin(u+v) + sin(u-v) = 2\,sin(u)\,cos(v) \qquad sin(x) + sin(y) = 2\,sin\left(\frac{x+y}{2}\right)\,cos\left(\frac{x-y}{2}\right) \tag{6.1a}$$

$$sin(u+v) - sin(u-v) = 2\,cos(u)\,sin(v) \qquad sin(x) - sin(y) = 2\,cos\left(\frac{x+y}{2}\right)\,sin\left(\frac{x-y}{2}\right) \tag{6.1b}$$

$$cos(u+v) + cos(u-v) = 2\,cos(u)\,cos(v) \qquad cos(x) + cos(y) = 2\,cos\left(\frac{x+y}{2}\right)\,cos\left(\frac{x-y}{2}\right) \tag{6.1c}$$

$$cos(u-v) - cos(u+v) = 2\,sin(u)\,sin(v) \qquad cos(y) - cos(x) = 2\,sin\left(\frac{x+y}{2}\right)\,sin\left(\frac{x-y}{2}\right) \tag{6.1d}$$

Proof: These identities follow immediately using the standard trigonometric sum and difference formulae

$$sin(u+v) = sin(u)\,cos(v) + cos(u)\,sin(v) \tag{6.2a}$$

$$sin(u-v) = sin(u)\,cos(v) - cos(u)\,sin(v) \tag{6.2b}$$

$$cos(u+v) = cos(u)\,cos(v) - sin(u)\,sin(v) \tag{6.2c}$$

$$cos(u-v) = cos(u)\,cos(v) + sin(u)\,sin(v) \tag{6.2d}$$

and the relations

$$u = \frac{x+y}{2} \qquad v = \frac{x-y}{2} \tag{6.3a}$$

□

Corollary 6.2:

$$\frac{sin(x) + sin(y)}{cos(x) + cos(y)} = tan\left(\frac{x+y}{2}\right) \tag{6.4}$$

Proof: Divide equation (6.1a) by equation (6.1c). □

7 Spherical Half-Angle Formulas

Given the spherical triangle ABC with opposing sides a, b, and c, use The Spherical Law of Cosines to compute $cos(A)$:

$$cos(A) = \frac{cos(a) - cos(b)cos(c)}{sin(b)sin(c)} \tag{7.1}$$

Thus, using equation (7.1), equation (6.1d), and setting $s = \frac{a+b+c}{2}$, we get

$$\begin{aligned} sin^2\left(\frac{A}{2}\right) &= \frac{1 - cos(A)}{2} && (7.2a) \\ &= \frac{sin(b)sin(c) + cos(b)cos(c) - cos(a)}{2\,sin(b)sin(c)} && (7.2b) \\ &= \frac{cos(b-c) - cos(a)}{2\,sin(b)sin(c)} && (7.2c) \\ &= \frac{sin(s-b)sin(s-c)}{sin(b)sin(c)} && (7.2d) \end{aligned}$$

Similarly, we have

$$\begin{aligned} cos^2\left(\frac{A}{2}\right) &= \frac{1 + cos(A)}{2} && (7.3a) \\ &= \frac{cos(a) - cos(b)cos(c) + sin(b)sin(c)}{2\,sin(b)sin(c)} && (7.3b) \\ &= \frac{cos(a) - cos(b+c)}{2\,sin(b)sin(c)} && (7.3c) \\ &= \frac{sin(s)sin(s-a)}{sin(b)sin(c)} && (7.3d) \end{aligned}$$

Using equation (7.2) and equation (7.3) and their analogs for B and C, we get

$$\begin{aligned} sin\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) &= \sqrt{\frac{sin(s-a)sin(s-b)sin^2(s-c)}{sin(a)sin(b)sin^2(c)}} && (7.4a) \\ &= sin\left(\frac{C}{2}\right)\frac{sin(s-c)}{sin(c)} && (7.4b) \end{aligned}$$

and

$$\begin{aligned} sin\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) &= \sqrt{\frac{sin(s)sin(s-c)sin^2(s-b)}{sin(a)sin(b)sin^2(c)}} && (7.5a) \\ &= cos\left(\frac{C}{2}\right)\frac{sin(s-b)}{sin(c)} && (7.5b) \end{aligned}$$

and

$$\begin{aligned} cos\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) &= \sqrt{\frac{sin(s)sin(s-c)sin^2(s-a)}{sin(a)sin(b)sin^2(c)}} && (7.6a) \\ &= cos\left(\frac{C}{2}\right)\frac{sin(s-a)}{sin(c)} && (7.6b) \end{aligned}$$

and

$$cos\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) = \sqrt{\frac{sin(s-a)sin(s-b)sin^2(s)}{sin(a)sin(b)sin^2(c)}} \tag{7.7a}$$

$$= sin\left(\frac{C}{2}\right)\frac{sin(s)}{sin(c)} \tag{7.7b}$$

Theorem 7.1: *Given the spherical triangle ABC with opposing sides a, b, and c, we have the following:*

$$\frac{cos\left(\frac{A+B}{2}\right)}{sin\left(\frac{C}{2}\right)} = \frac{cos\left(\frac{a+b}{2}\right)}{cos\left(\frac{c}{2}\right)} \tag{7.8a}$$

$$\frac{sin\left(\frac{A+B}{2}\right)}{cos\left(\frac{C}{2}\right)} = \frac{cos\left(\frac{a-b}{2}\right)}{cos\left(\frac{c}{2}\right)} \tag{7.8b}$$

$$\frac{cos\left(\frac{A-B}{2}\right)}{sin\left(\frac{C}{2}\right)} = \frac{sin\left(\frac{a+b}{2}\right)}{sin\left(\frac{c}{2}\right)} \tag{7.8c}$$

$$\frac{sin\left(\frac{A-B}{2}\right)}{cos\left(\frac{C}{2}\right)} = \frac{sin\left(\frac{a-b}{2}\right)}{sin\left(\frac{c}{2}\right)} \tag{7.8d}$$

Proof: We will use equations (6.1) and equations (6.2). We can write $cos\left(\frac{A+B}{2}\right)$ as

$$cos\left(\frac{A+B}{2}\right) = cos\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) - sin\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) \tag{7.9a}$$

$$= sin\left(\frac{C}{2}\right)\frac{sin(s) - sin(s-c)}{sin(c)} \tag{7.9b}$$

$$= sin\left(\frac{C}{2}\right)\frac{2\,cos\left(\frac{a+b}{2}\right)\,sin\left(\frac{c}{2}\right)}{sin(c)} \tag{7.9c}$$

$$= sin\left(\frac{C}{2}\right)\frac{cos\left(\frac{a+b}{2}\right)}{cos\left(\frac{c}{2}\right)} \tag{7.9d}$$

and we can write $sin\left(\frac{A+B}{2}\right)$ as

$$sin\left(\frac{A+B}{2}\right) = sin\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) + cos\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) \tag{7.10a}$$

$$= cos\left(\frac{C}{2}\right)\frac{sin(s-b)+sin(s-a)}{sin(c)} \tag{7.10b}$$

$$= cos\left(\frac{C}{2}\right)\frac{2\,cos\left(\frac{a-b}{2}\right)\;sin\left(\frac{c}{2}\right)}{sin(c)} \tag{7.10c}$$

$$= cos\left(\frac{C}{2}\right)\frac{cos\left(\frac{a-b}{2}\right)}{cos\left(\frac{c}{2}\right)} \tag{7.10d}$$

and we can write $cos\left(\frac{A-B}{2}\right)$ as

$$cos\left(\frac{A-B}{2}\right) = cos\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) + sin\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) \tag{7.11a}$$

$$= sin\left(\frac{C}{2}\right)\frac{sin(s)+sin(s-c)}{sin(c)} \tag{7.11b}$$

$$= sin\left(\frac{C}{2}\right)\frac{2\,sin\left(\frac{a+b}{2}\right)\;cos\left(\frac{c}{2}\right)}{sin(c)} \tag{7.11c}$$

$$= sin\left(\frac{C}{2}\right)\frac{sin\left(\frac{a+b}{2}\right)}{sin\left(\frac{c}{2}\right)} \tag{7.11d}$$

and we can write $sin\left(\frac{A-B}{2}\right)$ as

$$sin\left(\frac{A-B}{2}\right) = sin\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right) - cos\left(\frac{A}{2}\right)sin\left(\frac{B}{2}\right) \tag{7.12a}$$

$$= cos\left(\frac{C}{2}\right)\frac{sin(s-b)-sin(s-a)}{sin(c)} \tag{7.12b}$$

$$= cos\left(\frac{C}{2}\right)\frac{2\,cos\left(\frac{c}{2}\right)\;sin\left(\frac{a-b}{2}\right)}{sin(c)} \tag{7.12c}$$

$$= cos\left(\frac{C}{2}\right)\frac{sin\left(\frac{a-b}{2}\right)}{sin\left(\frac{c}{2}\right)} \tag{7.12d}$$

□

Corollary 7.2: *Given the spherical triangle ABC with opposing sides a, b, and c, we have the following:*

$$tan\left(\frac{A+B}{2}\right) tan\left(\frac{C}{2}\right) = \frac{cos\left(\frac{a-b}{2}\right)}{cos\left(\frac{a+b}{2}\right)} \tag{7.13a}$$

$$= \frac{1 + tan\left(\frac{a}{2}\right) tan\left(\frac{b}{2}\right)}{1 - tan\left(\frac{a}{2}\right) tan\left(\frac{b}{2}\right)} \tag{7.13b}$$

$$tan\left(\frac{A-B}{2}\right) tan\left(\frac{C}{2}\right) = \frac{sin\left(\frac{a-b}{2}\right)}{sin\left(\frac{a+b}{2}\right)} \tag{7.14a}$$

$$= \frac{tan\left(\frac{a}{2}\right) - tan\left(\frac{b}{2}\right)}{tan\left(\frac{a}{2}\right) + tan\left(\frac{b}{2}\right)} \tag{7.14b}$$

Proof: Dividing equation (7.8b) by equation (7.8a) yields equation (7.13). Dividing equation (7.8d) by equation (7.8c) yields equation (7.14). □

Corollary 7.3 (The Spherical Law of Tangents): *Given the spherical triangle ABC with opposing sides a, b, and c, we have the following:*

$$\frac{tan\left(\frac{A-B}{2}\right)}{tan\left(\frac{A+B}{2}\right)} = \frac{tan\left(\frac{a-b}{2}\right)}{tan\left(\frac{a+b}{2}\right)} \tag{7.15}$$

Proof: Divide equation (7.14a) by equation (7.13a). □

8 Spherical Excess

In plane triangles, the angles all sum to π radians. In spherical triangles, the sum of the angles is greater than π radians. This leads us to the following

Definition 8.1 (Spherical Excess): *The spherical excess of a spherical triangle is the sum of its angles minus π radians.*

Theorem 8.2 (Girard's Theorem): *The area of a spherical triangle is equal to its spherical excess.*

Proof: Consider the area between the two great circles which form $\angle A$ of $\triangle ABC$ in Figure 10. If $\angle A$ is π radians, the area of the double wedge would be 4π steradians, the area of the full sphere. The area of such a double wedge varies linearly with the angle of the wedge. Thus, the area of the wedge is 4 times $\angle A$.

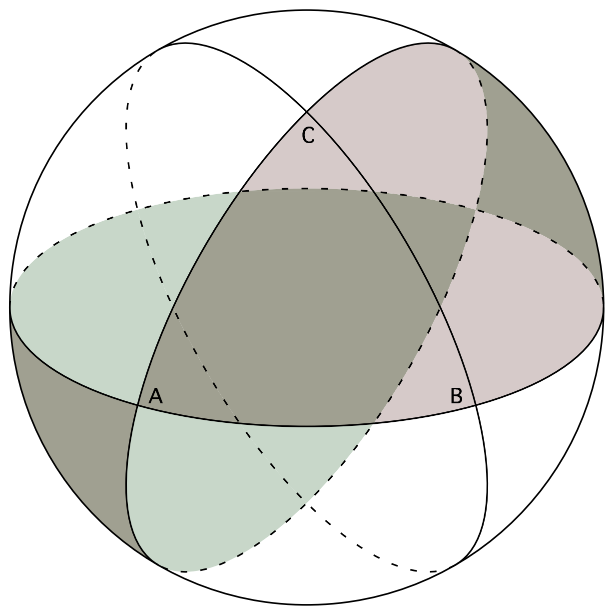

Figure 10

If we count the contribution of all the double wedges for all the angles of $\triangle ABC$, we see that all of the sphere is covered, but $\triangle ABC$ and its antipodal triangle are covered 3 times; that is, 2 times each more than needed to cover the sphere, or 4 times the area of $\triangle ABC$ more than 4π. Thus,

$$4A + 4B + 4C = 4\pi + 4 \times \text{area of } \triangle ABC \tag{8.1}$$

Whence we get the formula for the area of a spherical triangle:

$$\begin{aligned} \text{area of } \triangle ABC &= A + B + C - \pi && (8.2a) \\ &= \text{spherical excess of } \triangle ABC && (8.2b) \end{aligned}$$

□

9 Spherical Triangular Area

The following formula is the spherical equivalent of Heron's Formula for the area of a plane triangle.

Theorem 9.1 (L'Huilier's Formula): *Let E be the spherical excess of $\triangle ABC$, and a, b, and c be the sides opposite the corresponding angles. Let $s = \frac{a+b+c}{2}$. Then we have*

$$tan\left(\frac{E}{4}\right) = \sqrt{tan\left(\frac{s}{2}\right) tan\left(\frac{s-a}{2}\right) tan\left(\frac{s-b}{2}\right) tan\left(\frac{s-c}{2}\right)} \tag{9.1}$$

Proof:

$$tan\left(\frac{E}{4}\right) = tan\left(\frac{A+B+C-\pi}{4}\right) \tag{9.2a}$$

$$= \frac{sin\left(\frac{A+B}{2}\right) + sin\left(\frac{C-\pi}{2}\right)}{cos\left(\frac{A+B}{2}\right) + cos\left(\frac{C-\pi}{2}\right)} \tag{9.2b}$$

$$= \frac{sin\left(\frac{A+B}{2}\right) - cos\left(\frac{C}{2}\right)}{cos\left(\frac{A+B}{2}\right) + sin\left(\frac{C}{2}\right)} \tag{9.2c}$$

$$= \frac{\left(cos\left(\frac{a-b}{2}\right) - cos\left(\frac{c}{2}\right)\right) cos\left(\frac{C}{2}\right)}{\left(cos\left(\frac{a+b}{2}\right) + cos\left(\frac{c}{2}\right)\right) sin\left(\frac{C}{2}\right)} \tag{9.2d}$$

$$= \frac{2\, sin\left(\frac{s-a}{2}\right) sin\left(\frac{s-b}{2}\right)}{2\, cos\left(\frac{s}{2}\right) cos\left(\frac{s-c}{2}\right)} \sqrt{\frac{sin(s) sin(s-c)}{sin(s-a) sin(s-b)}} \tag{9.2e}$$

$$= \sqrt{tan\left(\frac{s}{2}\right) tan\left(\frac{s-a}{2}\right) tan\left(\frac{s-b}{2}\right) tan\left(\frac{s-c}{2}\right)} \tag{9.2f}$$

where equation (9.2b) follows by Corollary 6.2, equation (9.2d) by Theorem 7.1, and equation (9.2e) by Theorem 6.1 and the analogs for C of equation (7.2) and equation (7.3). □

The following formula is the spherical equivalent of the formula for the area of a plane right triangle: $\frac{1}{2}ab$.

Theorem 9.2: *Suppose* $\triangle ABC$ *is a right spherical triangle with right angle at* C *and sides* a, b, *and* c *opposite the corresponding angle. We have the following formula for the spherical excess,* E, *of* $\triangle ABC$*:*

$$tan\left(\frac{E}{2}\right) = tan\left(\frac{a}{2}\right) tan\left(\frac{b}{2}\right) \tag{9.3}$$

Proof: Use equation (7.13b) of Corollary 7.2, with $C = \frac{\pi}{2}$, to get

$$tan\left(\frac{E}{2}\right) = tan\left(\frac{A+B}{2} - \frac{\pi}{4}\right) \tag{9.4a}$$

$$= \frac{tan\left(\frac{A+B}{2}\right) - 1}{tan\left(\frac{A+B}{2}\right) + 1} \tag{9.4b}$$

$$= tan\left(\frac{a}{2}\right) tan\left(\frac{b}{2}\right) \tag{9.4c}$$

□